Filomat 32:9 (2018), 3381–3395
https://doi.org/10.2298/FIL1809381C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Monotone Iterative Technique for Impulsive Riemann-Liouville Fractional Differential Equations

Renu Chaudhary[a], **Dwijendra N Pandey**[b]

[a] *Department of Basic and Applied Sciences, School of Engineering, G D Goenka University, Sohna-122103, Gurugram, India*
[b] *Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee-247667, India.*

Abstract. In this article, Monotone iterative technique coupled with the method of lower and upper solutions is employed to discuss the existence and uniqueness of mild solution to an impulsive Riemann-Liouville fractional differential equation. The results are obtained using the concept of measure of noncompactness, semigroup theory and generalized Gronwall inequality for fractional differential equations. At last, an example is given to illustrate the applications of the main results.

1. Introduction

Fractional differential equations are generalizations of ordinary differential equations to an arbitrary order. Due to the nonlocal property fractional differential operators provide an appropriate tool for the description of hereditary properties of various materials and have lots of applications in science and engineering[4, 5, 17, 23]. Motivated by these facts, research in this area has grown significantly in the past few years and solutions of fractional differential equations in analytical and numerical senses have been discussed in large scale. For more details on fractional differential equations and applications, we refer the reader to the books [1, 11, 16, 30] and papers [7, 8, 15, 19, 33, 34, 37–41].

In recent years, the theory of impulsive differential equations has become an important area of investigation as it provide understanding of mathematical models to simulate the dynamics of processes in which sudden and discontinuous jumps occurs. Such processes are naturally observed in mechanics, electrical engineering, medicine, biology, ecology, etc. For a good introduction and applications to such equations we refer the reader to the books [20, 32] and papers [13, 21, 22, 40] and referencer therein.

On the other hand, the monotone iterative technique and its associated method of lower and upper solutions for nonlinear differential equations have been given considerable attention in recent years. In monotone iterative technique, starting from a pair of ordered lower and upper solutions, two monotone sequences are constructed such that they uniformly converge to the extremal solutions of the given problem in a closed set generated by upper and lower solutions. There has been a significant theoretical development in monotone iterative technique in recent years see [8, 10, 15, 18, 27–29, 35, 40, 41]. For details on upper and

2010 *Mathematics Subject Classification*. Primary 34G20, 34K30, 34K40, 34K50 (mandatory); Secondary 35K90
Keywords. Monotone iterative technique, Measure of noncompactness, Semigroup theory, Lower and upper solutions
Received: 04 July 2017; Revised: 03 May 2018; Accepted: 10 October 2018
Communicated by Naseer Shahzad
Email addresses: rrenu94@gmail.com (Renu Chaudhary), dwij.iitk@gmail.com (Dwijendra N Pandey)

lower solutions of fractional differential equations see [2, 24–26, 31] and paper cited therein.
In [33] Lakshmikantham and Vatsala discussed the monotone iterative technique for the differential equation

$$\begin{cases} {}^{L}D^{q}u(t) = f(t,u), & t \in (0,T]; \\ u(0) = u_0, \end{cases} \tag{1}$$

where ${}^{L}D^{q}$ is the Riemann-Liouville fractional derivative of order $0 < q < 1$. They prove some comparison results and global existence of solutions of (1). Later on, in [38] Shuqin discussed the monotone iterative method for the following initial value problem involving Riemann-Liouville fractional derivative

$$\begin{cases} {}^{L}D^{q}u(t) = f(t,u), & t \in (0,T]; \\ t^{1-\alpha}u(t)|_{t=0} = u_0, \end{cases} \tag{2}$$

where $0 < T < \infty$, and ${}^{L}D^{q}$ is Riemann-Liouville fractional derivative of order $0 < q < 1$. In [7] Wang studied monotone iterative technique for boundary value problems of a nonlinear fractional differential equation with deviating arguments. Recently in [39] authors studied (2) with a new condition on the nonlinear term f to guarantee the existence of solution of (2).
Motivated by the above work, this paper is concerned with the existence results for the following impulsive Riemann-Liouville fractional differential equations

$$\begin{cases} {}^{L}D_{t}^{q}u(t) = Ax(t) + F(t,u(t)), & t \in J = [0,a],\ t \neq t_i; \\ \Delta I_{t_i}^{1-q}u(t_i) = G_i(t_i,u(t_i)), & i = 1,2,\dots m; \\ I_{t}^{1-q}u(0) = u_0, \end{cases} \tag{3}$$

where ${}^{L}D_{t}^{q}$ denotes the Riemann-Liouville fractional derivative of order $q \in (0,1]$. A is a closed densely defined linear operator which generates a strongly continuous semigroup $\{T(t)\}_{t\geqslant 0}$ of bounded linear operators on a Banach space X and there exists $M \geqslant 1$ such that $\sup_{t\in J}\|T(t)\| \leqslant M$. $F : J \times X \to X$ and $G_i : J \times X \to X$ are given function to be specified later. $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = a$ are impulsive points. $\Delta I_{t_i}^{1-q}u(t_i)$ represent the jump of $u(t)$ at $t = t_i$ i.e. $\Delta I_{t_i}^{1-q}u(t_i) = I_{t_i^+}^{1-q}u(t_i^+) - I_{t_i^-}^{1-q}u(t_i^-) = \Gamma(q)[\lim_{t\to t_i^+}(t-t_i)^{1-q}u(t) - \lim_{t\to t_k^-}(t-t_i)^{1-q}u(t)]$(See [1, Lemma 3.2, Chapter 3]) where $I_{t_i^+}^{1-q}u(t_i^+)$ and $I_{t_i^-}^{1-q}u(t_i^-)$ represent the right and left limits of $I_t^{1-q}u(t)$ at $t = t_i$ respectively.

Monotone iterative technique for Riemann-Liouville fractional differential equations have been studied by many authors (see [6], [25] [33], [38], [39]) but a new semigroup theoretical approach to find the existence of solution to such problems has been introduced in this paper. Moreover, Most of the existing articles are only devoted to study the monotone iterative technique for Riemann-Liouville fractional differential equation, up until now monotone iterative technique for impulsive Riemann-Liouville fractional differential equation, has not been considered in the literature. Motivated by these facts, in this paper a new monotone iterative method has been established to find the existence and uniqueness of mild solutions to impulsive Riemann-Liouville fractional differential equations, which will provide an effective way to deal with such problems. The rest of the paper is organized as follows: In Section 2, we have some basic definitions, notations and lemmas which will be used later in this paper. In Section 3, we study the existence and uniqueness of extremal mild solution to the given system (3). At the end, in Section 4, we discuss an example to illustrate our results.

2. Preliminaries

Let X be an ordered Banach space with norm $\|\cdot\|$. Define a partial order $\leqslant$ in X with respect to positive cone $\mathcal{P} = \{u \in X : u \geqslant \delta\}$($\delta$ is the zero element of X). Here $u \leqslant v$ if and only if $v-u \in \mathcal{P}$. We symbolize $u < v$ to indicate $u \leqslant v$ but $u \neq v$. Let $AC(J,X)$ be the space of all absolutely continuous functions on J. Let $C(J,X)$ be the Banach space of all continuous X-valued functions on interval J with the norm $\|u\|_C = \sup\{\|u(t)\| : t \in J\}$.

Let $C_{1-q}(J,X) = \{u : t^{1-q}u(t) \in C(J,X)\}$ with the norm $\|u\|_{C_{1-q}} = \sup\{t^{1-q}\|u(t)\| : t \in J\}$. For investigation of impulsive conditions, consider the piecewise continuous Banach space $\mathcal{P}C_{1-q}(J,X) = \{u : (t-t_i)^{1-q}u(t) \in C((t_i,t_{i+1}],X)$ and $\lim_{t\to t_i^+}(t-t_i)^{1-q}u(t)$ exists, $i = 0,1,2,\ldots,m\}$, with the norm

$$\|u\|_{\mathcal{P}C_{1-q}} = \max\{\sup_{t\in(t_i,t_{i+1}]}(t-t_i)^{1-q}\|u(t)\| : i = 0,1,2,\ldots,m\}.$$

Definition 2.1. *Let X be an ordered Banach space with zero element δ. A cone $\mathcal{P} \subset X$ is called normal if there exists a real number $N > 0$ such that for all $u, v \in X$*

$$\delta \leqslant u \leqslant v \Rightarrow \|u\| \leqslant N\|v\|.$$

The smallest positive number N satisfying the above condition is called the normal constant of $\mathcal{P}$.

Definition 2.2. *Let X be an ordered Banach space. A cone $\mathcal{P} \subset X$ is called regular if every increasing sequence which is bounded from above is convergent i.e. if $\{u_n\}$ be a sequence such that*

$$u_1 \leqslant u_2 \leqslant \cdots \leqslant u_n \leqslant \cdots \leqslant v.$$

for some $v \in X$, then there is $u \in X$ with $\|u_n - u\| \to 0$ as $n \to \infty$. Equivalently, a cone $\mathcal{P} \subset X$ is called regular if every decreasing sequence which is bounded from below is convergent. Clearly, a regular cone is a normal cone.

Definition 2.3. *[30] The fractional integral of order q for a function F is defined by*

$$I_t^q F(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}F(s)ds, \quad t > 0, \quad q > 0.$$

provided the right hand side is pointwise defined on $[0,\infty)$. Here Γ is the gamma function.

Definition 2.4. *[1] The Riemaan-Liouville fractional derivative of order q for a function F is defined by*

$${}^L D_t^q F(t) = \frac{1}{\Gamma(n-q)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-q-1}F(s)ds,$$

provided the right hand side is pointwise defined on $[0,\infty)$. Here $n-1 < q < n$, $n = [q]+1$ and $[q]$ denotes the integral part of the real number q.

Lemma 2.5. *[1] Let $q \in (0,1]$. If $u \in \mathcal{P}C_{1-q}(J,X)$ and $I_t^{1-q}u(t) \in AC(J,X)$, then*

$$I_t^q D_t^q u(t) = \begin{cases} u(t) - I_t^{1-q}u(t)|_{t=0}\frac{t^{q-1}}{\Gamma(q)}, & t \in [0,t_1]; \\ u(t) - \displaystyle\sum_{j=1}^{i}\frac{\Delta I_t^{1-q}u(t_i)}{\Gamma(q)}(t-t_j)^{q-1} - I_t^{1-q}u(t)|_{t=0}\frac{t^{q-1}}{\Gamma(q)}, & t \in (t_i,t_{i+1}]. \end{cases}$$

where $\Delta I_t^{1-q}u(t_i) = I_t^{1-q}u(t_i^+) - I_t^{1-q}u(t_i^-)$, $i = 1,2,\ldots,m$.

Using the idea of [40], [41], we adopt the following definition of mild solution of (3).

Definition 2.6. *A function $u \in \mathcal{P}C_{1-q}(J,X)$ is called a mild solution of (3) if u satisfies the following integral equation*

$$u(t) = \begin{cases} t^{q-1}T_q(t)u_0 + \int_0^t (t-s)^{q-1}T_q(t-s)F(s,u(s))ds, & t \in [0,t_1]; \\ t^{q-1}T_q(t)u_0 + \displaystyle\sum_{j=1}^{i} T_q(t-t_j)(t-t_j)^{q-1}G_j(t_j,u(t_j)) & \\ + \int_0^t (t-s)^{q-1}T_q(t-s)F(s,u(s))ds, & t \in (t_i,t_{i+1}],\ i = 1,2,\ldots,m. \end{cases}$$

where

$$T_q(t) = q\int_0^\infty \theta\zeta_q(\theta)T(t^q\theta)d\theta,$$
$$\zeta_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}}\psi_q(\theta^{-\frac{1}{q}}),$$
$$\psi_q(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-nq-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), 0 < \theta < \infty.$$

ζ_q *is a probability density function defined on* $(0,\infty)$ *i.e.,* $\zeta_q(\theta) \geqslant 0$ *and* $\int_0^\infty \zeta_q(\theta)d\theta = 1$.

Lemma 2.7. *[37] The operator* $\{T_q(t), t \geqslant 0\}$ *is a bounded linear operator such that*

(i) $\|T_q(t)z\| \leqslant \frac{M}{\Gamma(q)}\|z\|$, *for any* $z \in X$.

(ii) *The operator* $\{T_q(t), t \geqslant 0\}$ *is strongly continuous i.e. for every* $z \in X$ *and* $0 < t' < t'' \leqslant a$, *we have*

$$\|T_q(t'')z - T_q(t')z\| \to 0, \quad \text{as } t'' \to t';$$

(iii) *If* $T(t)$ *is compact, then* $T_q(t)$ *is also compact operator for every* $t > 0$.

Definition 2.8. *A function* $u \in \mathcal{PC}_{1-q}(J, X)$ *is called a lower solution of (3) if it satisfies the following inequality*

$$\begin{cases} {}^LD_t^q u(t) \leqslant Au(t) + F(t, u(t)), \quad t \in J = (0, a],\ t \neq t_i; \\ \Delta I_{t_i}^{1-q}u(t_i) \leqslant G_i(t_i, u(t_i)), \quad i = 1, 2, \dots m; \\ I_t^{1-q}u(0) \leqslant u_0, \end{cases}$$

If all the inequalities are reversed, it is called an upper solution of (3).

Definition 2.9. *A* C_0- *semigroup* $\{T(t)\}_{t\geqslant 0}$ *in X is called a positive semigroup, if* $T(t)x \geqslant \delta$ *holds for all* $x \geqslant \delta$ *and* $t \geqslant 0$.

Let $\alpha(\cdot)$ denote the Kuratowski measure of noncompactness of the bounded set. For details of definition and properties of the measure of noncompactness, see [9, 14]. The following lemmas will be used in the proof of main results.

Lemma 2.10. *[12] For any* $B \subset \mathcal{PC}(J, X)$, *set* $B(t) = \{b(t) : b \in B\}$. *If B is bounded in* $C(J, X)$, *then* $B(t)$ *is bounded in X and* $\alpha(B) = \sup_{t\in J}\alpha(B(t))$.

Lemma 2.11. *[9] If* $\{b_n\}_{n=1}^\infty \subset L^1(J, X)$ *and there exists an* $c \in L^1(J, X)$ *such that* $\|b_n(t)\| \leqslant c(t)$, *a.e.* $t \in J$, *then* $\alpha(\{b_n(t)\}_{n=1}^\infty)$ *is integrable and*

$$\alpha\Big(\Big\{\int_0^t b_n(s)ds\Big\}_{n=1}^\infty\Big) \leqslant 2\int_0^t \alpha(\{b_n(s)\}_{n=1}^\infty ds.$$

Lemma 2.12. *[3] If B is bounded subset of X, then there exists* $\{b_n\}_{n=1}^\infty \subset B$, *such that* $\alpha(B) \leqslant 2\alpha(\{b_n\}_{n=1}^\infty)$.

Lemma 2.13. *[36](Generalized Gronwall inequality for fractional differential equation)Suppose* $a \geqslant 0$, $\beta > 0$, $c(t)$ *and* $z(t)$ *be the nonnegative locally integrable functions on* $0 \leqslant t < T < +\infty$ *with*

$$z(t) \leqslant c(t) + a\int_0^t (t-s)^{\beta-1}z(s)ds,$$

then

$$z(t) \leqslant c(t) + \int_0^t \Big[\sum_{n=1}^\infty \frac{(a\Gamma(\beta))^n}{\Gamma(n\beta)}(t-s)^{n\beta-1}c(s)\Big]ds, \quad 0 \leqslant t < T.$$

Evidently, $\mathcal{PC}_{1-q}(J,X)$ is also an ordered Banach space with partial order $\leqslant$ reduced by a positive cone $\bar{P} = \{u \in \mathcal{PC}_{1-q}(J,X) : u(t) \geqslant \delta, t \in J\}$ with normal constant N. For $x, y \in \mathcal{PC}_{1-q}(J,X)$ with $x \leqslant y$ we denote the ordered interval $[x,y] = \{u \in \mathcal{PC}_{1-q}(J,X), x \leqslant u \leqslant y\}$ in $\mathcal{PC}_{1-q}(J,X)$ and $[x(t),y(t)] = \{u \in X,\ x(t) \leqslant u(t) \leqslant y(t)\}$ in X.

3. Main Results

To prove our results, we will require the following assumptions:

(i) The function $F(t,\cdot) : X \to X$ is continuous for a.e. $t \in J$ and for all $v \in X$, the function $F(\cdot,v) : J \to X$ is strongly measurable.

(ii) For any upper and lower solutions $x_0, y_0 \in \mathcal{PC}_{1-q}(J,X)$ with $x_0 \leqslant y_0$ of the system (3), the function $F(t,\cdot) : X \to X$ satisfies

$$F(t,v_1) \leqslant F(t,v_2),$$

for any $t \in J$. Where $v_1, v_2 \in X$ with $x_0 \leqslant v_1 \leqslant v_2 \leqslant y_0$.

(iii) The function $G_i : J \times X \to X$ is increasing, continuous and compact and there exists a positive constant $L' > 0$ such that

$$\|G_i(t_1,v_1) - G_i(t_2,v_2)\| \leqslant L'[|t_1 - t_2| + \|v_1 - v_2\|],$$

for all $t_1, t_2 \in J$, $v_1, v_2 \in X$ and each $i \in \mathbb{N}$.

(iv) There exists a constant $L \geqslant 0$ for any bounded $U \subset \mathcal{PC}_{1-q}(J,X)$ such that

$$\alpha(F(t,U(t))) \leqslant L\alpha(U(t)), \quad \text{for a.e. } t \in J.$$

3.1. *The case that $T(t)$ is compact*

Theorem 3.1. *Let X be an ordered Banach space, whose positive cone $\mathcal{P}$ is normal with normal constant N. Assume that $T(t)(t \geqslant 0)$ is positive compact semigroup and the system (3) has upper and lower solutions $x_0, y_0 \in \mathcal{PC}_{1-q}(J,X)$ with $x_0 \leqslant y_0$ and the assumptions (i)-(iii) holds. Then the system (3) has minimal and maximal solutions between x_0 and y_0.*

Proof. Let $E = [x_0, y_0] = \{u \in \mathcal{PC}_{1-q}(J,X) : x_0 \leqslant u \leqslant y_0\}$. Define a map $\Theta : E \to \mathcal{PC}_{1-q}(J,X)$ by

$$(\Theta u)(t) = \begin{cases} t^{q-1}T_q(t)u_0 + \int_0^t (t-s)^{q-1}T_q(t-s)F(s,u(s))ds, & t \in [0,t_1]; \\ t^{q-1}T_q(t)u_0 + \sum\limits_{j=1}^{i} T_q(t-t_j)(t-t_j)^{q-1}G_j(t_j,u(t_j)) & \\ + \int_0^t (t-s)^{q-1}T_q(t-s)F(s,u(s))ds, & t \in (t_i,t_{i+1}],\ i = 1,2,\dots,m. \end{cases} \tag{4}$$

It is clear that Θ is well defined.
Using assumption (*ii*), for any $u \in E$, we have

$$F(t,x_0(t)) \leqslant F(t,u(t)) \leqslant F(t,y_0(t)).$$

Since the positive cone P is normal therefore there exists a constant $C > 0$ such that

$$\|F(t,u(t))\| \leqslant C, \quad \text{for any } u \in E.$$

The rest of the proof is divided into four steps:
Step 1: The map Θ is continuous in E.

Let $\{u_n\} \in E$ be a sequence such that $\{u_n\} \to u \in E$ as $n \to \infty$. Using assumptions (*i*) and (*iii*), for almost every $t \in J$, we get

$$F(t, u_n(t)) \to F(t, u(t)), \tag{5}$$
$$G_i(t, u_n(t)) \to G_i(t, u(t)), \tag{6}$$

as $n \to \infty$. For $t \in [0, t_1]$, using (5) together with Lebesgue dominated convergence theorem, we get

$$\begin{aligned} t^{1-q}\|(\Theta u_n)(t) - (\Theta u)(t)\| &\leqslant \frac{Mt^{1-q}}{\Gamma q}\int_0^t (t-s)^{q-1}\|F(s, u_n(s)) - F(s, u(s))\|ds \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Similarly, for $t \in (t_i, t_{i+1}]$, $i = 1, 2, \ldots$, we obtain

$$\begin{aligned} (t-t_i)^{1-q}\|(\Theta u_n)(t) - (\Theta u)(t)\| &\leqslant \frac{M(t-t_i)^{1-q}}{\Gamma q}\int_0^t (t-s)^{q-1}\|F(s, u_n(s)) - F(s, u(s))\|ds \\ &\quad + \sum_{j=1}^{i}(t-t_i)^{1-q}(t-t_j)^{q-1}\|T_q(t-t_j)\|\|G_j(t_j, u_n(t_j)) - G_j(t_j, u(t_j))\| \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Hence the map Θ is continuous in E.
Step 2: Θ is an increasing monotonic operator.
Since $T_q(t)$ is a positive operator, combine this with assumptions (*ii*) and (*iii*), we get Θ is an increasing operator in E.
Now show that $x_0 \leqslant \Theta x_0$ and $\Theta y_0 \leqslant y_0$.
For this, let $h(t) =^L D_t^q x_0(t) - Ax_0(t)$. Then by Definition 2.8, $h(t) \in \mathcal{P}C_{1-q}$ and $h(t) \leqslant F(t, x_0(t))$. Using Definition (2.6) and positivity of the operator $T_q(t)$, for $t \in [0, t_1]$, we get

$$\begin{aligned} x_0(t) &= t^{q-1}T_q(t)x_0(t)|_{t=0} + \int_0^t (t-s)^{q-1}T_q(t-s)h(s)ds \\ &\leqslant t^{q-1}T_q(t)x_0(t)|_{t=0} + \int_0^t (t-s)^{q-1}T_q(t-s)F(s, x_0(s))ds \\ &\leqslant \Theta x_0(t). \end{aligned}$$

For $t \in (t_i, t_{i+1}]$, $i = 1, 2, \ldots$, we obtain

$$\begin{aligned} x_0(t) &= t^{q-1}T_q(t)x_0(t)|_{t=0} + \sum_{j=1}^{i} T_q(t-t_j)(t-t_j)^{q-1}G_j(t_j, x_0(t_j)) + \int_0^t (t-s)^{q-1}T_q(t-s)h(s)ds \\ &\leqslant t^{q-1}T_q(t)x_0(t)|_{t=0} + \sum_{j=1}^{i} T_q(t-t_j)(t-t_j)^{q-1}G_j(t_j, x_0(t_j)) + \int_0^t (t-s)^{q-1}T_q(t-s)F(s, x_0(s))ds \\ &\leqslant \Theta x_0(t). \end{aligned}$$

Hence $x_0(t) \leqslant \Theta x_0(t)$ for all $t \in J$. Similarly, we can show that $\Theta y_0 \leqslant y_0$. Hence Θ is an increasing monotonic operator.
Step 3: $\Theta(E)$ is equicontinuous on J.

For any $u \in E$ and $s_1, s_2 \in [0, t_1]$ such that $0 < s_1 < s_2 \leqslant t_1$, we have

$$\begin{aligned}
\|s_2^{1-q}(\Theta u)(s_2) - s_1^{1-q}(\Theta u)(s_1)\| &\leqslant \|T_q(s_2)u_0 - T_q(s_1)u_0\| + \left\| s_2^{1-q}\int_{s_1}^{s_2}(s_2-s)^{q-1}T_q(s_2-s)F(s,u(s))ds \right\| \\
&\quad + \left\| \int_0^{s_1}[s_2^{1-q}(s_2-s)^{q-1} - s_1^{1-q}(s_1-s)^{q-1}]T_q(s_2-s)F(s,u(s))ds \right\| \\
&\quad + \left\| s_1^{1-q}\int_0^{s_1}(s_1-s)^{q-1}[T_q(s_2-s) - T_q(s_1-s)]F(s,u(s))ds \right\| \\
&\leqslant \|T_q(s_2)u_0 - T_q(s_1)u_0\| \\
&\quad + \frac{MC}{\Gamma(q)}\int_{s_1}^{s_2} s_2^{1-q}(s_2-s)^{q-1}ds \\
&\quad + \frac{MC}{\Gamma(q)}\int_0^{s_1} |s_2^{1-q}[(s_2-s)^{q-1} - (s_1-s)^{q-1}] + (s_1-s)^{q-1}[s_2^{1-q} - s_1^{1-q}]|ds \\
&\quad + C\int_0^{s_1} s_1^{1-q}(s_1-s)^{q-1}\|T_q(s_2-s) - T_q(s_1-s)\|ds \\
&= \sum_{i=1}^{4} J_i.
\end{aligned}$$

Using Lemma 2.7(ii), $J_1 \to 0$ as $s_2 \to s_1$. Moreover, it is easy to see that $J_2, J_3 \to 0$ as $s_2 \to s_1$. For any $\epsilon \in (0, s_1)$, we have

$$\begin{aligned}
J_4 &\leqslant C\int_0^{s_1-\epsilon} s_1^{1-q}(s_1-s)^{q-1}\|T_q(s_2-s) - T_q(s_1-s)\|ds \\
&\quad + C\int_{s_1-\epsilon}^{s_1} s_1^{1-q}(s_1-s)^{q-1}\|T_q(s_2-s) - T_q(s_1-s)\|ds \\
&\leqslant C\int_0^{s_1-\epsilon} s_1^{1-q}(s_1-s)^{q-1} \sup_{s\in[0,s_1-\epsilon]}\|T_q(s_2-s) - T_q(s_1-s)\|ds \\
&\quad + \frac{2MC}{\Gamma(q)}\int_{s_1-\epsilon}^{s_1} s_1^{1-q}(s_1-s)^{q-1}ds \\
&\leqslant C\int_0^{s_1-\epsilon} s_1^{1-q}(s_1-s)^{q-1} \sup_{s\in[0,s_1-\epsilon]}\|T_q(s_2-s) - T_q(s_1-s)\|ds \\
&\quad + \frac{2MCt_1^{1-q}}{\Gamma(q+1)}\epsilon^q ds \\
&\leqslant C\int_0^{s_1-\epsilon} s_1^{1-q}s^{q-1} \sup_{s\in[0,s_1-\epsilon]}\|T_q(s_2+s-s_1) - T_q(s)\|ds \\
&\quad + \frac{2MCt_1^{1-q}}{\Gamma(q+1)}\epsilon^q ds \\
&\to 0 \quad \text{as } s_2 \to s_1 \text{ and } \epsilon \to 0.
\end{aligned}$$

Similarly, for $t_i < s_1 < s_2 \leqslant t_{i+1}$, we can show that

$$\|(s_2-t_i)^{1-q}(\Theta u)(s_2) - (s_1-t_i)^{1-q}(\Theta u)(s_1)\| \to 0 \quad \text{as } s_2 \to s_1.$$

for every $i = 1, 2, \dots, m$. Hence $\Theta(E)$ is equicontinuous on J.

Step 4: The set $G(t) = \{(\Theta u)(t) : u \in E\}$, $t \in J$, is relatively compact in X.
Let

$$(\Theta u)(t) = (\Theta_1 u)(t) + (\Theta_2 u)(t),$$

where

$$(\Theta_1 u)(t) = t^{q-1}T_q(t)u_0 + \int_0^t (t-s)^{q-1}T_q(t-s)F(s,u(s))ds, \quad t \in [t_i, t_{i+1}], i = 0,1,2\ldots$$

$$(\Theta_2 u)(t) = \sum_{j=1}^{i} T_q(t-t_j)(t-t_j)^{q-1}G_j(t_j, u(t_j)) \quad t \in [t_i, t_{i+1}], i = 1,2\ldots.$$

For any $t \in [t_i, t_{i+1}]$, $i = 0,1,2\ldots$, choose $\epsilon \in (t_i, t)$ and $\nu > 0$ such that

$$\begin{aligned}(\Theta_1 u^{\epsilon,\nu})(t) &= qt^{q-1}\int_\nu^\infty \theta\zeta_q(\theta)T(t^q\theta)u_0 d\theta + q\int_0^{t-\epsilon}\int_\nu^\infty (t-s)^{q-1}\theta\zeta_q(\theta)T((t-s)^q\theta)F(s,u(s))d\theta ds \\ &\leqslant T(\epsilon^q\nu)\Big[qt^{1-q}\int_\nu^\infty \theta\zeta_q(\theta)T(t^q\theta - \epsilon^q\nu)u_0 d\theta \\ &\quad + q\int_0^{t-\epsilon}\int_\nu^\infty (t-s)^{q-1}\theta\zeta_q(\theta)T((t-s)^q\theta - \epsilon^q\nu)F(s,u(s))d\theta ds\Big].\end{aligned}$$

Note that $\theta \geqslant \nu$ and $t - \epsilon \geqslant s$ so $(t-s)^q\theta - \epsilon^q\nu \geqslant 0$. Therefore from Lemma 2.7(iii), The operators $T(\epsilon^q\nu)$ and $T(t^q\theta - \epsilon^q\nu)$ are compact. Hence $(\Theta_1 u^{\epsilon,\nu})(t)$ is relatively compact in X.
Now, we have

$$\begin{aligned}t^{1-q}\|(\Theta_1 u)(t) - (\Theta_1 u^{\epsilon,\nu})(t)\| &= \|q\int_0^\nu \theta\zeta_q(\theta)T(t^q\theta)u_0 d\theta\| \\ &\quad + \|qt^{1-q}\int_0^t\int_0^\nu (t-s)^{q-1}\theta\zeta_q(\theta)T((t-s)^q\theta)F(s,u(s))d\theta ds\| \\ &\quad + \|qt^{1-q}\int_{t-\epsilon}^t\int_\nu^\infty (t-s)^{q-1}\theta\zeta_q(\theta)T((t-s)^q\theta)F(s,u(s))d\theta ds\| \\ &\leqslant qM\|u_0\|\int_0^\nu \theta\zeta_q(\theta)d\theta \\ &\quad + qCMa^{1-q}\int_0^t (t-s)^{q-1}ds\int_0^\nu \theta\zeta_q(\theta)d\theta \\ &\quad + qCMa^{1-q}\int_{t-\epsilon}^t (t-s)^{q-1}ds\int_0^\infty \theta\zeta_q(\theta)d\theta \\ &\to 0 \quad \text{as } \epsilon \to 0,\ \nu \to 0.\end{aligned}$$

i.e. relatively compact sets $(\Theta_1 u^{\epsilon,\nu})(t)$ are arbitrarily close to the set $\{(\Theta_1 u)(t) : u \in E\}$. Hence the set $\{(\Theta_1 u)(t) : u \in E\}$ is relatively compact in X.
Moreover, for $t \in [t_j, t_{j+1}]$, $j = 1,2\ldots$, using assumption (iii) and Lemma 2.7(iii), we get $\{(\Theta_2 u)(t) : u \in E\}$ is relatively compact in X. Hence $G(t) = \{(\Theta u)(t) : u \in E\}$, $t \in J$, is relatively compact in X. From Arzela-Ascoli theorem, we get $\Theta : E \to E$ is relatively compact.
Now define two sequences $\{x_n\}$ and $\{y_n\}$, by the iterative scheme

$$x_n = \Theta x_{n-1} \quad \text{and} \quad y_n = \Theta y_{n-1}, \quad n = 1,2,\ldots. \tag{7}$$

Since Θ is an increasing monotonic operator, we have

$$x_0 \leqslant x_1 \leqslant \cdots \leqslant x_n \leqslant \cdots \leqslant y_n \leqslant \cdots \leqslant y_1 \leqslant y_0. \tag{8}$$

Since $\Theta : E \to E$ is relatively compact therefore there exists a convergent subsequence $\{x_{n_\kappa}\}$ of $\{x_n\}$. Let $\{x_{n_\kappa}\}$ converges to x^*. Therefore for each $\epsilon > 0$ there exists an n_κ such that

$$\|x_{n_\kappa} - x^*\| < \frac{\epsilon}{1+N}.$$

For $n_\kappa \leqslant n$, we have

$$x_{n_\kappa} \leqslant x_n \leqslant x^*,$$

i.e.

$$\delta \leqslant x_n - x_{n_\kappa} \leqslant x^* - x_{n_\kappa}.$$

Using the normality of positive cone P, we get

$$\|x_n - x_{n_\kappa}\| \leqslant N\|x^* - x_{n_\kappa}\|.$$

Thus

$$\begin{aligned}\|x_n - x^*\| &\leqslant \|x_n - x_{n_\kappa}\| + \|x_{n_\kappa} - x^*\| \\ &\leqslant (N+1)\|x_{n_\kappa} - x^*\| \\ &\leqslant \epsilon.\end{aligned}$$

Hence $x_n \to x^*$. Now using equation (4) and (7), we get

$$x_n(t) = \begin{cases} t^{q-1}T_q(t)u_0 + \int_0^t (t-s)^{q-1}T_q(t-s)F(s, x_{n-1}(s))ds, & t \in [0, t_1]; \\ t^{q-1}T_q(t)u_0 + \sum_{j=1}^{i} T_q(t-t_j)(t-t_j)^{q-1}G_j(t_j, x_{n-1}(t_j)) & \\ + \int_0^t (t-s)^{q-1}T_q(t-s)F(s, x_{n-1}(s))ds, & t \in (t_i, t_{i+1}],\ i = 1, 2, \dots, m. \end{cases}$$

as $n \to \infty$ and using Lebesgue dominated convergence theorem, we have

$$x^*(t) = \begin{cases} t^{q-1}T_q(t)u_0 + \int_0^t (t-s)^{q-1}T_q(t-s)F(s, x^*(s))ds, & t \in [0, t_1]; \\ t^{q-1}T_q(t)u_0 + \sum_{j=1}^{i} T_q(t-t_j)(t-t_j)^{q-1}G_j(t_j, x^*(t_j)) & \\ + \int_0^t (t-s)^{q-1}T_q(t-s)F(s, x^*(s))ds, & t \in (t_i, t_{i+1}],\ i = 1, 2, \dots, m. \end{cases}$$

Here $x^* \in PC_{1-q}(J, X)$ and $x^* = \Theta x^*$. Hence x^* is a fixed point of Θ. Similarly, we can show that there exists $y^* \in PC_{1-q}(J, X)$ such that $y_n \to y^*$ as $n \to \infty$ and $y^* = \Theta y^*$. If $u \in E$ and u is a fixed point of Θ then by using monotonic increasing property of Θ, we get $x_1 \leqslant \Theta x_0 \leqslant \Theta u = u \leqslant \Theta y_0 \leqslant y_1$. By induction $x_n \leqslant u \leqslant y_n$. Using (8) and taking limit $n \to \infty$, we get $x_0 \leqslant x^* \leqslant u \leqslant y^* \leqslant y_0$. Hence x^*, y^* are the minimal and maximal mild solutions of (4) on $[x_0, y_0]$ respectively. □

3.2. The case that $T(t)$ is not compact

Theorem 3.2. *Let X be an ordered Banach space, whose positive cone P is normal with normal constant N. Assume that $T(t)(t \geqslant 0)$ is positive semigroup and the system (3) has upper and lower solutions $x_0, y_0 \in PC_{1-q}(J, X)$ with $x_0 \leqslant y_0$ and the assumptions (i)-(iv) holds. Then the system (3) has minimal and maximal solutions between x_0 and y_0.*

Proof. From Theorem 3.1, we have that $\Theta : E \to E$ is a continuous increasing monotonic operator. Now, define the sequences $\{x_n\}$ and $\{y_n\}$ as defined in Theorem 3.1, which satisfies equations (7) and (8).
We prove that $\{x_n\}$ and $\{y_n\}$ are uniformly convergent in J.
Let $S = \{x_n : n \in \mathbb{N}\}$ and $S_0 = \{x_{n-1} : n \in \mathbb{N}\}$. By normality of cone, S and S_0 are bounded. From $S_0 = S \cup \{x_0\}$ it follows that $\alpha(S_0(t)) = \alpha(S(t))$ for $t \in J$. Let

$$\varphi(t) := \alpha(S_0(t)) = \alpha(S(t)), \quad t \in J.$$

Since $S = \Theta(S_0)$, we have

$$\alpha(S(t)) = \alpha(\Theta S_0(t)).$$

For $t \in [0, t_1]$, we have,

$$\begin{aligned}
\varphi(t) &= \alpha\Big(T_q(t)\bar{x}_0 + t^{1-q}\int_0^t (t-s)^{q-1}T_q(t-s)F(s, x_{n-1}(s))ds\Big)\\
&\leqslant \frac{2Mt_1^{1-q}}{\Gamma(q)}\int_0^t (t-s)^{q-1}\alpha(F(s, x_{n-1}(s)))ds\\
&\leqslant \frac{2MLt_1^{1-q}}{\Gamma(q)}\int_0^t (t-s)^{q-1}\alpha(x_{n-1}(s))ds\\
&\leqslant \frac{2MLt_1^{1-q}}{\Gamma(q)}\int_0^t (t-s)^{q-1}\varphi(s)ds
\end{aligned}$$

Using Lemma 2.13, $\varphi(t) = 0$ for $t \in [0, t_1]$.
For $t \in (t_i, t_{i+1}]$, $i = 1, 2, \ldots, m$, we have

$$\begin{aligned}
\varphi(t) &= \alpha\Big((t-t_i)^{1-q}t^{q-1}T_q(t)\bar{x}_0 + (t-t_i)^{1-q}\sum_{j=1}^{i} T_q(t-t_j)(t-t_j)^{q-1}G_j(t_j, x_{n-1}(t_j))\\
&\quad + (t-t_i)^{1-q}\int_0^t (t-s)^{q-1}T_q(t-s)F(s, x_{n-1}(s))ds\Big)\\
&\leqslant \frac{2Ma^{1-q}}{\Gamma(q)}\int_0^t (t-s)^{q-1}\alpha(F(s, x_{n-1}(s)))ds\\
&\leqslant \frac{2Ma^{1-q}}{\Gamma(q)}\int_0^t (t-s)^{q-1}\alpha(x_{n-1}(s))ds\\
&\leqslant \frac{2MLa^{1-q}}{\Gamma(q)}\int_0^t (t-s)^{q-1}\varphi(s)ds
\end{aligned}$$

Using Lemma 2.13, $\varphi(t) = 0$ for $t \in [t_j, t_{j+1}]$. Hence, for any $t \in J$, $\varphi(t) = 0$ i.e. $\alpha(\Theta(S)) = 0$.
Thus the set $\{x_n : n \in \mathbb{N}\}$ is precompact in E. Therefore $\{x_n\}$ has a convergent sequence in E. From (8) we can see that $\{x_n\}$ is itself a convergent sequence. Therefore there exists $x^* \in E$ such that $x_n \to x^*$ as $n \to \infty$. Similarly there exists $y^* \in E$ such that $y_n \to y^*$ as $n \to \infty$. Using same argument as in Theorem 3.1, we get there exists x^* and y^* which are the minimal and maximal mild solutions of Riemann-Liouville fractional differential equation (3) in $[x_0, y_0]$ respectively. □

Corollary 3.3. *Let X be an ordered Banach space with regular positive cone $\mathcal{P}$. Assume that $T(t)(t \geqslant 0)$ is positive semigroup and the system (3) has upper and lower solutions $x_0, y_0 \in \mathcal{P}C_{1-q}(J, X)$ with $x_0 \leqslant y_0$, and the assumptions (i)-(iii) holds. Then the system (3) has minimal and maximal solutions between x_0 and y_0.*

Proof. Since the assumptions $(i)-(iii)$ holds, therefore equation (8) is satisfied. Let $\{x_n\}$ and $\{y_n\}$ be two increasing or decreasing sequences in E. Then using Definition (2.2) and assumption (ii), $\{f(t,x_n)\}$ is convergent. Therefore $\alpha(\{f(t,x_n)\}) = \alpha(\{x_n\}) = 0$. Hence assumption (iv) holds. Then from Theorem 3.2, the proof is complete. □

Corollary 3.4. *Let X be an ordered and weakly sequentially complete Banach space with normal positive cone P. Assume that $T(t)(t \geqslant 0)$ is positive semigroup and the system (3) has upper and lower solutions $x_0, y_0 \in \mathcal{P}C_{1-q}(J,X)$ with $x_0 \leqslant y_0$, and the assumptions (i)-(iii) holds. Then the system (3) has minimal and maximal solutions between x_0 and y_0.*

Proof. In an ordered and weakly sequentially complete Banach space, the normal cone P is regular. Then using Corollary 3.3, the proof may be completed. □

Now we shall prove the uniqueness of the solution of the system (3). For this we use the following assumptions

(v) The function $F : J \times X \to X$ is continuous and there exists $c \geqslant 0$ such that

$$F(t,x_2) - F(t,x_1) \leqslant c(x_2 - x_1),$$

for any $t \in J$, $x_1, x_2 \in X$ with $x_0 \leqslant x_1 \leqslant x_2 \leqslant y_0$.

(vi) The function $G_n : (t_j, t_{j+1}] \to X$ is continuous and there exists a constant $b > 0$ such that

$$G_n(t,x_2) - G_n(t,x_1) \leqslant b(x_2 - x_1),$$

for $x_1, x_2 \in X$ with $x_0 \leqslant x_1 \leqslant x_2 \leqslant y_0$.

Theorem 3.5. *Let X be an ordered Banach space, whose positive cone $\mathcal{P}$ is normal with normal constant N. Assume that $T(t)$ is a positive operator. Also assume that the system (3) has upper and lower solutions $x_0, y_0 \in \mathcal{P}C_{1-q}(J,X)$ with $x_0 \leqslant y_0$. If the assumptions (ii), (iii), (v) and (vi) hold with*

$$\frac{bMN}{\Gamma(q)} \sum_{j=1}^{i} (t_j - t_{j-1})^{q-1} + \frac{MNca^q}{\Gamma(q+1)} < 1, \quad \text{for } i = 1, 2, \ldots m. \tag{9}$$

Then the system (3) has a unique mild solution between x_0 and y_0.

Proof. Let $\{x_n\} \subset [x_0(t), y_0(t)]$ be a monotonic increasing sequence. Then for any $m, p = 1, 2, \ldots$ with $m > p$, using (ii), (v) and (vi), we have

$$\delta \leqslant F(t,x_m) - F(t,x_p) \leqslant c(x_m - x_p).$$

Using the normality of positive cone P, we get

$$||F(t,x_m) - F(t,x_p)|| \leqslant Nc||x_m - x_p||. \tag{10}$$

By the definition of measure of noncompactness, we obtain

$$\alpha(\{F(t,x_m)\}) \leqslant L\alpha(\{x_m\}),$$

where $L = Nc$. Thus the assumptions $(i)-(iv)$ are satisfied. Therefore by Theorem 3.2, there exists x^* and y^* which are the minimal and maximal mild solutions of (3) between x_0 and y_0 in E respectively.

Now, we will show that $x^* = y^*$ for every $[t_i, t_{i+1}]$, $i = 0, 1, \dots, m$.
For $t \in [0, t_1]$, we have

$$\begin{aligned}
\|x^* - y^*\|_{\mathcal{P}C_{1-q}} &= \|t^{1-q} \int_0^t (t-s)^{q-1} T_q(t-s)[F(s, x^*(s)) - F(s, y^*(s))]ds\| \\
&\leqslant t^{1-q} \int_0^t (t-s)^{q-1} \|T_q(t-s)\| \|F(s, x^*(s)) - F(s, y^*(s))\| ds \\
&\leqslant \frac{cNMt^{1-q}}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|x^*(s) - y^*(s)\| ds \\
&\leqslant \frac{cNMt_1^q}{\Gamma(q+1)} \|x^* - y^*\|_{\mathcal{P}C_{1-q}}.
\end{aligned}$$

Using equation (9), $\frac{cNMt_1^q}{\Gamma(q+1)} < 1$, so we obtain $\|x^* - y^*\|_{\mathcal{P}C_{1-q}} = 0$, i.e. $x^*(t) = y^*(t)$ for $t \in [0, t_1]$.
For $t \in (t_i, t_{i+1}]$, $i = 1, 2, \dots, m$, we have

$$\begin{aligned}
\|x^* - y^*\|_{PC_{1-q}} &= (t-t_i)^{1-q} \sum_{j=1}^{i} (t-t_j)^{q-1} \|T_q(t-t_j)\| G_j(t_j, x^*(t_j)) - G_j(t_j, y^*(t_j))\| \\
&\quad + (t-t_i)^{1-q} \int_0^t (t-s)^{q-1} \|T_q(t-s)\| \|F(s, x^*(s)) - F(s, y^*(s))\| ds \\
&\leqslant \frac{bMN}{\Gamma(q)} \sum_{j=1}^{i} (t_j - t_{j-1})^{q-1} (t_j - t_{j-1})^{1-q} \|x^*(t_j) - y^*(t_j)\| \\
&\quad + \frac{MNc(t-t_i)^{1-q}}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|x^*(s) - y^*(s)\| ds \\
&\leqslant \frac{bMN}{\Gamma(q)} \sum_{j=1}^{i} (t_j - t_{j-1})^{q-1} \|x^* - y^*\|_{PC_{1-q}} + \frac{MNca^q}{\Gamma(q+1)} \|x^* - y^*\|_{PC_{1-q}} \\
&= \Big[\frac{bMN}{\Gamma(q)} \sum_{j=1}^{i} (t_j - t_{j-1})^{q-1} + \frac{MNca^q}{\Gamma(q+1)} \Big] \|x^* - y^*\|_{PC_{1-q}}
\end{aligned}$$

Using equation (9), we obtain $\|x^* - y^*\|_{\mathcal{P}C_{1-q}} = 0$, i.e. $x^*(t) = y^*(t)$ for $t \in (t_i, t_{i+1}]$, $i = 1, 2, \dots, m$. Thus, we obtain $x^*(t) = y^*(t)$ for $t \in [0, a]$. Hence $x^* = y^*$ is the unique mild solution of the system (3), which can be acquired by the monotone iterative procedure beginning from x_0 and y_0. □

4. Discussions

In this paper, monotone iterative technique coupled with the method of lower and upper solution has been applied to show the existence and uniqueness of mild solution to impulsive Riemann-Liouville fractional differential equation (3). Here we have proved two existence results. In the first existence result, the semigroup $T(t)$ generated by the linear operator A is assumed to be compact. While in the second existence result, we relax the condition of compactness of the semigroup $T(t)$ that is we have assumed that the semigroup $T(t)$ is non-compact and the existence of the mild solution is shown using the theory of measure of noncompactness. Moreover, if the functions F and G satisfies Lipschitz type condition (*v*)-(*vi*), then the solution will be unique.
In some applications of partial differential equations (for example neutron transport equations, reaction diffusion equations, population models), the linear part generates a positive analytic semigroup in weakly sequentially complete Banach space. Therefore, if the assumptions (*i*)-(*iii*) are satisfied, one can easily apply Corollary (3.4) to these partial differential equations.

5. Example

Consider the following Riemann-Liouville fractional impulsive differential equation in an ordered Banach space $X = L^2[0,\pi]$:

$$\begin{cases} D^{1/2}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + \frac{e^t|u(t,x)|}{1+|u(t,x)|}, \quad t \in [0,\frac{1}{2}) \cup (\frac{1}{2},1], x \in [0,\pi]; \\ \Delta I_t^{1/2}u(\frac{1}{2},x) = \frac{|u(t,x)|}{1+|u(t,x)|}, \\ u(t,0) = u(t,\pi) = 0, \\ I_t^{1/2}u(t,x)|_{t=0} = u_0(x), \end{cases} \tag{11}$$

where $t \in [0,1]$ and $x \in [0,\pi]$. Let $\mathcal{P} = \{u \in X : u(\nu) \geqslant 0 \text{ a.e. } \nu \in [0,\pi]\}$. Then $\mathcal{P}$ is normal cone in Banach space X with normal constant $N = 1$. Define an operator $A : D(A) \subset X \to X$ by $Au = u''$ with domain

$$D(A) = \{u \in X : u, u' \text{ are absolutely continuous } u'' \in X, u(0) = u(\pi) = 0\}.$$

Clearly, A has a discrete spectrum with the eigenvalues of the form $-n^2$ for $n \in \mathbb{N}$, whose corresponding(normalized) eigenfunctions are given by $u_n(x) = \sqrt{2/\pi}\sin nx$ and can be written as

$$Au = -\sum_{n=1}^{\infty} n^2(u,u_n)u_n, \quad u \in D(A).$$

Then A generates an analytic semigroup of uniformly bounded linear operator $\{T(t)\}_{t\geqslant 0}$ in X given by

$$T(t)u = \sum_{n=1}^{\infty} e^{-n^2t}(u,u_n)u_n, \quad u \in X.$$

and

$$\|T(t)\| \leqslant e^{-1} < 1 = M.$$

Now define

$$\begin{aligned} y(t) &= u(t,x), \\ F(t,y(t)) &= \frac{e^t y(t)}{1+y(t)}, \\ G(t,y(t))|_{t=1/2} &= \frac{y(t)}{1+y(t)}. \end{aligned}$$

Thus, the aforementioned equation (11) can be written in the form

$$\begin{cases} {}^LD_t^{1/2}y(t) = Ay(t) + F(t,y(t)), \quad t \in J = [0,1],\ t \neq 1/2; \\ \Delta I_{t=1/2}^{1/2}y(1/2) = G(1/2, y(1/2)), \\ I_t^{1/2}y(0) = y_0, \end{cases} \tag{12}$$

Let $y_0(x) \geqslant 0$, $x \in [0,\pi]$ and there exists a function $\xi(t) > 0$ such that

$$\begin{cases} {}^LD_t^{1/2}\xi(t) \geqslant A\xi(t)(t) + F(t,\xi(t)), \quad t \in J = [0,1],\ t \neq 1/2; \\ \Delta I_{t=1/2}^{1/2}\xi(1/2) \geqslant G(1/2, \xi(1/2)), \\ I_t^{1/2}\xi(0) \geqslant y_0, \end{cases} \tag{13}$$

From (13), we get that $x_0 = 0$ and $y_0 = \xi(t)$ are the lower and upper solutions of the system (12) respectively. We can easily check that the functions F and G satisfies all the assumptions $(i) - (vi)$. Hence using Theorem 3.1 or 3.2 and 3.5, we conclude that, the given system (11) has the unique mild solution lying between the lower solution 0 and the upper solution $\xi(t)$.

References

[1] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, Theory and applications of fractional differential equations, North-Holland Math. Stud. 204, Elservier Science B.V., Amsterdam, 2006.
[2] C. C. Tisdell, Maximal solutions to fractional differential equations, Frac. Diff. Calculus 1 (2015), Vol. 5, 79-85.
[3] D. Bothe, Multivalued perturbations of m-accretive differential inclusions, Israel J. Math., 108(1998), 109-138.
[4] F. Metzler, W. Schick, H. G. Kilian and T. F. Nonnenmacher, Relaxation in filled polymers: A fractional calculus approach, J. Chem. Phys. 103 (1995), 7180-7186.
[5] F. Mainardi, Fractional calculus: Some basic problems in continuum and statistical mechanics, Fractals and Fractional Calculus in Continuum Mechanics, Springer-Verlag, Wien, 1997, 291-348.
[6] F. A. McRae, Monotone iterative technique and existence results for fractional differential equations, Nonlinear Anal. 71(2009), No. 12, 6093-6096.
[7] G. Wang, Monotone iterative technique for boundary value problems of a nonlinear fractional differential equation with deviating arguments, Jour. Comp. Appl. Mathe., No. 236(2012), 2425-2430.
[8] G. Wang, R.P. Agarwal and A. Cabada, Existence results and the monotone iterative technique for systems of nonlinear fractional differential equations, Appl. Math. Lett., No. 25(2012), 1019-1024.
[9] H. Heinz, On the behaviour of measures of noncompactness with respect to differentiation and integration of vector valued functions, Nonlinear Anal. 7(1983), No. 12, 1351-1371.
[10] H. Jian, B. Liu and S. Xie, Monotone iterative solutions for nonlinear fractional differential systems with deviating arguments, Appl. Math. and Comp. 262 (2015), 1-14.
[11] I. Podlubny, An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications, Mathematics in Science and Engineering, Vol. 198, Academic Press, San Diego, 1999.
[12] J. Banas and K. Goebel, Measure of noncompactness in Banach space, Marcal Dekker Inc., New York and Basel, 1980.
[13] Jin Rong Wang, Michal Fečkan, Yong Zhou, On the new concept of solution and existence results for impulsive fractional evolution equations, Dynamics of PDE, 8 (2011), 345-361.
[14] K. Deimling, Nonlinear functional analysis, Springer-Verlag, Berlin, 1985.
[15] Kamaljeet and D. Bahuguna, Monotone iterative technique for nonlocal fractional differential equations with finite delay in a Banach space, Elec. Jour. Qual. Theo. Diff. Eqn., No. 3(2015), 1-16.
[16] K. S. Miller and B. Ross, An introduction to the fractional calculus and fractional differential equations, A Wiley-Interscience Publication, Wiley, New York, 1993.
[17] L. Gaul, P. Klein and S. Kempfle, Damping description involving fractional operators, Mech. Systems Signal Processing 5 (1991), 81-88.
[18] L. Zhang, B. Ahmed and G. Wang, Monotone iterative method for a class of nonlinear fractional differential equations on unbounded domains in Banach Spaces, Filomat, Vol. 31, no. 5(2017), 1331-1338.
[19] M. El-Borai, Some probability densities and fundamental solutions of fractional evolution equations, Chaos Solitons Fractals **14** (2002), No. 3, 433-440.
[20] M. Benchohra, J. Henderson, S.K. Ntouyas, Impulsive Differential Equations and Inclusions, Contemporary Mathematics and its Applications, vol. 2, Hindawi Publishing Corporation, New York (2006).
[21] P. Chen and J. Mu, Monotone iterative method for semilinear impulsive evolution equations of mixed type in Banach spaces, Electron. J. Diff. Equ., No. 14(2010), 1-13.
[22] P. Chen and Y. Li, Mixed monotone iterative technique for a class of semilinear impulsive evolution equations in Banach spaces, Nonlinear Analysis, No. 74(2011), 3578-3588.
[23] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.
[24] R. W. Ibrahim and S. Momani, Upper and lower bounds of solutions for fractional integral equations, Survays in Math. and its Appl. 2 (2007), 145- 156.
[25] R. W. Ibrahim, Extremal solutions for certain type of fractional differential equations with maxima, Adv. in Diff. Eqn. 7 (2012), 1-8.
[26] R. Darzi, B. Mohammadzadeh, A. Neamaty and D. Baleanu, Lower and upper solutions method for positive solutions of fractional boundary value problems, Abstract and Applied Analysis, vol. 2013, Article ID 847184, 7 pages, 2013. https://doi.org/10.1155/2013/847184.
[27] R. Agarwal, S. Hristova and D. O'Regan, Iterative techniques for the initial value problem for Caputo fractional differential equations with non-instantaneous impulses, Appl. Math. Comp. 298 (2017) 45-56.
[28] R. Agarwal, D. O'Regan and S. Hristova, Monotone iterative technique for the initial value problem for differential equations with non-instantaneous impulses, Appl. Math. Comp. 334 (2018) 407-421.
[29] R. Chaudhary and D. N. Pandey, Monotone iterative technique for neutral fractional differential equation with infinite delay, Math. Meth. Appl. Sci., vol 39, no. 15(2016), 4642-4653.
[30] S. G. Samko, A.A. Kilbas and O.I. Marichev, Fractional Integrals and Derivatives, Theory and Applications, Gordon and Breach, Yverdon, 1993.
[31] S. Abbas, M Benchohra, J. J. Trujillo, Upper and lower solutions method for partial fractional differential inclusions with non-instantaneous impulses, Progr. Fract. Differ. Appl., vol. 1, no. 1(2015), 11-22.
[32] V. Lakshmikantham, D. Baînov and P.S. Simeonov, Theory of Impulsive Differential Equations. World Scientific, Singapore (1989).
[33] V. Lakshmikantham and A. S. Vatsala, Basic theory of fractional differential equations, Nonlinear Anal. 69(8)(2008), 2677-2682.
[34] X. H. Liu, Z.H. Liu and M.J. Bin, Approximate controllability of impulsive fractional neutral evolution equations with Riemann-Liouville fractional derivatives, J. Comp. Anal. Appl. 17 (2014), 468-485.

[35] X. B. Shu and F. Xu, Upper and lower solution method for fractional evolution equations with order $1 < \alpha < 2$, J. Korean Math. Soc. 51 (2014), No. 6, 1123-1139.

[36] Y. Haiping, G. Jianming and D. Yongsheng, A generalized Gronwall inequality and its application to a fractional differential equation, J. Math. Anal. Appl. 328 (2007), No. 2, 1075-1081.

[37] Y. Zhou, F. Jiao, Existence of mild solutions for fractional neutral evolution equations, Comput. Math. Appl. 59(2010), No. 3, 1063-1077.

[38] Z. Shuqin, Monotone iterative method for initial value problem involving Riemann-Liouville fractional derivatives, Nonlinear Anal. 71(2009), 2087-2093.

[39] Z. H. Liu, J. Sun and I. Szanto, Monotone iterative technoque for Riemann-Liouville fractional integro-differential equations with advanced arguments, Results. Math. 63(2013), 1277-1287.

[40] Z. H. Liu and M.J. Bin, Approximate controllability of impulsive Riemann-Liouville fractional equations in Banach spaces, J. Integral Eqn. Appl. Vol. 26, No. 4(2014), 527-551.

[41] Z. Bai, S. Zhang, S. Sun and C. Yin, Monotone iterative method for fractional differential equations, Elec. J. Diff. Eqn., Vol. 2016 (2016), No. 06, 1-8.