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A New Note on Generalized Absolute Cesàro Summability Factors

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Abstract. Quite recently, in [10], we have proved a theorem dealing with the generalized absolute Cesàro summability factors of infinite series by using quasi monotone sequences and quasi power increasing sequences. In this paper, we generalize this theorem for the more general summability method. This new theorem also includes some new and known results.

1. Introduction

A positive sequence (b_n) is said to be an almost increasing sequence if there exists a positive increasing sequence (c_n) and two positive constants M and N such that $Mc_n \leq b_n \leq Nc_n$ (see [3]). A sequence (d_n) is said to be δ-quasi monotone, if $d_n \to 0$, $d_n > 0$ ultimately, and $\Delta d_n \geq -\delta_n$, where $\Delta d_n = d_n - d_{n+1}$ and δ= (δ_n) is a sequence of positive numbers (see [4]). A positive sequence $X = (X_n)$ is said to be a quasi-f-power increasing sequence, if there exists a constant $K = K(X, f) \geq 1$ such that $Kf_nX_n \geq f_mX_m$ for all $n \geq m \geq 1$, where $f = \{f_n(\sigma,\gamma)\} = \{n^\sigma(\log n)^\gamma, \gamma \geq 0, 0 < \sigma < 1\}$(see [16]). If we take γ=0, then we get a quasi-σ-power increasing sequence (see [15]). Let $\sum a_n$ be a given infinite series. We denote by $t_n^{\alpha,\beta}$ the nth Cesàro mean of order (α,β), with $\alpha+\beta > -1$, of the sequence (na_n), that is (see [11])

$$t_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v, \tag{1}$$

where

$$A_n^{\alpha+\beta} = O(n^{\alpha+\beta}), \quad A_0^{\alpha+\beta} = 1 \quad \texttt{and} \quad A_{-n}^{\alpha+\beta} = 0 \quad \texttt{for} \quad n > 0. \tag{2}$$

Let $(\theta_n^{\alpha,\beta})$ be a sequence defined by (see [5])

$$\theta_n^{\alpha,\beta} = \begin{cases} \left|t_n^{\alpha,\beta}\right|, & \alpha = 1, \beta > -1, \\ \max_{1\leq v\leq n} \left|t_v^{\alpha,\beta}\right|, & 0 < \alpha < 1, \beta > -1. \end{cases} \tag{3}$$

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Let (φ_n) be a sequence of complex numbers. The series $\sum a_n$ is said to be summable $\varphi - \mid C, \alpha, \beta \mid_k$, $k \geq 1$, if (see [6])

$$\sum_{n=1}^{\infty} n^{-k} \mid \varphi_n t_n^{\alpha,\beta} \mid^k < \infty. \tag{4}$$

In the special case when $\varphi_n = n^{1-\frac{1}{k}}$, $\varphi - \mid C, \alpha, \beta \mid_k$ summability is the same as $\mid C, \alpha, \beta \mid_k$ summability (see [12]). Also, if we take $\varphi_n = n^{\sigma+1-\frac{1}{k}}$), then $\varphi - \mid C, \alpha, \beta \mid_k$ summability reduces to $\mid C, \alpha, \beta; \sigma \mid_k$ summability (see [7]). If we take $\beta = 0$, then we have $\varphi - \mid C, \alpha \mid_k$ summability (see [2]). If we take $\varphi_n = n^{1-\frac{1}{k}}$ and $\beta = 0$, then we get $\mid C, \alpha \mid_k$ summability (see [13]). Finally, if we take $\varphi_n = n^{\sigma+1-\frac{1}{k}}$ and $\beta = 0$, then we obtain $\mid C, \alpha; \sigma \mid_k$ summability (see [14]).

2. Known result

The following theorems is known dealing with the $\mid C, \alpha, \beta \mid_k$ summability method involving δ-quasi monotone sequence and power increasing sequence.

Theorem 2.1 ([10]). Let $(\theta_n^{\alpha,\beta})$ be a sequence defined as in (3). Let (X_n) be a quasi-f-power increasing sequence and $\lambda_n \to 0$ as $n \to \infty$. Suppose that there exists a sequence of numbers (B_n) such that it is δ-quasi-monotone with $\Delta B_n \leq \delta_n$, $\sum n\delta_n X_n < \infty$, $\sum B_n X_n$ is convergent, and $\mid \Delta\lambda_n \mid \leq \mid B_n \mid$ for all n. If the condition

$$\sum_{n=1}^{m} \frac{(\theta_n^{\alpha,\beta})^k}{n {X_n}^{k-1}} = O(X_m) \quad as \quad m \to \infty \tag{5}$$

holds, then the series $\sum a_n \lambda_n$ is summable $\mid C, \alpha, \beta \mid_k$, $0 < \alpha \leq 1$, $(\alpha + \beta - 1) > 0$, and $k \geq 1$.

3. Main result

The aim of this paper is to generalize Theorem 2.1 for the more general summability method. We shall prove the following theorem.

Theorem 3. 1 Let $(\theta_n^{\alpha,\beta})$ be a sequence defined as in (3). Let (X_n) be a quasi-f-power increasing sequence and $\lambda_n \to 0$ as $n \to \infty$. Suppose that there exists a sequence of numbers (B_n) such that it is δ-quasi-monotone with $\Delta B_n \leq \delta_n$, $\sum n\delta_n X_n < \infty$, $\sum B_n X_n$ is convergent, and $\mid \Delta\lambda_n \mid \leq \mid B_n \mid$ for all n. If there exists an $\epsilon > 0$ such that the sequence $(n^{\epsilon-k} \mid \varphi_n \mid^k)$ is non-increasing and if the condition

$$\sum_{n=1}^{m} \frac{(\mid \varphi_n \mid \theta_n^{\alpha,\beta})^k}{n^k {X_n}^{k-1}} = O(X_m) \quad \text{as} \quad m \to \infty \tag{6}$$

holds, then the series $\sum a_n \lambda_n$ is summable $\varphi- \mid C, \alpha, \beta \mid_k$, $k \geq 1$, $0 < \alpha \leq 1$, $\beta > -1$, and $(\alpha + \beta - 1)k + \epsilon > 1$.

We need the following lemmas for the proof of our theorem.

Lemma 3. 2 (Abel transformation) ([1]). Let (a_k), (b_k) be complex sequences, and write $s_n = a_1 + a_2 + ... + a_n$. Then

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n-1} s_k \Delta b_k + s_n b_n. \tag{7}$$

Lemma 3. 3 ([5]). If $0 < \alpha \leq 1$, $\beta > -1$, and $1 \leq v \leq n$, then

$$\mid \sum_{p=0}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} a_p \mid \leq \max_{1 \leq m \leq v} \mid \sum_{p=0}^{m} A_{m-p}^{\alpha-1} A_p^{\beta} a_p \mid . \tag{8}$$

Lemma 3. 4 ([8]). Let (X_n) be a quasi-f-power increasing sequence. If (B_n) is a δ-quasi-monotone sequence with $\Delta B_n \leq \delta_n$ and $\sum n\delta_n X_n < \infty$, then we have the following

$$\sum_{n=1}^{\infty} nX_n \mid \Delta B_n \mid < \infty, \tag{9}$$

$$nB_nX_n = O(1) \quad as \quad n \to \infty. \tag{10}$$

Lemma 3. 5 ([8]). Under the conditions regarding (λ_n) and (X_n) of the theorem, we have

$$\mid \lambda_n \mid X_n = O(1) \quad as \quad n \to \infty. \tag{11}$$

4. Proof of Theorem 3. 1 Let $(T_n^{\alpha,\beta})$ be the nth (C,α,β) mean of the sequence $(na_n\lambda_n)$. Then, by (1), we have

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \lambda_v.$$

Applying Abel's transformation first and then using Lemma 3. 3, we have that

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} p a_p + \frac{\lambda_n}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v,$$

$$\begin{aligned}
\mid T_n^{\alpha,\beta} \mid &\leq \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \mid \Delta\lambda_v \mid \mid \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} p a_p \mid + \frac{\mid \lambda_n \mid}{A_n^{\alpha+\beta}} \mid \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \mid \\
&\leq \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} A_v^{(\alpha+\beta)} \theta_v^{\alpha,\beta} \mid \Delta\lambda_v \mid + \mid \lambda_n \mid \theta_n^{\alpha,\beta} = T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta}.
\end{aligned}$$

To complete the proof of Theorem 3. 1, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{-k} \mid \varphi_n T_{n,r}^{\alpha,\beta} \mid^k < \infty, \quad \text{for} \quad r = 1, 2.$$

Now, when $k > 1$, applying Hölder's inequality with indices k and k', where $\frac{1}{k} + \frac{1}{k'} = 1$, we get that

$$\begin{aligned}
\sum_{n=2}^{m+2} n^{-k} \mid \varphi_n T_{n,1}^{\alpha,\beta} \mid^k &\leq \sum_{n=2}^{m+1} n^{-k} (A_n^{\alpha+\beta})^{-k} \mid \varphi_n \mid^k \{ \sum_{v=1}^{n-1} A_v^{\alpha+\beta} \theta_v^{\alpha,\beta} \mid \Delta\lambda_v \mid \}^k \\
&\leq \sum_{n=2}^{m+1} n^{-k} n^{-(\alpha+\beta)k} \mid \varphi_n \mid^k \sum_{v=1}^{n-1} (A_v^{\alpha+\beta})^k (\theta_v^{\alpha,\beta})^k \mid B_v \mid^k \times \left\{ \sum_{v=1}^{n-1} 1 \right\}^{k-1} \\
&= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} (\theta_v^{\alpha,\beta})^k \mid B_v \mid^k \sum_{n=v+1}^{m+1} \frac{n^{\epsilon-k} \mid \varphi_n \mid^k}{n^{1+(\alpha+\beta-1)k+\epsilon}} \\
&= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} (\theta_v^{\alpha,\beta})^k \mid B_v \mid^k v^{\epsilon-k} \mid \varphi_v \mid^k \sum_{n=v+1}^{m+1} \frac{1}{n^{1+(\alpha+\beta-1)k+\epsilon}} \\
&= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} (\theta_v^{\alpha,\beta})^k v^{\epsilon-k} \mid \varphi_v \mid^k \mid B_v \mid^k \int_v^{\infty} \frac{dx}{x^{1+(\alpha+\beta-1)k+\epsilon}} \\
&= O(1) \sum_{v=1}^{m} \mid B_v \mid \mid B_v \mid^{k-1} (\theta_v^{\alpha,\beta} \mid \varphi_v \mid)^k = O(1) \sum_{v=1}^{m} \mid B_v \mid \frac{(\theta_v^{\alpha,\beta} \mid \varphi_v \mid)^k}{v^{k-1} X_v^{k-1}} \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v \mid B_v \mid) \sum_{r=1}^{v} \frac{(\theta_r^{\alpha,\beta} \mid \varphi_r \mid)^k}{r^k X_r^{k-1}} + O(1) m \mid B_m \mid \sum_{v=1}^{m} \frac{(\theta_v^{\alpha,\beta} \mid \varphi_v \mid)^k}{v^k X_v^{k-1}} \\
&= O(1) \sum_{v=1}^{m-1} \mid \Delta(v \mid B_v \mid) \mid X_v + O(1) m \mid B_m \mid X_m
\end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m-1} \mid (v+1)\Delta \mid B_v \mid - \mid B_v \mid\mid X_v + O(1)m\mid B_m \mid X_m \\
&= O(1)\sum_{v=1}^{m-1} v \mid \Delta B_v \mid X_v + O(1)\sum_{v=1}^{m-1} \mid B_v \mid X_v \\
&\quad + O(1)m\mid B_m \mid X_m = O(1) \quad \text{as} \quad m \to \infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3. 4. Finally, we have

$$
\begin{aligned}
\sum_{n=1}^{m} n^{-k} \mid \varphi_n T_{n,2}^{\alpha,\beta} \mid^k &= \sum_{n=1}^{m} \mid \lambda_n \mid \mid \lambda_n \mid^{k-1} \frac{(\theta_n^{\alpha,\beta} \mid \varphi_n \mid)^k}{n^k} = \sum_{n=1}^{m} \mid \lambda_n \mid \frac{(\theta_n^{\alpha,\beta} \mid \varphi_n \mid)^k}{n^k X_n^{k-1}} = \sum_{n=1}^{m-1} \Delta \mid \lambda_n \mid \sum_{v=1}^{n} \frac{(\theta_v^{\alpha,\beta} \mid \varphi_v \mid)^k}{v^k X_v^{k-1}} \\
&\quad + O(1)|\lambda_m| \sum_{n=1}^{m} \frac{(\theta_n^{\alpha,\beta} \mid \varphi_n \mid)^k}{n^k X_n^{k-1}} = O(1)\sum_{n=1}^{m-1} \mid \Delta\lambda_n \mid X_m + O(1) \mid \lambda_m \mid X_m \\
&= O(1)\sum_{n=1}^{m-1} \mid B_n \mid X_n + O(1) \mid \lambda_m \mid X_m = O(1) \quad \text{as} \quad m \to \infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3. 5. This completes the proof of Theorem 3. 1.

5. Conclusions. If we take $\epsilon = 1$ and $\varphi_n = n^{1-\frac{1}{k}}$, then we obtain Theorem 2.1. If we set $\epsilon = 1$, $\beta = 0$ and $\varphi_n = n^{\sigma+1-\frac{1}{k}}$, then we have a new result dealing with the $\mid C, \alpha; \sigma \mid_k$ summability factors of infinite series. Also, if we take $\beta = 0$, then we obtain another new result dealing with $\varphi - \mid C, \alpha \mid_k$ summability factors of infinite series. If we take $\beta = 0$, $\epsilon = 1$, and $\varphi_n = n^{1-\frac{1}{k}}$, then we obtain a new result concerning the $\mid C, \alpha \mid_k$ summability factors of infinite series. Furthermore, if we take $\beta = 0$, $\epsilon = 1$, $\varphi_n = n^{1-\frac{1}{k}}$, and $\alpha = 1$, then we obtain a result dealing with $\mid C, 1 \mid_k$ summability factors. If we take (X_n) as an almost increasing sequence such that $\mid \Delta X_n \mid = O(\frac{X_n}{n})$ and $\varphi_n = n^{1-\frac{1}{k}}$, then we obtain a known result dealing with $\mid C, \alpha, \beta \mid_k$ summability factors of infinite series, in this case the condition $'\Delta B_n \leq \delta_n'$ is not needed (see [9]). Finally, if we take γ=0, then we get another new result dealing with quasi-σ-power increasing sequences.

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