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Frames Associated with Shift-invariant Spaces on Local Fields

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Abstract. In this paper, we present a unified approach to the study of shift-invariant systems to be frames on local fields of positive characteristic. We establish a necessary condition and three sufficient conditions under which the shift-invariant systems on local fields constitute frames for $L^2(K)$. As an application of these results, we obtain some known conclusions about the Gabor frames and wavelet frames on local fields.

1. Introduction

A shift-invariant space is a space of functions that is invariant under all translations. A variety of such spaces have been used successfully in both pure and applied mathematics. They play an important role in modern analysis because of their rich underlying theory and their applications in many areas of contemporary mathematics, such as wavelets, spline systems, Gabor systems, theory of frames, approximation theory, numerical analysis, digital signal and image processing, nonuniform sampling problems, and so on.

The concept of shift-invariant subspace of $L^2(\mathbb{R})$ was introduced by Helson [13]. In fact, he introduced range functions and used this notion to completely characterize shift invariant spaces. Later on, a considerable amount of research has been conducted using this framework in order to describe and characterize frames and bases of these spaces. For example, de Boor et al.[7] gave the general structure of these spaces in $L^2(\mathbb{R}^n)$ using the machinery of fiberization based on range functions. This has been further developed in the work of Ron and Shen [21] with the introduction of the technique of Gramians and dual Gramians. Bownik [8] gave a characterization of shift-invariant subspaces of $L^2(\mathbb{R}^n)$ following an idea from Helsons book [13]. The invariance properties of shift-invariant spaces under non-integer translations were completely characterized by Aldroubi et al.[3] and they showed that the principal shift-invariant spaces generated by a compactly supported function is not invariant under such translations. In [19], authors constructed p-frames for the weighted shift-invariant spaces and investigated their frame properties under some mild technical conditions on the frame generators. On the other side, the study of shift-invariant spaces and frames have been extended to locally compact Abelian groups in [9], nilpotent Lie groups in

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[10] and non-abelian compact groups in [20]. The results of Aldroubi et al.[3] were further generalized to the context of LCA groups by Anastasio et al.[4]. They provide necessary and sufficient conditions for an H-invariant space to be M-invariant space by means of range functions, where H is a countable uniform lattice in G and M is any closed subgroup of G containing H. Shift-invariant spaces for local fields were first introduced and investigated by Ahmadi et al.[2]. More precisely, they studied shift-invariant spaces of $L^2(G)$, where G is a locally compact abelian group, or in general a local field, with a compact open subgroup. In his recent paper, Behera [5] showed that every closed shift-invariant subspace of $L^2(K)$ is generated by the Λ-translates of a countable number of functions, where K is the local field of positive characteristic and Λ is the associated translation set.

The notion of frames was first introduced by Duffin and Shaeffer [12] in connection with some deep problems in non-harmonic Fourier series, and more particularly with the question of determining when a family of exponentials $\left\{e^{i\alpha_n t} : n \in \mathbb{Z}\right\}$ is complete for $L^2[a,b]$. Frames did not generate much interest outside non- harmonic Fourier series until the seminal work by Daubechies et al.[11]. After their pioneer work, the theory of frames began to be studied widely and deeply, particularly in the more specialized context of wavelet frames and Gabor frames. Frames provide a useful model to obtain signal decompositions in cases where redundancy, robustness, over-sampling, and irregular sampling play a role. Today, the theory of frames has become an interesting and fruitful field of mathematics with abundant applications in signal processing, image processing, harmonic analysis, Banach space theory, sampling theory, wireless sensor networks, optics, filter banks, quantum computing, and medicine and so on. Recall that a countable collection $\{f_k : k \in \mathbb{Z}\}$ in an infinite-dimensional separable Hilbert space $\mathcal{H}$ is called a *frame* if there exist positive constants A and B such that

$$A\left\|f\right\|^2 \leq \sum_{k\in\mathbb{Z}} \left|\left\langle f, f_k\right\rangle\right|^2 \leq B\left\|f\right\|^2, \tag{1.1}$$

holds for every $f \in \mathcal{H}$ and we call the optimal constants A and B the lower frame bound and the upper frame bound, respectively. If we only require the second inequality to hold in (1.1), then $\{f_k : k \in \mathbb{Z}\}$ is called a *Bessel collection*. A frame is tight if $A = B$ in (1.1) and if $A = B = 1$ it is called a *Parseval frame* or a *normalized tight frame*.

A field K equipped with a topology is called a local field if both the additive K^+ and multiplicative groups K^* of K are locally compact Abelian groups. For example, any field endowed with the discrete topology is a local field. For this reason we consider only non-discrete fields. The local fields are essentially of two types (excluding the connected local fields $\mathbb{R}$ and $\mathbb{C}$). The local fields of characteristic zero include the p-adic field $\mathbb{Q}_p$. Examples of local fields of positive characteristic are the Cantor dyadic group and the Vilenkin p-groups. There is a substantial body of work that has been concerned with the construction and characterization of wavelet frames and Gabor frames on K, or more generally, on local fields of positive characteristic. For example, Li and Jiang [15] have constructed Gabor frames on local fields of positive characteristic using the basic concepts of operator theory and have established a necessary and sufficient conditions for the system $\left\{\psi_{m,n} =: \chi_m(bx)\psi\left(x - u(n)a\right) : m, n \in \mathbb{N}_0\right\}$ to be a frame for $L^2(K)$. Later on, they have obtained a necessary condition and a set of sufficient conditions for the wavelet system $\left\{\psi_{j,k} =: q^{j/2}\psi\left(\mathfrak{p}^{-j}x - u(k)\right) : j, k \in \mathbb{N}_0\right\}$ to be a tight wavelet frame on local fields in the frequency domain in [16]. The characterizations of tight wavelet frames on local fields were completely established by Shah and Abdullah [24] by virtue of two basic equations in the Fourier domain. These studies were continued by Shah and his colleagues in [1, 22, 23, 25–27], where they have described some algorithms for constructing periodic wavelet frames, wave packet frames and semi-orthogonal wavelet frames on local fields of positive characteristic.

The general results in Euclidean spaces to characterize tight frame generators for the shift-invariant subspaces were studied by Labate in [14]. Some applications of this general result are then obtained, among which are the characterization of tight wavelet frames and tight Gabor frames [17, 18]. As for the corresponding counterparts for a local field K, such results are not yet reported. So in this paper, we extend these concepts to local fields of positive characteristic. We establish some necessary and sufficient conditions under which shift-invariant systems constitute frames for $L^2(K)$ and, we use these results to give some necessary conditions and sufficient conditions for Gabor frames and wavelet frames on local fields of positive characteristic.

The paper is organized as follows. In Section 2, we give a brief introduction to the local fields and Fourier analysis including the definition of shift-invariant spaces on such a field. Section 3 establishes a necessary condition for the shift-invariant system to be a frame for $L^2(K)$. Section 4 is devoted to the discussion of sufficient conditions for shift-invariant systems to be frames. Sections 5 and 6 contain applications of the main results to Gabor frames and wavelet frames, respectively.

2. Preliminaries and Shift-invariant Spaces on Local Fields

Local fields have attracted the attention of several mathematicians, and have found innumerable applications not only in the number theory, but also in the representation theory, division algebras, quadratic forms and algebraic geometry. As a result, local fields are now consolidated as a part of the standard repertoire of contemporary mathematics.

Let K be a fixed local field with the ring of integers $\mathfrak{D} = \{x \in K : |x| \le 1\}$. Since K^+ is a locally compact Abelian group, we choose a Haar measure dx for K^+. The field K is locally compact, non-trivial, totally disconnected and complete topological field endowed with non–Archimedean norm $|\cdot| : K \to \mathbb{R}^+$ satisfying

(a) $|x| = 0$ if and only if $x = 0$;

(b) $|x\,y| = |x||y|$ for all $x, y \in K$;

(c) $|x + y| \le \max\{|x|, |y|\}$ for all $x, y \in K$.

Property (c) is called the ultrametric inequality. Let $\mathfrak{B} = \{x \in K : |x| < 1\}$ be the prime ideal of the ring of integers $\mathfrak{D}$ in K. Then, the residue space $\mathfrak{D}/\mathfrak{B}$ is isomorphic to a finite field $GF(q)$, where $q = p^c$ for some prime p and $c \in \mathbb{N}$. Since K is totally disconnected and $\mathfrak{B}$ is both prime and principal ideal, so there exist a prime element $\mathfrak{p}$ of K such that $\mathfrak{B} = \langle \mathfrak{p} \rangle = \mathfrak{p}\mathfrak{D}$. Let $\mathfrak{D}^* = \mathfrak{D} \setminus \mathfrak{B} = \{x \in K : |x| = 1\}$. Clearly, $\mathfrak{D}^*$ is a group of units in K^* and if $x \neq 0$, then we can write $x = \mathfrak{p}^n y, y \in \mathfrak{D}^*$. Moreover, if $\mathcal{U} = \{a_m : m = 0, 1, \ldots, q-1\}$ denotes the fixed full set of coset representatives of $\mathfrak{B}$ in $\mathfrak{D}$, then every element $x \in K$ can be expressed uniquely as $x = \sum_{\ell=k}^{\infty} c_\ell\, \mathfrak{p}^\ell$ with $c_\ell \in \mathcal{U}$. Recall that $\mathfrak{B}$ is compact and open, so each fractional ideal $\mathfrak{B}^k = \mathfrak{p}^k\mathfrak{D} = \left\{x \in K : |x| < q^{-k}\right\}$ is also compact, open and is a subgroup of K^+. We use the notation in Taibleson's book [28]. In the rest of this paper, we use the symbols $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{Z}$ to denote the sets of natural numbers, non-negative integers and integers, respectively.

Let χ be a fixed character on K^+ that is trivial on $\mathfrak{D}$ but non-trivial on $\mathfrak{B}^{-1}$. Therefore, χ is constant on cosets of $\mathfrak{D}$ so if $y \in \mathfrak{B}^k$, then $\chi_y(x) = \chi(y, x), x \in K$. Suppose that χ_u is any character on K^+, then the restriction $\chi_u|\mathfrak{D}$ is a character on $\mathfrak{D}$. Moreover, as characters on $\mathfrak{D}$, $\chi_u = \chi_v$ if and only if $u - v \in \mathfrak{D}$. Hence, if $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representative of $\mathfrak{D}$ in K^+, then, as it was proved in [28], the set $\left\{\chi_{u(n)} : n \in \mathbb{N}_0\right\}$ of distinct characters on $\mathfrak{D}$ is a complete orthonormal system on $\mathfrak{D}$.

Definition 2.1. The Fourier transform of $f \in L^1(K)$ is denoted by $\hat{f}(\xi)$ and defined by

$$\hat{f}(\xi) = \int_K f(x)\overline{\chi_\xi(x)}\, dx. \tag{2.1}$$

Note that

$$\hat{f}(\xi) = \int_K f(x)\,\overline{\chi_\xi(x)}\, dx = \int_K f(x)\chi(-\xi x)\, dx.$$

The properties of Fourier transform on local field K are much similar to those of on the real line. In fact, the Fourier transform has the following properties:

- The map $f \to \hat{f}$ is a bounded linear transformation of $L^1(K)$ into $L^\infty(K)$, and $\|\hat{f}\|_\infty \le \|f\|_1$.
- If $f \in L^1(K)$, then $\hat{f}$ is uniformly continuous.
- If $f \in L^1(K) \cap L^2(K)$, then $\|\hat{f}\|_2 = \|f\|_2$.

As usual, the Fourier transform is extended to a unitary operator on $L^2(K)$ as

$$\hat{f}(\xi) = \lim_{k\to\infty} \hat{f}_k(\xi) = \lim_{k\to\infty} \int_{|x|\le q^k} f(x)\overline{\chi_\xi(x)}\,dx, \tag{2.2}$$

where $f_k = f\,\Phi_{-k}$ and Φ_k is the characteristic function of $\mathfrak{B}^k$. Furthermore, if $f \in L^2(\mathfrak{D})$, then we define the Fourier coefficients of f as

$$\hat{f}\big(u(n)\big) = \int_{\mathfrak{D}} f(x)\overline{\chi_{u(n)}(x)}\,dx. \tag{2.3}$$

The series $\sum_{n\in\mathbb{N}_0} \hat{f}\big(u(n)\big)\chi_{u(n)}(x)$ is called the Fourier series of f. From the standard L^2-theory for compact Abelian groups, we conclude that the Fourier series of f converges to f in $L^2(\mathfrak{D})$ and Parseval's identity holds:

$$\|f\|_2^2 = \int_{\mathfrak{D}} \big|f(x)\big|^2 dx = \sum_{n\in\mathbb{N}_0} \Big|\hat{f}\big(u(n)\big)\Big|^2. \tag{2.4}$$

We now impose a natural order on the sequence $\{u(n)\}_{n=0}^\infty$. We have $\mathfrak{D}/\mathfrak{B} \cong GF(q)$ where $GF(q)$ is a c-dimensional vector space over the field $GF(p)$. We choose a set $\{1 = \zeta_0, \zeta_1, \zeta_2, \dots, \zeta_{c-1}\} \subset \mathfrak{D}^*$ such that $\operatorname{span}\{\zeta_j\}_{j=0}^{c-1} \cong GF(q)$. For $n \in \mathbb{N}_0$ satisfying

$$0 \le n < q, \;\; n = a_0 + a_1 p + \cdots + a_{c-1}p^{c-1}, \;\; 0 \le a_k < p, \;\; \text{and } k = 0, 1, \dots, c-1,$$

we define

$$u(n) = (a_0 + a_1\zeta_1 + \cdots + a_{c-1}\zeta_{c-1})\,\mathfrak{p}^{-1}. \tag{2.5}$$

Also, for $n = b_0 + b_1 q + b_2 q^2 + \cdots + b_s q^s$, $n \in \mathbb{N}_0$, $0 \le b_k < q, k = 0, 1, 2, \dots, s$, we set

$$u(n) = u(b_0) + u(b_1)\mathfrak{p}^{-1} + \cdots + u(b_s)\mathfrak{p}^{-s}. \tag{2.6}$$

This defines $u(n)$ for all $n \in \mathbb{N}_0$. In general, it is not true that $u(m+n) = u(m) + u(n)$. But, if $r, k \in \mathbb{N}_0$ and $0 \le s < q^k$, then $u(rq^k + s) = u(r)\mathfrak{p}^{-k} + u(s)$. Further, it is also easy to verify that $u(n) = 0$ if and only if $n = 0$ and $\{u(\ell) + u(k) : k \in \mathbb{N}_0\} = \{u(k) : k \in \mathbb{N}_0\}$ for a fixed $\ell \in \mathbb{N}_0$. Hereafter, we use the notation $\chi_n = \chi_{u(n)}$, $n \ge 0$.

Let the local field K be of characteristic $p > 0$ and $\zeta_0, \zeta_1, \zeta_2, \dots, \zeta_{c-1}$ be as above. We define a character χ on K as follows:

$$\chi(\zeta_\mu \mathfrak{p}^{-j}) = \begin{cases} \exp(2\pi i/p), & \mu = 0 \text{ and } j = 1, \\ 1, & \mu = 1, \dots, c-1 \text{ or } j \ne 1. \end{cases} \tag{2.7}$$

Since $\bigcup_{j\in\mathbb{Z}} \mathfrak{p}^{-j}\mathfrak{D} = K$, we can regard $\mathfrak{p}^{-1}$ as the dilation and since $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representatives of $\mathfrak{D}$ in K, the set $\Delta = \{u(n) : n \in \mathbb{N}_0\}$ can be treated as the translation set. Note that Λ is a subgroup of K^+ and unlike the standard wavelet theory on the real line, the translation set is not a group.

We also denote the test function space on K by Ω, that is, each function f in Ω is a finite linear combination of functions of the form $\mathbf{\Phi}_k(x-h), h \in K, k \in \mathbb{Z}$, where $\mathbf{\Phi}_k$ is the characteristic function of $\mathfrak{B}^k$. This class of functions can also be described in the following way. A function $g \in \Omega$ if and only if there exist integers k, ℓ such that g is constant on cosets of $\mathfrak{B}^k$ and is supported on $\mathfrak{B}^\ell$. It follows that Ω is closed under Fourier transform and is an algebra of continuous functions with compact support, which is dense in $C_0(K)$ as well as in $L^p(K), 1 \leq p < \infty$. Thus, we will consider the following set of functions:

$$\Omega_0 = \left\{ f \in \Omega : \text{supp}\hat{f} \subset K \backslash \{0\} \text{ and } \left\|\hat{f}\right\|_\infty < \infty \right\}. \tag{2.8}$$

For any prime $\mathfrak{p}$ and $m, n \in \mathbb{N}_0$, we define the unitary operators on $L^2(K)$ as:

$$\begin{aligned} T_{u(n)}f(x) &= f\big(x-u(n)\big), && \text{(Translation)} \\ E_{u(m)}f(x) &= \chi\big(u(m)x\big)f(x), && \text{(Modulation)} \\ D_{\mathfrak{p}}f(x) &= \sqrt{q}f\left(\mathfrak{p}^{-1}x\right), && \text{(Dilation).} \end{aligned}$$

Then for any $f \in L^2(K)$, the following results can be easily verified:

$$\begin{aligned} \mathcal{F}\Big\{T_{u(n)}f(x)\Big\} &= E_{-u(n)}\mathcal{F}\Big\{f(x)\Big\}, \\ \mathcal{F}\Big\{E_{u(m)}f(x)\Big\} &= T_{u(m)}\mathcal{F}\Big\{f(x)\Big\}, \\ \mathcal{F}\Big\{D_{\mathfrak{p}^j}f(x)\Big\} &= D_{\mathfrak{p}^{-j}}\mathcal{F}\Big\{f(x)\Big\}. \end{aligned}$$

Definition 2.2. A closed subspace S of $L^2(K)$ is called a *shift invariant space* if $T_{u(k)}f_\alpha(x) \in S$, for every $f_\alpha \in S, k \in \mathbb{N}_0$, and $\alpha \in \Lambda$, where $T_{u(k)}$ is the translation operator and Λ is a countable indexing set.

A closed shift-invariant subspace S of $L^2(K)$ is said to be *generated* by $\Psi \subset L^2(K)$ if

$$S = \overline{\text{span}}\left\{T_{u(k)}\psi_\alpha(x) := \psi_\alpha\big(x-u(k)\big) : k \in \mathbb{N}_0, \psi_\alpha \in \Psi\right\}.$$

The cardinality of a smallest generating set Ψ for S is called the *length* of S which is denoted by $|S|$. If $|S| =$ finite, then S is called a *finite shift-invariant space* (FSI) and if $|S| = 1$, then S is called a *principal shift-invariant space* (PSI). Moreover, the *spectrum* of a shift-invariant space is defined to be

$$\sigma(S) = \left\{\xi \in \mathfrak{D} : \hat{S}(\xi) \neq \{0\}\right\}, \tag{2.9}$$

where $\hat{S}(\xi) = \left\{\hat{f}_\alpha\big(\xi + u(k)\big) \in l^2(\mathbb{N}_0) : f_\alpha \in S, k \in \mathbb{N}_0, \alpha \in \Lambda\right\}$.

It is easy to verify that the system

$$\mathcal{W}(f, \alpha, k) = \left\{T_{u(k)}f_\alpha(x) =: f_\alpha\big(x-u(k)\big) : k \in \mathbb{N}_0, \alpha \in \Lambda\right\}, \tag{2.10}$$

is a shift-invariant system with respect to lattice $\mathbb{N}_0$, where $f_\alpha(x) \in L^2(K)$.

Definition 2.3. The shift-invariant system $\mathcal{W}(f, \alpha, k)$ defined by (2.10) is called *shift-invariant frame* if there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A\left\|f\right\|_2^2 \leq \sum_{\alpha\in\Lambda}\sum_{n\in\mathbb{N}_0}\left|\Big\langle f, T_{u(n)}f_\alpha\Big\rangle\right|^2 \leq B\left\|f\right\|_2^2, \tag{2.11}$$

holds for every $f \in L^2(K)$. The largest constant A and the smallest constant B satisfying (2.11) are called the *optimal lower* and *upper frame bounds*, respectively. A frame is a *tight frame* if A and B are chosen so that $A = B$ and is a *Parseval frame* if $A = B = 1$.

Since the set Ω is dense in $L^2(K)$ and is closed under the Fourier transform, the set Ω_0 defined by (2.8) is also dense in $L^2(K)$. Therefore, it is sufficient to prove that the shift-invariant system $\mathcal{W}(f, \alpha, k)$ given by (2.10) is a frame for $L^2(K)$ if the inequalities in (2.11) holds for all $f \in \Omega_0$.

3. Necessary Condition for $\mathcal{W}(f,\alpha,k)$ to be Frame for $L^2(K)$

In this section, we shall derive the necessary condition for the shift-invariant system $\mathcal{W}(f,\alpha,k)$ defined by (2.10) to be frame for $L^2(K)$.

Theorem 3.1. *If the shift-invariant system $\mathcal{W}(f,\alpha,k)$ defined by (2.10) is a frame for $L^2(K)$ with bounds A and B, then*

$$A \le S_f(\xi) \le B, \quad a.e.\ \xi \in K \tag{3.1}$$

where $S_f(\xi) = \sum_{\alpha\in\Lambda} \left|\hat{f}_\alpha(\xi)\right|^2$.

Proof. Since the shift-invariant system $\mathcal{W}(f,\alpha,k)$ is a frame for $L^2(K)$ with bounds A and B, we have

$$A\|f\|_2^2 \le \sum_{\alpha\in\Lambda}\sum_{n\in\mathbb{N}_0} \left|\left\langle f, T_{u(n)} f_\alpha\right\rangle\right|^2 \le B\|f\|_2^2, \quad \text{for all } f \in L^2(K). \tag{3.2}$$

Notice that for all $f \in L^2(K)$ and $m \in \mathbb{N}_0$, we have $\left\|T_{u(m)} f\right\|_2^2 = \|f\|_2^2$. Hence, equation (3.2) can be rewritten as

$$A\|f\|_2^2 \le \sum_{\alpha\in\Lambda}\sum_{n\in\mathbb{N}_0} \left|\left\langle T_{u(m)} f, T_{u(n)} f_\alpha\right\rangle\right|^2 \le B\|f\|_2^2.$$

Or, equivalently

$$A\|f\|_2^2 \le \int_{\mathfrak{D}} \sum_{\alpha\in\Lambda}\sum_{n\in\mathbb{N}_0} \left|\left\langle T_{u(m)} f, T_{u(n)} f_\alpha\right\rangle\right|^2 dx \le B\|f\|_2^2, \text{ for all } f \in L^2(K). \tag{3.3}$$

Since $K = \bigcup_{n\in\mathbb{N}_0} \left(\mathfrak{D} + u(n)\right)$ is a disjoint union, it follows from the Plancherel theorem that

$$\begin{aligned}
\int_{\mathfrak{D}} \sum_{\alpha\in\Lambda}\sum_{n\in\mathbb{N}_0} \left|\left\langle T_{u(m)} f, T_{u(n)} f_\alpha\right\rangle\right|^2 dx &= \sum_{\alpha\in\Lambda}\sum_{n\in\mathbb{N}_0} \int_{\mathfrak{D}} \left|\left\langle f, T_{u(n)-u(m)} f_\alpha\right\rangle\right|^2 dx \\
&= \sum_{\alpha\in\Lambda} \int_K \left|\left\langle f, T_{u(m)} f_\alpha\right\rangle\right|^2 dx \\
&= \sum_{\alpha\in\Lambda} \int_K \left|\left\langle \hat{f}, \left(T_{u(m)} f_\alpha\right)^\wedge\right\rangle\right|^2 dx \\
&= \sum_{\alpha\in\Lambda} \int_K \left|\left\langle \hat{f}, E_{-u(m)} \hat{f}_\alpha\right\rangle\right|^2 dx \\
&= \sum_{\alpha\in\Lambda} \int_K \left|\int_K \hat{f}(\xi)\overline{\hat{f}_\alpha(\xi)}\, \chi_m(\xi)\, d\xi\right|^2 dx.
\end{aligned} \tag{3.4}$$

Clearly, for all $f \in \Omega_0$, we have $\hat{f}(\xi)\overline{\hat{f}_\alpha(\xi)} \in L^2(K)$. Therefore, it follows from (3.4) and the Plancherel theorem that for all $f \in \Omega_0$,

$$\begin{aligned}
\int_{\mathfrak{D}} \sum_{\alpha\in\Lambda}\sum_{n\in\mathbb{N}_0} \left|\left\langle T_{u(m)} f, T_{u(n)} f_\alpha\right\rangle\right|^2 dx &= \sum_{\alpha\in\Lambda} \int_K \left|\left(\hat{f}\overline{\hat{f}_\alpha}\right)^\vee(\xi)\right|^2 dx \\
&= \sum_{\alpha\in\Lambda} \int_K \left|\hat{f}(\xi)\overline{\hat{f}_\alpha}(\xi)\right|^2 d\xi \\
&= \int_K \left|\hat{f}(\xi)\right|^2 \sum_{\alpha\in\Lambda} \left|\hat{f}_\alpha(\xi)\right|^2 d\xi.
\end{aligned} \tag{3.5}$$

Combining (3.3) and (3.5), we observe that for all $f \in \Omega_0$, we have

$$A\|f\|_2^2 \le \int_K \left|\hat{f}(\xi)\right|^2 \sum_{\alpha\in\Lambda} \left|\hat{f_\alpha}(\xi)\right|^2 d\xi \le B\|f\|_2^2. \tag{3.6}$$

Making appropriate choices of $f \in \Omega_0$ in(3.6), we obtain

$$A \le S_f(\xi) \le B, \quad a.e.\, \xi \in K.$$

This is what we desired. The proof of Theorem 3.1 is complete.

4. Sufficient Conditions for $\mathcal{W}(f, \alpha, k)$ to be Frame for $L^2(K)$

In this section, we derive three sufficient conditions for the shift-invariant system $\mathcal{W}(f, \alpha, k)$ to be a frame for $L^2(K)$.

In order to prove our results, we need the following lemma.

Lemma 4.1. *Suppose that shift-invariant system $\mathcal{W}(f, \alpha, k)$ is defined by (2.10). If $f \in \Omega_0$ and*

$$ess \sup \left\{ \sum_{\alpha\in\Lambda} \left|\hat{f_\alpha}(\xi)\right|^2 : \xi \in \mathfrak{B}^{-1} \setminus \mathfrak{D} \right\} < \infty$$

, then

$$\sum_{\alpha\in\Lambda} \sum_{n\in\mathbb{N}_0} \left|\left\langle f, T_{u(n)} f_\alpha \right\rangle\right|^2 = \int_K \left|\hat{f}(\xi)\right|^2 \sum_{\alpha\in\Lambda} \left|\hat{f_\alpha}(\xi)\right|^2 d\xi + R(f), \tag{4.1}$$

where

$$R(f) = \sum_{\alpha\in\Lambda} \sum_{m\in\mathbb{N}} \int_K \hat{f}(\xi)\overline{\hat{f_\alpha}(\xi)}\hat{f}\big(\xi + u(m)\big)\overline{\hat{f_\alpha}\big(\xi + u(m)\big)}\, d\xi. \tag{4.2}$$

Furthermore, the iterated series in (4.2) is absolutely convergent.

Proof. By Parseval formula, we have

$$\begin{aligned}
\sum_{n\in\mathbb{N}_0} \left|\left\langle f, T_{u(n)} f_\alpha \right\rangle\right|^2 &= \sum_{n\in\mathbb{N}_0} \left|\left\langle \hat{f}, \big(T_{u(n)} f_\alpha\big)^\wedge \right\rangle\right|^2 \\
&= \sum_{n\in\mathbb{N}_0} \left| \int_K \hat{f}(\xi)\overline{\hat{f_\alpha}(\xi)}\, \chi_n(\xi)\, d\xi \right|^2 \\
&= \sum_{n\in\mathbb{N}_0} \int_K \left\{ \sum_{m\in\mathbb{N}_0} \int_{\mathfrak{D}} \hat{f}\big(\xi + u(m)\big)\overline{\hat{f_\alpha}\big(\xi + u(m)\big)}\, \chi_n\big(\xi + u(m)\big) d\xi \right\} \overline{\hat{f}(\xi)}\hat{f_\alpha}(\xi)\, \overline{\chi}_n(\xi)\, d\xi.
\end{aligned}$$

Notice that $\hat{f}$ has compact support and $\chi_n\big(u(m)\big) \equiv 1$ for all $m, n \in \mathbb{N}_0$. Therefore, by the convergence theorem of Fourier series on $\mathfrak{D}$, we obtain

$$\sum_{n\in\mathbb{N}_0} \left|\left\langle f, T_{u(n)} f_\alpha \right\rangle\right|^2 = \int_K \overline{\hat{f}(\xi)}\hat{f_\alpha}(\xi) \left\{ \sum_{m\in\mathbb{N}_0} \hat{f}\big(\xi + u(m)\big)\overline{\hat{f_\alpha}\big(\xi + u(m)\big)} \right\} d\xi. \tag{4.3}$$

We claim that

$$\sum_{\alpha\in\Lambda} \sum_{n\in\mathbb{N}_0} \left|\left\langle f, T_{u(n)} f_\alpha \right\rangle\right|^2 = \sum_{\alpha\in\Lambda} \int_K \left|\hat{f}(\xi)\right|^2 \left|\hat{f_\alpha}(\xi)\right|^2 d\xi + R(f), \tag{4.4}$$

hold for all $f \in \Omega_0$. In fact, by (4.3), we have

$$\sum_{\alpha\in\Lambda}\sum_{n\in\mathbb{N}_0}\left|\left\langle f, T_{u(n)}f_\alpha\right\rangle\right|^2 = \sum_{\alpha\in\Lambda}\int_K\left\{\left|\hat{f}(\xi)\right|^2\left|\hat{f}_\alpha(\xi)\right|^2 + \overline{\hat{f}(\xi)}\hat{f}_\alpha(\xi)\sum_{m\in\mathbb{N}_0}\hat{f}\big(\xi+u(m)\big)\overline{\hat{f}_\alpha\big(\xi+u(m)\big)}\right\}$$
$$= \sum_{\alpha\in\Lambda}\int_K\left|\hat{f}(\xi)\right|^2\left|\hat{f}_\alpha(\xi)\right|^2 d\xi + R(f).$$

This is just (4.4). Finally, by the condition that $\operatorname{ess\,sup}\left\{\sum_{\alpha\in\Lambda}\left|\hat{f}_\alpha(\xi)\right|^2 : \xi \in \mathfrak{B}^{-1}\setminus\mathfrak{D}\right\} < \infty$, and Levi Lemma, we have

$$\sum_{\alpha\in\Lambda}\sum_{n\in\mathbb{N}_0}\left|\left\langle f, T_{u(n)}f_\alpha\right\rangle\right|^2 = \int_K\left|\hat{f}(\xi)\right|^2\sum_{\alpha\in\Lambda}\left|\hat{f}_\alpha(\xi)\right|^2 d\xi + R(f).$$

We now turn to prove that the iterated series in (4.2) is absolutely convergent. Note that

$$\left|\overline{\hat{f}_\alpha(\xi)}\hat{f}_\alpha\big(\xi+u(m)\big)\right| \le \frac{1}{2}\left[\left|\hat{f}_\alpha(\xi)\right|^2 + \left|\hat{f}_\alpha\big(\xi+u(m)\big)\right|^2\right].$$

We have

$$\left|R(f)\right| = \left|\sum_{\alpha\in\Lambda}\sum_{m\in\mathbb{N}}\int_K \hat{f}(\xi)\overline{\hat{f}_\alpha(\xi)}\hat{f}\big(\xi+u(m)\big)\overline{\hat{f}_\alpha\big(\xi+u(m)\big)}\,d\xi\right|$$
$$\le \sum_{\alpha\in\Lambda}\sum_{m\in\mathbb{N}}\int_K\left|\hat{f}(\xi)\overline{\hat{f}_\alpha(\xi)}\hat{f}\big(\xi+u(m)\big)\overline{\hat{f}_\alpha\big(\xi+u(m)\big)}\,d\xi\right|$$
$$\le \frac{1}{2}\sum_{\alpha\in\Lambda}\sum_{m\in\mathbb{N}}\int_K\left|\hat{f}(\xi)\hat{f}\big(\xi+u(m)\big)\right|\left|\hat{f}_\alpha(\xi)\right|^2 d\xi + \frac{1}{2}\sum_{\alpha\in\Lambda}\sum_{m\in\mathbb{N}}\int_K\left|\hat{f}(\xi)\hat{f}\big(\xi+u(m)\big)\right|\left|\hat{f}_\alpha\big(\xi+u(m)\big)\right|^2 d\xi.$$

Hence it suffices to verify that the series

$$\sum_{\alpha\in\Lambda}\sum_{m\in\mathbb{N}}\int_K\left|\hat{f}(\xi)\hat{f}\big(\xi+u(m)\big)\right|\left|\hat{f}_\alpha(\xi)\right|^2 d\xi$$

is convergent. In fact,

$$\sum_{\alpha\in\Lambda}\sum_{m\in\mathbb{N}}\int_K\left|\hat{f}(\xi)\hat{f}\big(\xi+u(m)\big)\right|\left|\hat{f}_\alpha(\xi)\right|^2 d\xi \le \operatorname{ess}\sup_{\xi\in\mathfrak{B}^{-1}\setminus\mathfrak{D}}\sum_{\alpha\in\Lambda}\left|\hat{f}_\alpha(\xi)\right|^2\sum_{m\in\mathbb{N}}\int_K\left|\hat{f}(\xi)\hat{f}\big(\xi+u(m)\big)\right|d\xi. \tag{4.5}$$

Since $u(m) \neq 0 (m \in \mathbb{N})$ and $f \in \Omega_0$, hence, only finite terms of the iterated series in (4.5) are non-zero. Consequently, (4.5) becomes

$$\sum_{\alpha\in\Lambda}\sum_{m\in\mathbb{N}}\int_K\left|\hat{f}(\xi)\hat{f}\big(\xi+u(m)\big)\right|\left|\hat{f}_\alpha(\xi)\right|^2 d\xi \le \operatorname{ess}\sup_{\xi\in\mathfrak{B}^{-1}\setminus\mathfrak{D}}\sum_{\alpha\in\Lambda}\left|\hat{f}_\alpha(\xi)\right|^2 C\left\|f\right\|^2, \tag{4.6}$$

where C is a constant. Using the assumption $\operatorname{ess\,sup}_{\xi\in\mathfrak{B}^{-1}\setminus\mathfrak{D}}\sum_{\alpha\in\Lambda}\left|\hat{f}_\alpha(\xi)\right|^2 < \infty$ and equation (4.6), it follows that the series (4.2) is absolutely convergent for all $f \in \Omega_0$. The proof of the lemma is completed. □

To establish the first sufficient condition of shift-invariant frame for $L^2(K)$, we put

$$\underline{S}_f = \operatorname{ess\,inf}\left\{S_f(\xi) : \xi \in \mathfrak{B}^{-1}\setminus\mathfrak{D}\right\},$$
$$\overline{S}_f = \operatorname{ess\,sup}\left\{S_f(\xi) : \xi \in \mathfrak{B}^{-1}\setminus\mathfrak{D}\right\},$$
$$\beta_f\big(u(m)\big) = \operatorname{ess\,sup}\left\{\sum_{\alpha\in\Lambda}\left|\hat{f}_\alpha(\xi)\hat{f}_\alpha\big(\xi+u(m)\big)\right| : \xi \in \mathfrak{B}^{-1}\setminus\mathfrak{D}\right\},$$

where $S_f(\xi)$ is the same as the one in (3.1).

Theorem 4.2. *Suppose* $\left\{f_\alpha(x) : \alpha \in \Lambda\right\} \subset L^2(K)$ *such that*

$$A_1 = \underline{S}_f - \sum_{m\in\mathbb{N}} \left[\beta_f\big(u(m)\big)\cdot\beta_f\big(-u(m)\big)\right]^{1/2} > 0, \tag{4.7}$$

$$B_1 = \overline{S}_f + \sum_{m\in\mathbb{N}} \left[\beta_f\big(u(m)\big)\cdot\beta_f\big(-u(m)\big)\right]^{1/2} < +\infty. \tag{4.8}$$

Then $\left\{T_{u(n)}f_\alpha(x) : n \in \mathbb{N}_0, \alpha \in \Lambda\right\}$ *is a frame for* $L^2(\mathbb{R})$ *with bounds* A_1 *and* B_1.

Proof. By Lemma 4.1 and (4.8), equation (4.1) holds, where

$$R(f) = \sum_{m\in\mathbb{N}} \int_K \hat{f}(\xi)\overline{\hat{f}\big(\xi + u(m)\big)} \sum_{\alpha\in\Lambda} \overline{\hat{f_\alpha}(\xi)}\hat{f_\alpha}\big(\xi + u(m)\big)\,d\xi. \tag{4.9}$$

Using the CauchySchwarz inequality, the change of variables $\eta = \xi + u(m)$ and (4.9), we obtain

$$\begin{aligned}
\left|R(f)\right| &\le \sum_{m\in\mathbb{N}} \left\{\int_K \left|\hat{f}(\xi)\right|^2 \left|\sum_{\alpha\in\Lambda} \overline{\hat{f_\alpha}(\xi)}\hat{f_\alpha}\big(\xi + u(m)\big)\right| d\xi\right\}^{1/2} \left\{\int_K \left|\hat{f}\big(\xi + u(m)\big)\right|^2 \left|\sum_{\alpha\in\Lambda} \overline{\hat{f_\alpha}(\xi)}\hat{f_\alpha}\big(\xi + u(m)\big)\right| d\xi\right\}^{1/2} \\
&= \sum_{m\in\mathbb{N}} \left\{\int_K \left|\hat{f}(\xi)\right|^2 \left|\sum_{\alpha\in\Lambda} \overline{\hat{f_\alpha}(\xi)}\hat{f_\alpha}\big(\xi + u(m)\big)\right| d\xi\right\}^{1/2} \left\{\int_K \left|\hat{f}(\eta)\right|^2 \left|\sum_{\alpha\in\Lambda} \overline{\hat{f_\alpha}\big(\eta - u(m)\big)}\hat{f_\alpha}(\eta)\right| d\eta\right\}^{1/2} \\
&\le \int_K \left|\hat{f}(\xi)\right|^2 d\xi \sum_{m\in\mathbb{N}} \left[\beta_f\big(u(m)\big)\cdot\beta_f\big(-u(m)\big)\right]^{1/2} \\
&\le \|f\|_2^2 \sum_{m\in\mathbb{N}} \left[\beta_f\big(u(m)\big)\cdot\beta_f\big(-u(m)\big)\right]^{1/2}.
\end{aligned} \tag{4.10}$$

Consequently, it follows from equations (4.1), (4.7), (4.8) and (4.10) that

$$A_1\|f\|_2^2 \le \sum_{\alpha\in\Lambda}\sum_{n\in\mathbb{N}_0} \left|\left\langle f, T_{u(n)}f_\alpha\right\rangle\right|^2 \le B_1\|f\|_2^2, \quad \text{for all } f \in \Omega_0.$$

The proof of Theorem 4.2 is complete. □

Remark: It is not difficult to show that

$$\sum_{m\in\mathbb{N}} \beta_f\big(u(m)\big) = \sum_{m\in\mathbb{N}} \beta_f\big(-u(m)\big).$$

Set $\beta_f = \sum_{m\in\mathbb{N}} \beta_f\big(u(m)\big)$. Then by (4.9) and the CauchySchwarz inequality, we have

$$\left|R(f)\right| \le \sum_{m\in\mathbb{N}} \left[\beta_f\big(u(m)\big)\cdot\beta_f\big(-u(m)\big)\right]^{1/2}\|f\|_2^2 \le \beta_f\|f\|_2^2.$$

As a consequence, the following second sufficient condition is obtained.

Theorem 4.3. *Suppose* $\left\{f_\alpha(x) : \alpha \in \Lambda\right\} \subset L^2(K)$ *such that*

$$A_2 = \underline{S}_f - \beta_f > 0, \tag{4.11}$$

$$B_2 = \overline{S}_f + \beta_f < +\infty. \tag{4.12}$$

Then $\left\{T_{u(n)} f_\alpha(x) : n \in \mathbb{N}_0, \alpha \in \Lambda\right\}$ *is a frame for* $L^2(\mathbb{R})$ *with bounds* A_2 *and* B_2*.*

The proof is similar to that of Theorem 4.2.

Using another estimation technique, we are able to give the third sufficient condition for the shift-invariant system $\mathcal{W}(f, \alpha, k)$ to be frame of $L^2(K)$ as follows:

Theorem 4.4. *Suppose* $\left\{f_\alpha(x) : \alpha \in \Lambda\right\} \subset L^2(K)$ *such that*

$$A_3 = \operatorname{ess} \inf_{\xi \in \mathfrak{B}^{-1} \setminus \mathfrak{D}} \left[\sum_{\alpha \in \Lambda} \left| \hat{f}_\alpha(\xi) \right|^2 - \sum_{m \in \mathbb{N}} \left| \sum_{\alpha \in \Lambda} \overline{\hat{f}_\alpha(\xi)} \hat{f}_\alpha\big(\xi + u(m)\big) \right| \right] > 0, \tag{4.13}$$

$$B_3 = \operatorname{ess} \sup_{\xi \in \mathfrak{B}^{-1} \setminus \mathfrak{D}} \left[\sum_{m \in \mathbb{N}_0} \left| \sum_{\alpha \in \Lambda} \overline{\hat{f}_\alpha(\xi)} \hat{f}_\alpha\big(\xi + u(m)\big) \right| \right] < +\infty. \tag{4.14}$$

Then $\left\{T_{u(n)} f_\alpha(x) : n \in \mathbb{N}_0, \alpha \in \Lambda\right\}$ *is a frame for* $L^2(\mathbb{R})$ *with bounds* A_3 *and* B_3*.*

Proof. We estimate $R(f)$ in (4.9) by another technique. Using the Cauchy-Schwarz inequality, the change of variables $\eta = \xi + u(m)$ and the Levi Lemma, we have

$$\begin{aligned}
\left|R(f)\right| &\le \sum_{m \in \mathbb{N}} \left\{ \int_K \left|\hat{f}(\xi)\right|^2 \left| \sum_{\alpha \in \Lambda} \overline{\hat{f}_\alpha(\xi)} \hat{f}_\alpha\big(\xi + u(m)\big) \right| d\xi \right\}^{1/2} \left\{ \int_K \left|\hat{f}\big(\xi + u(m)\big)\right|^2 \left| \sum_{\alpha \in \Lambda} \overline{\hat{f}_\alpha(\xi)} \hat{f}_\alpha\big(\xi + u(m)\big) \right| d\xi \right\}^{1/2} \\
&\le \left\{ \int_K \left|\hat{f}(\xi)\right|^2 \sum_{m \in \mathbb{N}} \left| \sum_{\alpha \in \Lambda} \overline{\hat{f}_\alpha(\xi)} \hat{f}_\alpha\big(\xi + u(m)\big) \right| d\xi \right\}^{1/2} \left\{ \int_K \left|\hat{f}(\eta)\right|^2 \sum_{m \in \mathbb{N}} \left| \sum_{\alpha \in \Lambda} \overline{\hat{f}_\alpha(\eta)} \hat{f}_\alpha\big(\eta - u(m)\big) \right| d\eta \right\}^{1/2} \\
&= \int_K \left|\hat{f}(\xi)\right|^2 \sum_{m \in \mathbb{N}} \left| \sum_{\alpha \in \Lambda} \overline{\hat{f}_\alpha(\xi)} \hat{f}_\alpha\big(\xi + u(m)\big) \right| d\xi.
\end{aligned} \tag{4.15}$$

Consequently, it follows from equations (4.1), (4.13), (4.14) and (4.15) that

$$A_3 \|f\|_2^2 \le \sum_{\alpha \in \Lambda} \sum_{n \in \mathbb{N}_0} \left| \left\langle f, T_{u(n)} f_\alpha \right\rangle \right|^2 \le B_3 \|f\|_2^2, \quad \text{for all } f \in \Omega_0.$$

The proof of Theorem 4.4 is complete.

5. Shift-invariant Systems as Gabor Frames on Local Fields

In this section, we apply Theorems 3.1, 4.2 and 4.4 to the Gabor systems and obtain some results on Gabor frames on local fields of positive characteristic.

Gabor systems are the collection of functions

$$\mathcal{G}(g, m, n) = \left\{ M_{u(m)} T_{u(n)} g(x) =: \chi_m(x) g\big(x - u(n)\big) : m, n \in \mathbb{N}_0 \right\} \tag{5.1}$$

which are built by the action of modulations and translations of a single function $g \in L^2(K)$. If we change the order of the translation and modulation operators, we have the system

$$\mathcal{G}'(g, m, n) = \left\{ T_{u(n)} M_{u(m)} g(x) =: \chi_m(x) g\big(x - u(n)\big) : m, n \in \mathbb{N}_0 \right\}. \tag{5.2}$$

It is immediate to see that the system $\mathcal{G}(g,m,n)$ is a frame of $L^2(K)$ if and only if $\mathcal{G}'(a,b,g)$ is a frame of $L^2(K)$, and the frame bounds are the same in the two cases. It is evident that Gabor system (5.2) is shift-invariant. So, the main results in Sections 3 and 4 can apply directly to the Gabor systems.

By setting $\Lambda = \{u(m) : m \in \mathbb{N}_0\}$, and for all $\alpha \in \Lambda$, we take $f_\alpha = E_{u(m)}g(x)$. Then the system $\mathcal{W}(f,\alpha,m)$ given by (2.10) reduces to the Gabor system $\mathcal{G}(g,m,n)$ defined by (5.1). Notice that for all $\alpha \in \Lambda$,

$$\hat{f_\alpha}(\xi) = \hat{g}\Big(\xi - u(m)\Big).$$

Therefore, for all $n \in \mathbb{N}_0$, we have

$$\sum_{\alpha\in\Lambda} \overline{\hat{f_\alpha}(\xi)}\hat{f_\alpha}\Big(\xi + u(n)\Big) = \sum_{m\in\mathbb{N}_0} \overline{\hat{g}\Big(\xi - u(m)\Big)}\hat{g}\Big(\xi - u(m) + u(n)\Big).$$

Using Theorems 3.1, 4.2 and 4.4, we obtain

Theorem 5.1. *If the Gabor system $\mathcal{G}(g,m,n)$ defined by (5.1) is a frame for $L^2(K)$ with bounds A_4 and B_4, then*

$$A_4 \le \sum_{m\in\mathbb{N}_0} \left|\hat{g}\Big(\xi - u(m)\Big)\right|^2 \le B_4, \quad a.e.\ \xi \in K. \tag{5.3}$$

Theorem 5.2. *Suppose $g \in L^2(K)$ such that*

$$A_5 = \operatorname{ess}\inf_{\xi\in\mathfrak{B}^{-1}\setminus\mathfrak{D}} \sum_{m\in\mathbb{N}_0} \left|\hat{g}\Big(\xi - u(m)\Big)\right|^2 - \sum_{n\in\mathbb{N}} \Big[\beta_g\Big(u(n)\Big)\cdot\beta_g\Big(-u(n)\Big)\Big]^{1/2} > 0, \tag{5.4}$$

$$B_5 = \operatorname{ess}\sup_{\xi\in\mathfrak{B}^{-1}\setminus\mathfrak{D}} \sum_{m\in\mathbb{N}_0} \left|\hat{g}\Big(\xi - u(m)\Big)\right|^2 + \sum_{n\in\mathbb{N}} \Big[\beta_g\Big(u(n)\Big)\cdot\beta_g\Big(-u(n)\Big)\Big]^{1/2} < +\infty, \tag{5.5}$$

then $\Big\{M_{u(m)}T_{u(n)}g(x) : m,n \in \mathbb{N}_0\Big\}$ is a Gabor frame for $L^2(K)$ with bounds A_5 and B_5, where

$$\beta_g\Big(u(n)\Big) = \operatorname{ess\,sup}\left\{\sum_{m\in\mathbb{N}_0}\left|\hat{g}\Big(\xi - u(m)\Big)\hat{g}\Big(\xi - u(m) + u(n)\Big)\right| : \xi \in \mathfrak{B}^{-1}\setminus\mathfrak{D}\right\}.$$

Theorem 5.3. *Suppose $g \in L^2(K)$ such that*

$$A_6 = \operatorname{ess}\inf_{\xi\in\mathfrak{B}^{-1}\setminus\mathfrak{D}} \left\{\sum_{m\in\mathbb{N}_0}\left|\hat{g}\Big(\xi - u(m)\Big)\right|^2 - \sum_{n\in\mathbb{N}}\left|\sum_{m\in\mathbb{N}_0}\overline{\hat{g}\Big(\xi - u(m)\Big)}\hat{g}\Big(\xi - u(m) + u(n)\Big)\right|\right\} > 0, \tag{5.6}$$

$$B_6 = \operatorname{ess}\sup_{\xi\in\mathfrak{B}^{-1}\setminus\mathfrak{D}} \left\{\sum_{n\in\mathbb{N}_0}\left|\sum_{m\in\mathbb{N}_0}\overline{\hat{g}\Big(\xi - u(m)\Big)}\hat{g}\Big(\xi - u(m) + u(n)\Big)\right|\right\} < +\infty. \tag{5.7}$$

Then $\Big\{M_{u(m)}T_{u(n)}g(x) : m,n \in \mathbb{N}_0\Big\}$ is a Gabor frame for $L^2(K)$ with bounds A_6 and B_6.

Remark: Since

$$\Big\langle f, T_{u(n)}M_{u(m)}g\Big\rangle = \Big\langle f^\vee, \Big(T_{u(n)}M_{u(m)}g\Big)^\vee\Big\rangle = \Big\langle f^\vee, T_{-u(m)}M_{u(n)}g^\vee\Big\rangle$$

by the Plancherel Theorem, similarly to the case in the frequency domain, we able to present similar results in the time domain. They were omitted.

6. Shift-invariant Systems as Wavelet Frames on Local Fields

In this section, we apply Theorems 3.1, 4.2 and 4.4 to the wavelet systems and obtain some results on wavelet frames on local fields of positive characteristic.

For a given $\psi \in L^2(K)$, define the wavelet system

$$\mathcal{F}(\psi, j, k) = \left\{\psi_{j,k}(x) : j \in \mathbb{Z}, k \in \mathbb{N}_0\right\} \tag{6.1}$$

where $\psi_{j,k}(x) = q^{j/2}\psi\left(\mathfrak{p}^{-j}x - u(k)\right)$. In general, the wavelet system $\mathcal{F}(\psi, j, k)$ is not shift-invariant, and thus the main results in Sections 3 and 4 do not apply directly to a wavelet system. But we can use a quasi-wavelet system to investigate the wavelet system. The *quasi-wavelet system* generated by $\psi \in L^2(K)$ is defined by

$$\tilde{\mathcal{F}}\left(\tilde{\psi}, j, k\right) = \left\{\tilde{\psi}_{j,k}(x) : j \in \mathbb{Z}, k \in \mathbb{N}_0\right\} \tag{6.2}$$

where

$$\tilde{\psi}_{j,k}(x) = \begin{cases} D_{\mathfrak{p}^j} T_{u(k)}\psi(x) = q^{j/2}\psi\left(\mathfrak{p}^{-j}x - u(k)\right), & j \geq 0, k \in \mathbb{N}_0, \\ \\ q^{j/2} T_{u(k)} D_{\mathfrak{p}^j}\psi(x) = q^{j}\psi\left(\mathfrak{p}^{-j}\left(x - u(k)\right)\right), & j < 0, k \in \mathbb{N}_0. \end{cases} \tag{6.3}$$

It is not difficult to see that the quasi-wavelet system is shift-invariant. There is some sort of equivalence between wavelet and quasi-wavelet systems. Indeed, Behera and Jehan [6] proved in full generality the following result on local fields of positive characteristic.

Theorem 6.1. *Let $\psi \in L^2(K)$. Then*

(a) *$\mathcal{F}(\psi, j, k)$ is a Bessel family if and only if $\tilde{\mathcal{F}}\left(\tilde{\psi}, j, k\right)$ is a Bessel family. Furthermore, their exact upper bounds are equal.*

(b) *$\mathcal{F}(\psi, j, k)$ is a frame for $L^2(K)$ if and only if $\tilde{\mathcal{F}}\left(\tilde{\psi}, j, k\right)$ is a frame for $L^2(K)$. Furthermore, their lower and upper exact bounds are equal.*

For $j \in \mathbb{N}_0$, let $\mathcal{N}_j$ denotes a full collection of coset representatives of $\mathbb{N}_0/q^j\mathbb{N}_0$, i.e.,

$$\mathcal{N}_j = \begin{cases} \left\{0, 1, 2, \ldots, q^j - 1\right\}, & j \geq 0 \\ \{0\}, & j < 0. \end{cases} \tag{6.4}$$

Then, $\mathbb{N}_0 = \bigcup_{n \in \mathcal{N}_j}\left(n + q^j\mathbb{N}_0\right)$, and for any distinct $n_1, n_2 \in \mathcal{N}_j$, we have $\left(n_1 + q^j\mathbb{N}_0\right) \cap \left(n_2 + q^j\mathbb{N}_0\right) = \emptyset$. Thus, every non-negative integer k can uniquely be written as $k = rq^j + s$, where $r \in \mathbb{N}_0, s \in \mathcal{N}_j$.

We now show that the quasi-wavelet system $\tilde{\mathcal{F}}\left(\tilde{\psi}, j, k\right)$ given by (6.2) is invariant under translations by $u(k), k \in \mathbb{N}_0$. In fact

$$T_{u(k)}\tilde{\psi}_{j,n}(x) = \tilde{\psi}_{j,0}\left(x - u(k)\right) = q^j\psi\left(\mathfrak{p}^{-j}\left(x - u(k)\right)\right) = \tilde{\psi}_{j,k}, \quad \text{if } j < 0,$$

and for $j \geq 0, n \in \mathcal{N}_j$, we have

$$\begin{aligned} T_{u(k)}\tilde{\psi}_{j,n}(x) &= \tilde{\psi}_{j,n}\left(x - u(k)\right) \\ &= \psi_{j,d}\left(x - u(k)\right) \\ &= q^{j/2}\psi\left(\mathfrak{p}^{-j}\left(x - u(k)\right) - u(n)\right) \end{aligned}$$

$$= q^{j/2}\psi\Big(\mathfrak{p}^{-j}x - \big(\mathfrak{p}^{-j}u(k) + u(n)\big)\Big)$$
$$= q^{j/2}\psi\Big(\mathfrak{p}^{-j}x - u(kq^j + n)\Big)$$
$$= \psi_{j,kq^j+n}(x).$$

Therefore, the quasi-wavelet system can also be represented as

$$\tilde{\mathcal{F}}\Big(\tilde{\psi}, j, k\Big) = \Big\{T_{u(k)}\tilde{\psi}_{j,n}(x) : j \in \mathbb{Z}, k \in \mathbb{N}_0, n \in \mathcal{N}_j\Big\}. \tag{6.5}$$

Suppose that $\Lambda = \Big\{(j, n) : j \in \mathbb{Z}, n \in \mathcal{N}_j\Big\}$. Then, for all $\alpha \in \Lambda$, we set

$$f_\alpha(x) = \begin{cases} q^{j/2}\psi\Big(\mathfrak{p}^{-j}x - u(n)\Big), & \text{if } j \geq 0, \\ q^{j}\psi\Big(\mathfrak{p}^{-j}\big(x - u(n)\big)\Big), & \text{if } j < 0. \end{cases} \tag{6.6}$$

Therefore, one can easily see that $f_\alpha \in L^2(K)$ and consequently, the system $\Big\{T_{u(k)}f_\alpha : k \in \mathbb{N}_0, \alpha \in \Lambda\Big\}$ is the quasi-wavelet system $\tilde{\mathcal{F}}\Big(\tilde{\psi}, j, k\Big)$. Moreover, the Fourier transform of (6.6) yields

$$\hat{f_\alpha}(\xi) = \begin{cases} q^{-j/2}\hat{\psi}\Big(\mathfrak{p}^j\xi\Big)\overline{\chi_n\Big(\mathfrak{p}^j\xi\Big)}, & \text{if } j \geq 0, \\ \hat{\psi}\Big(\mathfrak{p}^j\xi\Big)\overline{\chi_n\Big(\mathfrak{p}^j\xi\Big)}, & \text{if } j < 0. \end{cases} \tag{6.7}$$

Thus, for all $m \in \mathbb{N}_0$, we have

$$\begin{aligned} \sum_{\alpha\in\Lambda}\Big|\hat{f_\alpha}(\xi)\Big|\Big|\hat{f_\alpha}\Big(\xi + u(m)\Big)\Big| &= \sum_{j<0}\Big|\hat{\psi}\Big(\mathfrak{p}^j\xi\Big)\Big|\Big|\hat{\psi}\Big(\mathfrak{p}^j\Big(\xi + u(m)\Big)\Big)\Big| + \sum_{j\geq 0}\sum_{n\in\mathcal{N}_j} q^{-j/2}\Big|\hat{\psi}\Big(\mathfrak{p}^j\xi\Big)\Big|\Big|\hat{\psi}\Big(\mathfrak{p}^j\Big(\xi + u(m)\Big)\Big)\Big| \\ &= \sum_{j\in\mathbb{Z}}\Big|\hat{\psi}\Big(\mathfrak{p}^j\xi\Big)\Big|\Big|\hat{\psi}\Big(\mathfrak{p}^j\Big(\xi + u(m)\Big)\Big)\Big|. \end{aligned}$$

As a consequence of Theorems 3.1, 4.2, 4.4 and Theorem 6.1, a necessary condition and two sufficient conditions for wavelet frames on local fields of positive characteristic are obtained. These results are known (see [16, 17]).

Theorem 6.2. *If the quasi-wavelet system* $\Big\{\tilde{\psi}_{j,k} : j \in \mathbb{Z}, k \in \mathbb{N}_0\Big\}$ *defined by (6.2) is a frame for* $L^2(K)$ *with bounds* A_7 *and* B_7*, then*

$$A_7 \leq \sum_{j\in\mathbb{Z}}\Big|\hat{\psi}\Big(\mathfrak{p}^j\xi\Big)\Big|^2 \leq B_7, \quad a.e.\ \xi \in K. \tag{6.8}$$

Theorem 6.3. *Let* $\psi \in L^2(K)$*. If*

$$A_8 = \operatorname{ess}\inf_{\xi\in\mathfrak{B}^{-1}\setminus\mathfrak{D}}\sum_{j\in\mathbb{Z}}\Big|\hat{\psi}\Big(\mathfrak{p}^j\xi\Big)\Big|^2 - \sum_{n\in\mathbb{N}}\Big[\beta_\psi\Big(u(n)\Big)\cdot\beta_\psi\Big(-u(n)\Big)\Big]^{1/2} > 0, \tag{6.9}$$

$$B_8 = \operatorname{ess}\sup_{\xi\in\mathfrak{B}^{-1}\setminus\mathfrak{D}}\sum_{j\in\mathbb{Z}}\Big|\hat{\psi}\Big(\mathfrak{p}^j\xi\Big)\Big|^2 + \sum_{n\in\mathbb{N}}\Big[\beta_\psi\Big(u(n)\Big)\cdot\beta_\psi\Big(-u(n)\Big)\Big]^{1/2} < +\infty. \tag{6.10}$$

Then $\Big\{\tilde{\psi}_{j,k}(x) : j \in \mathbb{Z}, k \in \mathbb{N}_0\Big\}$ *is a wavelet frame for* $L^2(K)$ *with bounds* A_8 *and* B_8*, where*

$$\beta_\psi\Big(u(n)\Big) = \operatorname{ess\,sup}\left\{\sum_{j\in\mathbb{Z}}\Big|\hat{\psi}\Big(\mathfrak{p}^j\xi\Big)\hat{\psi}\Big(\mathfrak{p}^j\Big(\xi + u(n)\Big)\Big)\Big| : \xi \in \mathfrak{B}^{-1}\setminus\mathfrak{D}\right\}.$$

Theorem 6.4. *Let $\psi \in L^2(K)$. If*

$$A_9 = \operatorname{ess}\inf_{\xi\in\mathfrak{B}^{-1}\setminus\mathfrak{D}} \left\{ \sum_{j\in\mathbb{Z}} \left|\hat{\psi}\left(\mathfrak{p}^j\xi\right)\right|^2 - \sum_{j\in\mathbb{Z}}\sum_{n\in\mathbb{N}} \left|\hat{\psi}\left(\mathfrak{p}^j\xi\right)\hat{\psi}\left(\mathfrak{p}^j\left(\xi+u(n)\right)\right)\right| \right\} > 0, \tag{6.11}$$

$$B_9 = \operatorname{ess}\sup_{\xi\in\mathfrak{B}^{-1}\setminus\mathfrak{D}} \left\{ \sum_{j\in\mathbb{Z}}\sum_{n\in\mathbb{N}_0} \left|\hat{\psi}\left(\mathfrak{p}^j\xi\right)\hat{\psi}\left(\mathfrak{p}^j\left(\xi+u(n)\right)\right)\right| \right\} < +\infty. \tag{6.12}$$

Then $\left\{\tilde{\psi}_{j,k}(x) : j \in \mathbb{Z}, k \in \mathbb{N}_0\right\}$ is a wavelet frame for $L^2(K)$ with bounds A_9 and B_9.

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