Filomat 32:9 (2018), 3123–3128
https://doi.org/10.2298/FIL1809123E

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Concavity in Fractional Calculus

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Abstract. We discuss a concavity like property for functions u satisfying $D_{0^+}^{\alpha}u \in C[0,b]$ with $u(0) = 0$ and $-D_{0^+}^{\alpha}u(t) \geq 0$ for all $t \in [0,b]$. We develop the property for $\alpha \in (1,2]$, where $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative. We observe the property is also valid in the case $\alpha = 1$. Finally, we show that under certain conditions, $-D_{0^+}^{\alpha}u(t) \geq 0$ implies u is concave in the classical sense.

1. Introduction

In this paper, we show that Green's functions for fractional boundary value problems with order $\alpha \in (1,2]$ satisfy a concavity like property. Using this property, it is shown that if $D_{0^+}^{\alpha}u \in C[0,b]$, $u(0) = 0$, and $-D_{0^+}^{\alpha}u(t) \geq 0$ for all $t \in [0,b]$, then u also satisfies this concavity like property. This property gives a geometric meaning to sign properties of fractional derivatives of order $\alpha \in (1,2]$, similar to the geometric meaning of sign properties of the first and second derivative. Interestingly, this property provides a geometric link between monotonicity and concavity. Finally, we show that if $-D_{0^+}^{\alpha}u(t) \geq 0$ and if u satisfies other conditions, u is concave in the classical sense.

There has been limited work done on concavity properties of Caputo and Riemann Liouville fractional derivatives. In [1], Al-Refai shows that if $f \in C^{(2)}[0,1]$ attains its minimum at $t_0 \in (0,1)$ and $f'(0) \leq 0$, $D_C^{\delta}f(t_0) \geq 0$ for all $\delta \in (1,2)$. Here D_C^{δ} is the Caputo fractional derivative. This is similar to a classical result related to concavity. However, the sign of the Riemann-Liouville fractional derivative depends on the sign of $f(t)$ on $[0,1]$. There has also been work done on monotonicity, convexity, and concavity related to fractional differences. For a few examples, see [3, 8, 9].

2. A Geometric Property of Concave Functions

A function $u \in C^{(2)}[a,b]$ is concave on $[a,b]$ if the graphs of secant lines connecting $(c,u(c))$ and $(d,u(d))$ lie below the graph of the function u for all $c,d \in [a,b]$; i.e., if

$$u(\lambda c + (1-\lambda)d) \geq \lambda u(c) + (1-\lambda)u(d) \text{ for } \lambda \in [0,1] \text{ and for all } c,d \in [a,b].$$

Concavity is equivalent to slopes of secant lines from $(a,u(a))$ to $(t,u(t))$ being decreasing as a function of t. This gives the following lemma.

2010 *Mathematics Subject Classification*. Primary 26A33
Keywords. Fractional derivatives, concavity
Received: 14 August 2017; Accepted: 12 November 2017
Communicated by Jelena Manojlović
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Lemma 2.1. *Let $u \in C^{(2)}[a,b]$. Then $-u''(t) \geq 0$ for all $t \in [a,b]$ if and only if*

$$(y-a)u(w) \leq (w-a)u(y) \tag{1}$$

for all $w, y \in [a,b]$ with $y \leq w$.

Proof. Suppose $-u''(t) \geq 0$ for all $t \in [a,b]$. We approach this portion of the proof by utilizing the Green's function for $-u''(t) = 0$, $u(a) = 0$, $u(b) = 0$, which is given by

$$G(t,s) = \frac{1}{b-a}\begin{cases} (t-a)(b-s), & a \leq t \leq s \leq b, \\ (s-a)(b-t), & a \leq s \leq t \leq b. \end{cases}$$

We show $(y-a)G(w,s) \leq (w-a)G(y,s)$ for all $w, y \in [a,b]$ with $y \leq w$. If $y = a$ or $w = b$, the proof is trivial. Notice if $a < y \leq w \leq s < b$,

$$\frac{G(w,s)}{G(y,s)} = \frac{(w-a)(b-s)}{(y-a)(b-s)} = \frac{w-a}{y-a},$$

if $a < y \leq s \leq w < b$,

$$\frac{G(w,s)}{G(y,s)} = \frac{(s-a)(b-w)}{(y-a)(b-s)} \leq \frac{(w-a)(b-w)}{(y-a)(b-w)} = \frac{w-a}{y-a},$$

and if $a < s \leq y \leq w < b$,

$$\frac{G(w,s)}{G(y,s)} = \frac{(s-a)(b-w)}{(s-a)(b-y)} \leq 1 \leq \frac{w-a}{y-a}.$$

So $(y-a)G(w,s) \leq (w-a)G(y,s)$ for all $w, y \in [a,b]$ with $y \leq w$.

By the properties of the Green's function, since $u \in C^{(2)}[a,b]$,

$$u(t) = z(t) + \int_a^b G(t,s)(-u''(s))ds,$$

where $z(t) = u(a) + \frac{t-a}{b-a}(u(b) - u(a))$. Notice

$$\begin{aligned}(y-a)z(w) &= (y-a)u(a) + \frac{(y-a)(w-a)}{b-a}(u(b)-u(a)) \\ &\leq (w-a)u(a) + \frac{(w-a)(y-a)}{b-a}(u(b)-u(a)) \\ &= (w-a)z(y).\end{aligned}$$

So

$$\begin{aligned}(y-a)u(w) &= (y-a)z(w) + \int_a^b (y-a)G(w,s)(-u''(s))ds \\ &\leq (w-a)z(y) + \int_a^b (w-a)G(y,s)(-u''(s))ds \\ &= (w-a)u(y).\end{aligned}$$

That completes the proof of $-u''(t) \geq 0$ for all $t \in [a,b]$ implies (1).

Now, assume (1). Let

$$s(t) = \frac{u(w) - u(y)}{w-y}(t-w) + u(w)$$

be the secant line connecting the points $(y, u(y))$ and $(w, u(w))$. If $t = y$ or $t = w$, then $s(t) = u(t)$. If $t \in (y, w)$, then $\frac{y-a}{t-a}u(t) \leq u(y)$ and $u(w) \leq \frac{w-a}{t-a}u(t)$. So

$$\begin{aligned} s(t) &= \frac{u(w)-u(y)}{w-y}(t-w)+u(w) \\ &\leq \frac{(w-a)u(t)-(y-a)u(t)}{(t-a)(w-y)}(t-w)+\frac{w-a}{t-a}u(t) \\ &\leq \frac{u(t)}{t-a}(t-w)+\frac{w-a}{t-a}u(t) \\ &= u(t). \end{aligned}$$

So by definition, u is concave. Since $u \in C^{(2)}[a, b]$, $-u''(t) \geq 0$ for all $t \in [a, b]$. □

The inequality (1) and related inequalities are useful when applying Avery type fixed point theorems to prove the existence of positive solutions of second order boundary value problems satisfying Dirichlet, right focal, periodic, and other boundary conditions. For some examples, see [2, 4, 5, 10].

In this paper, we show if $D^{\alpha}_{0^+}u \in C[0, b]$ with $-D^{\alpha}_{0^+}u \geq 0$, then u satisfies a concavity like property similar to (1).

3. Concavity in the Fractional Case

Definition 3.1. *Let $\nu > 0$. The Riemann-Liouville fractional integral of a function u of order ν, denoted $I^{\nu}_{0^+}u$, is defined as*

$$I^{\nu}_{0^+}u(t) = \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}u(s)ds,$$

provided the right-hand side exists.

Definition 3.2. *Let n denote a positive integer and assume $n-1 < \alpha \leq n$. The Riemann-Liouville fractional derivative of order α of the function $u : [0, 1] \to \mathbb{R}$, denoted $D^{\alpha}_{0^+}u$, is defined as*

$$D^{\alpha}_{0^+}u(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\alpha-1}u(s)ds = D^n I^{n-\alpha}_{0+}u(t),$$

provided the right-hand side exists.

Let $\alpha \in (1, 2]$. The Green's function for the the differential equation $-D^{\alpha}_{0^+}u = 0$ satisfying the boundary conditions

$$u(0) = 0, \quad D^{\beta}_{0^+}u(b) = 0, \tag{2}$$

where $\beta \in [0, 1]$, is given by

$$G(\beta; t, s) = \begin{cases} \frac{t^{\alpha-1}(b-s)^{\alpha-1-\beta}}{b^{\alpha-1-\beta}\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s < t \leq b, \\ \frac{t^{\alpha-1}(b-s)^{\alpha-1-\beta}}{b^{\alpha-1-\beta}\Gamma(\alpha)}, & 0 \leq t \leq s \leq b. \end{cases} \tag{3}$$

The Green's function, (3), has been constructed by many authors, and we refer the reader to [6]. Therefore, if $D^{\alpha}_{0^+}u \in C[0, b]$ and u satisfies the boundary conditions (2), then

$$u(t) = \int_0^b G(\beta; t, s)(-D^{\alpha}_{0^+}u(s))ds, \quad t \in [0, b].$$

In [11], it was shown that for $b = 1$, if $0 \le s < 1$, $G(1; t, s)$ has the property that

$$y^{\alpha-1} G(1; w, s) \le w^{\alpha-1} G(1; y, s)$$

for all $y, w \in [0, 1]$ with $y \le w$. Later, in [7] it was shown that for $b = 1$, if $0 \le s \le 1$, $G(0; t, s)$ also has the property that

$$y^{\alpha-1} G(0; w, s) \le w^{\alpha-1} G(0; y, s)$$

for all $y, w \in [0, 1]$ with $y \le w$.

Here, we generalize this inequality for $G(\beta; t, s)$, $0 \le \beta \le 1$.

Lemma 3.3. *Assume* $1 < \alpha \le 2$, $0 \le \beta \le 1$. *For* $0 \le s < b$, $G(\beta; t, s)$ *has the property that*

$$y^{\alpha-1} G(\beta; w, s) \le w^{\alpha-1} G(\beta; y, s) \tag{4}$$

for all $y, w \in [0, b]$ *with* $y \le w$.

Proof. If $y = 0$, the property holds trivially, and so we assume $y > 0$. We have three cases to consider, when $0 < y \le w \le s < b$, when $0 < y \le s < w \le b$, and when $0 \le s < y \le w \le b$. First, consider the case $0 < y \le w \le s < b$. So

$$\frac{G(\beta; w, s)}{G(\beta; y, s)} = \frac{w^{\alpha-1}(b-s)^{\alpha-\beta-1}}{y^{\alpha-1}(b-s)^{\alpha-\beta-1}} = \frac{w^{\alpha-1}}{y^{\alpha-1}}.$$

Next, suppose $0 < y \le s < w \le b$. Then

$$\begin{aligned} \frac{G(\beta; w, s)}{G(\beta; y, s)} &= \frac{w^{\alpha-1}(b-s)^{\alpha-\beta-1} - b^{\alpha-1-\beta}(w-s)^{\alpha-1}}{y^{\alpha-1}(b-s)^{\alpha-\beta-1}} \\ &\le \frac{w^{\alpha-1}(b-s)^{\alpha-\beta-1}}{y^{\alpha-1}(b-s)^{\alpha-\beta-1}} = \frac{w^{\alpha-1}}{y^{\alpha-1}}. \end{aligned}$$

Finally, let $0 \le s < y \le w \le b$. Thus

$$\begin{aligned} \frac{G(\beta; w, s)}{G(\beta; y, s)} &= \frac{w^{\alpha-1}(b-s)^{\alpha-\beta-1} - b^{\alpha-1-\beta}(w-s)^{\alpha-1}}{y^{\alpha-1}(b-s)^{\alpha-\beta-1} - b^{\alpha-1-\beta}(y-s)^{\alpha-1}} \\ &= \frac{w^{\alpha-1}\left((b-s)^{\alpha-\beta-1} - b^{\alpha-1-\beta}(1-\frac{s}{w})^{\alpha-1}\right)}{y^{\alpha-1}\left((b-s)^{\alpha-\beta-1} - b^{\alpha-1-\beta}(1-\frac{s}{y})^{\alpha-1}\right)} \\ &\le \frac{w^{\alpha-1}\left((b-s)^{\alpha-\beta-1} - b^{\alpha-1-\beta}(1-\frac{s}{y})^{\alpha-1}\right)}{y^{\alpha-1}\left((b-s)^{\alpha-\beta-1} - b^{\alpha-1-\beta}(1-\frac{s}{y})^{\alpha-1}\right)} \\ &= \frac{w^{\alpha-1}}{y^{\alpha-1}}. \end{aligned}$$

□

Lemma 3.3 gives the following result.

Theorem 3.4. *Assume* $1 < \alpha \le 2$. *Let* $D_{0^+}^{\alpha} u \in C[0, b]$ *with* $u(0) = 0$. *If* $-D_{0^+}^{\alpha} u(t) \ge 0$ *for all* $t \in [0, b]$, u *satisfies the concavity like property*

$$y^{\alpha-1} u(w) \le w^{\alpha-1} u(y) \tag{5}$$

for all $y, w \in [0, b]$ *with* $y \le w$.

Proof. By the properties of the Green's function,

$$u(t) = \left(\frac{t}{b}\right)^{\alpha-1} u(b) + \int_0^b G(0;t,s)(-D^{\alpha}_{0^+}u(s))ds.$$

Thus, if $y, w \in [0,1]$ with $y \leq w$,

$$\begin{aligned} y^{\alpha-1}u(w) &= y^{\alpha-1}\left(\frac{w}{b}\right)^{\alpha-1} u(b) + \int_0^b y^{\alpha-1}G(0;w,s)(-D^{\alpha}_{0^+}u(s))ds \\ &\leq w^{\alpha-1}\left(\frac{y}{b}\right)^{\alpha-1} u(b) + \int_0^b w^{\alpha-1}G(0;y,s)(-D^{\alpha}_{0^+}u(s))ds \\ &= w^{\alpha-1}u(y). \end{aligned}$$

□

Note that at $\alpha = 1$, (5) implies that u is monotone decreasing and note that Theorem 3.4 is valid at $\alpha = 1$. Thus, we obtain a stronger version of Theorem 3.4 that contains both concavity and monotonicity.

Theorem 3.5. *Assume* $1 \leq \alpha \leq 2$. *Let* $D^{\alpha}_{0^+}u \in C[0,b]$ *with* $u(0) = 0$. *If* $-D^{\alpha}_{0^+}u(t) \geq 0$ *for all* $t \in [0,b]$, *u satisfies the concavity like property*

$$y^{\alpha-1}u(w) \leq w^{\alpha-1}u(y)$$

for all $y, w \in [0,b]$ *with* $y \leq w$.

The following corollary gives a similar result for functions with $D^{\alpha}_{0^+}u(t) \geq 0$.

Corollary 3.6. *Assume* $1 \leq \alpha \leq 2$. *Let* $D^{\alpha}_{0^+}u \in C[0,b]$ *with* $u(0) = 0$. *If* $D^{\alpha}_{0^+}u(t) \geq 0$ *for all* $t \in [0,b]$, *then u satisfies the convexity like property*

$$y^{\alpha-1}u(w) \geq w^{\alpha-1}u(y) \tag{6}$$

$y, w \in [0,b]$ *with* $y \leq w$.

We close with two results showing that under certain conditions, $-D^{\alpha}_{0^+}u(t) \geq 0$ implies concavity in the classical sense.

Theorem 3.7. *Assume* $1 < \alpha < 2$. *Let* $D^{\alpha}_{0^+}u \in C[0,b]$ *with* $u(0) = 0$ *and* $u(b) \geq 0$. *Assume* $-D^{\alpha}_{0^+}u(t) \geq 0$ *for all* $t \in [0,b]$, *and assume* $u \in C^{(2)}(0,b]$. *Then* $-u''(t) \geq 0$ *for all* $t \in (0,b]$.

Proof. Define

$$v(t) = u(t) - \left(\frac{t}{b}\right)^{\alpha-1} u(b) = \int_0^1 G(0;t,s)(-D^{\alpha}_{0^+}u(s))ds.$$

Note that $v \in C^{(2)}(0,b]$, $-D^{\alpha}_{0^+}v(t) = -D^{\alpha}_{0^+}u(t) \geq 0$, and $v(0) = 0$. So Theorem 3.4 applies to v and

$$y^{\alpha-1}v(w) \leq w^{\alpha-1}v(y)$$

for all $y, w \in [0,b]$ with $y \leq w$. Also note that the sign properties of $G(0;t,s)$ and $-D^{\alpha}_{0^+}u$, imply $v(t) \geq 0$ on $[0,b]$. Thus,

$$y\,[v(w)]^{\frac{1}{\alpha-1}} \leq w\,[v(y)]^{\frac{1}{\alpha-1}},$$

for all $y, w \in [0,b]$ with $y \leq w$. By Theorem 2.1, $v^{\frac{1}{\alpha-1}}$ is concave.

Since $v^{\frac{1}{\alpha-1}}$ is concave, $-\left(v^{\frac{1}{\alpha-1}}\right)'' \geq 0$ on $(0, b]$. But

$$\left(v^{\frac{1}{\alpha-1}}\right)'' = \frac{2-\alpha}{(\alpha-1)^2} v^{\frac{3-2\alpha}{\alpha-1}} (v')^2 + \frac{1}{\alpha-1} v^{\frac{2-\alpha}{\alpha-1}} v'',$$

implying

$$\frac{2-\alpha}{\alpha-1} v^{\frac{3-2\alpha}{\alpha-1}} (v')^2 \leq -v^{\frac{2-\alpha}{\alpha-1}} v'',$$

which, since $v \geq 0$ on $(0, b]$, implies $-v'' \geq 0$ on $(0, b]$.

By the definition of v,

$$v''(t) = u''(t) - \frac{1}{b^2}(\alpha-1)(\alpha-2)\left(\frac{t}{b}\right)^{\alpha-3} u(b).$$

Since $\alpha - 2 < 0$, $\frac{1}{b^2}(\alpha-1)(\alpha-2)\left(\frac{t}{b}\right)^{\alpha-3} u(b) \leq 0$ for $t \in (0, b]$. So $-u''(t) \geq 0$ for all $t \in (0, b]$. □

Corollary 3.8. *Assume* $1 < \alpha < 2$. *Let* $D_{0^+}^{\alpha} u \in C[0, b]$ *with* $u(0) = 0$ *and assume* u *satisfies a boundary condition* $D_{0^+}^{\beta} u(b) = 0$ *for some* $\beta \in [0, 1]$. *Assume* $-D_{0^+}^{\alpha} u(t) \geq 0$ *for all* $t \in [0, b]$, *and assume* $u \in C^{(2)}(0, b]$. *Then* $-u''(t) \geq 0$ *for all* $t \in (0, b]$.

Proof. Since u satisfies boundary conditions (2), u has the representation

$$u(t) = \int_0^b G(\beta; t, s)(-D_{0^+}^{\alpha} u(s))ds.$$

Then, $u(b) \geq 0$ and Theorem 3.7 applies. □

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