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Topological Structures of a Type of Granule Based Covering Rough Sets

Yan-lan Zhang[a,b], Chang-qing Li[c]

[a]College of Computer, Minnan Normal University, Zhangzhou, Fujian 363000, China
[b]Key Laboratory of Data Science and Intelligence Application, Fujian Province University, Fujian 363000, China
[c]School of Mathematics and Statistics, Minnan Normal University, Zhangzhou, Fujian 363000, China

Abstract. Rough set theory is one of important models of granular computing. Lower and upper approximation operators are two important basic concepts in rough set theory. The classical Pawlak approximation operators are based on partition and have been extended to covering approximation operators. Covering is one of the fundamental concepts in the topological theory, then topological methods are useful for studying the properties of covering approximation operators. This paper presents topological properties of a type of granular based covering approximation operators, which contains seven pairs of approximation operators. Then, topologies are induced naturally by the seven pairs of covering approximation operators, and the topologies are just the families of all definable subsets about the covering approximation operators. Binary relations are defined from the covering to present topological properties of the topological spaces, which are proved to be equivalence relations. Moreover, connectedness, countability, separation property and Lindelöf property of the topological spaces are discussed. The results are not only beneficial to obtain more properties of the pairs of covering approximation operators, but also have theoretical and actual significance to general topology.

1. Introduction

Rough set theory, proposed by Pawlak, has been successfully applied in many fields such as granular computing, pattern recognition, data mining, and knowledge discovery. The lower and upper approximation operators are two important basic concepts in the rough set theory. In the Pawlak's rough set model [20], the lower and upper approximation operators are induced by equivalence relations or partitions. However, the requirement of an equivalence relation or partition in the Pawlak's rough set model may limit the applications of the rough set model. Then, many authors have generalized the Pawlak's rough set model by using more general binary relations [4, 30, 31, 37, 38], by employing coverings [1, 3, 41, 46], by utilizing adjoint operators [19], or by considering the fuzzy environment [10, 18, 35]. The covering rough set model is one of the most important extensions of the classical Pawlak rough set model.

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Email addresses: zyl_1983_2004@163.com (Yan-lan Zhang), helen_smile0320@163.com (Chang-qing Li)

Topology is a theory with many applications not only in almost all branches of mathematics, but also in many real life applications. There exist near connections between topology and rough set theory. Many authors explored topological structures of rough sets [13, 15, 16, 21–24, 26, 27, 29, 33, 34, 38, 42, 46]. Wiweger extended the Pawlak rough sets to topological rough sets [33]. Skowron investigated the topic of topology in information systems [29]. Lin and Liu explored axioms for approximation operations within the framework of topological spaces [16]. Yao discussed topological properties of the Pawlak approximation operations [38]. Wu and Mi examined topological structure of generalized rough sets in infinite universes of discourse [34]. Polkowski defined the hit-or-miss topology on rough sets and proposed a scheme to approximate mathematical morphology within the general paradigm of soft computing [22, 23]. Kondo [13], Qin et al. [27], Zhang et al. [42], Li et al. [15], Yang and Xu [36], Zhang et al. [44] studied topological properties of relation-based rough sets.

There also exists much result on relationships between topology and covering rough sets. Pomykala discussed topological properties of two pairs of covering approximations [24]. Zhu explored a type of covering rough sets by topological approach [46]. Zhao constructed a topology in a covering approximation space and explored topological properties of the topological space [45]. Chen and Li defined open sets, closed sets, rough inclusion, rough equality on covering rough sets and studied some of their properties [7–9]. Ge, Bai, Yun, Bian and Wang gave topological characterizations of the covering C for covering upper approximation operators to be closure operators [2, 12]. Restrepo and Gómez investigated properties of covering approximation operators being closure and topological closure in a framework of sixteen pairs of dual approximation operators [28]. Thuan discussed covering rough sets from a topological view [32]. Liu et al. discussed the topological structures induced by covering lower approximations and established the relationship among these topologies [17].

The purpose of this paper is to discuss topological properties of a type of granular based covering approximation operators, which contains seven pairs of covering approximation operators. In Section 2, we present definitions and properties of the covering approximation operators and some basic concepts in the topological theory. In Section 3, we investigate topological properties of the covering approximation operators, and obtain that the topologies induced from the covering approximation operators are just the families of all definable sets about the covering approximation operators. Then equivalence relations are constructed from the covering to present connectedness, countability, separate property and Lindelöf property of the topological spaces. In Section 4, we explore relationships among the topologies induced by the covering approximation operators and the topology induced by the covering in [45]. Section 5 concludes the paper.

2. Concepts and Properties

In this section, we introduce some basic definitions and properties of covering rough sets and some basic concepts of topological spaces.

2.1. Definitions and Properties of Covering Approximation Operators

Suppose that U is a non-empty set called the universe, and $\mathcal{P}(U)$ is the power set of U. For $X \subseteq U$, $-X$ is the complement of X in U. We do not restrict the universe to be finite.

We now review the first pair of granular based covering approximation operators in which the upper approximation operator was first defined by Zakowski [41]. The approximation operators were also studied by Pomykala [24], Yao [38, 39], Li [14], Zhu and Wang [47], and Zhang et al. [43].

Definition 2.1. Let C be a covering of the universe U. The pair of dual covering approximation operators $\underline{apr''_C} : \mathcal{P}(U) \to \mathcal{P}(U)$ and $\overline{apr''_C} : \mathcal{P}(U) \to \mathcal{P}(U)$ is defined as follows: for any $X \subseteq U$,

$$\underline{apr''_C}(X) = \{x | \forall K \in C, x \in K \Rightarrow K \subseteq X\},$$
$$\overline{apr''_C}(X) = \cup\{K | K \in C, K \cap X \neq \emptyset\}.$$

There are also six pairs of this type of granular based covering approximation operators defined in [40].

Definition 2.2. ([40]) Let C be a covering of U. Define
$Md(C,x) = \{K \in C | x \in K \wedge (\forall S \in C \wedge x \in S \wedge S \subseteq K \Rightarrow S = K)\}$,
$MD(C,x) = \{K \in C | x \in K \wedge (\forall S \in C \wedge x \in S \wedge K \subseteq S \Rightarrow K = S)\}$,
$(x)_C = \cap\{K \in C | x \in K\}$,
$st(x,C) = \cup\{K \in C | x \in K\}$,
$\cap - reduct(C) = C - \{K | K \in C, \exists C_1 \subseteq (C - \{K\})[K = \cap C_1]\}$,
$\cup - reduct(C) = C - \{K | K \in C, \exists C_1 \subseteq (C - \{K\})[K = \cup C_1]\}$.

In [40], six coverings induced by C are defined as follows:
$C_1 = \cup\{Md(C,x) | x \in U\}$,
$C_2 = \cup\{MD(C,x) | x \in U\}$,
$C_3 = \{(x)_C | x \in U\}$,
$C_4 = \{st(x,C) | x \in U\}$,
$C_\cap = \cap - reduct(C)$,
$C_\cup = \cup - reduct(C)$.

From Definitions 2.1 and 2.2, six pairs of covering approximation operators are obtained in [40]: $(\underline{apr}''_{C_1}, \overline{apr}''_{C_1})$, $(\underline{apr}''_{C_2}, \overline{apr}''_{C_2})$, $(\underline{apr}''_{C_3}, \overline{apr}''_{C_3})$, $(\underline{apr}''_{C_4}, \overline{apr}''_{C_4})$, $(\underline{apr}''_{C_\cap}, \overline{apr}''_{C_\cap})$, $(\underline{apr}''_{C_\cup}, \overline{apr}''_{C_\cup})$. The pair of covering approximation operators $(\underline{apr}''_{C_3}, \overline{apr}''_{C_3})$ was also discussed in [25]. The upper approximation operator $\overline{apr}''_{C_3}$ was also introduced in [6].

Some basic properties of the pairs of covering approximation operators $(\underline{apr}''_{C}, \overline{apr}''_{C})$, $(\underline{apr}''_{C_1}, \overline{apr}''_{C_1})$, $(\underline{apr}''_{C_2}, \overline{apr}''_{C_2})$, $(\underline{apr}''_{C_3}, \overline{apr}''_{C_3})$, $(\underline{apr}''_{C_4}, \overline{apr}''_{C_4})$, $(\underline{apr}''_{C_\cap}, \overline{apr}''_{C_\cap})$, $(\underline{apr}''_{C_\cup}, \overline{apr}''_{C_\cup})$ are enumerated in the following propositions.

Proposition 2.3. ([24, 41, 43]) *Let C be a covering of U. Then, for any $X, Y \subseteq U$, $x, y \in U$,*
(1) $\underline{apr}''_C(U) = U$, $\overline{apr}''_C(\emptyset) = \emptyset$,
(2) $\underline{apr}''_C(X) \subseteq X \subseteq \overline{apr}''_C(X)$,
(3) $\underline{apr}''_C(X \cap Y) = \underline{apr}''_C(X) \cap \underline{apr}''_C(Y)$, $\overline{apr}''_C(X \cup Y) = \overline{apr}''_C(X) \cup \overline{apr}''_C(Y)$,
(4) $x \in \overline{apr}''_C(\{y\}) \Longleftrightarrow y \in \overline{apr}''_C(\{x\})$,
(5) $X \subseteq Y \subseteq U \Longrightarrow \underline{apr}''_C(X) \subseteq \underline{apr}''_C(Y)$ *and* $\overline{apr}''_C(X) \subseteq \overline{apr}''_C(Y)$,
(6) $\overline{apr}''_C(X) = \cup_{x \in X} \overline{apr}''_C(\{x\})$,
(7) $\underline{apr}''_C(X) = X \Leftrightarrow \overline{apr}''_C(X) = X$.

Proof. We only prove (7). "$\Rightarrow$". For any $x \in \overline{apr}''_C(X)$, there exists a $K \in C$ such that $x \in K$ and $K \cap X \neq \emptyset$. Let $y \in K \cap X$. Since $\underline{apr}''_C(X) = X$, we get $y \in \underline{apr}''_C(X)$. Then $K \subseteq X$, which follows that $x \in X$. Hence, $\overline{apr}''_C(X) \subseteq X$. By (2), $X \subseteq \overline{apr}''_C(X)$. Therefore, $X = \overline{apr}''_C(X)$.

"$\Leftarrow$". For any $x \in X$ and any $K \in C$ with $x \in K$, we have $\{x\} \subseteq K \cap X \neq \emptyset$. Then $K \subseteq \overline{apr}''_C(X)$. Since $\overline{apr}''_C(X) = X$, we obtain $K \subseteq X$. It follows that $x \in \underline{apr}''_C(X)$. Thus $X \subseteq \underline{apr}''_C(X)$. Therefore, $X = \underline{apr}''_C(X)$. □

Corollary 2.4. *Let C be a covering of U. Then, for any $X, Y \subseteq U$, $x, y \in U$, $i \in \{1, 2, 3, 4, \cap, \cup\}$,*
(1) $\underline{apr}''_{C_i}(U) = U$, $\overline{apr}''_{C_i}(\emptyset) = \emptyset$,
(2) $\underline{apr}''_{C_i}(X) \subseteq X \subseteq \overline{apr}''_{C_i}(X)$,
(3) $\underline{apr}''_{C_i}(X \cap Y) = \underline{apr}''_{C_i}(X) \cap \underline{apr}''_{C_i}(Y)$, $\overline{apr}''_{C_i}(X \cup Y) = \overline{apr}''_{C_i}(X) \cup \overline{apr}''_{C_i}(Y)$,
(4) $x \in \overline{apr}''_{C_i}(\{y\}) \Longleftrightarrow y \in \overline{apr}''_{C_i}(\{x\})$,
(5) $X \subseteq Y \subseteq U \Longrightarrow \underline{apr}''_{C_i}(X) \subseteq \underline{apr}''_{C_i}(Y)$ *and* $\overline{apr}''_{C_i}(X) \subseteq \overline{apr}''_{C_i}(Y)$,
(6) $\overline{apr}''_{C_i}(X) = \cup_{x \in X} \overline{apr}''_{C_i}(\{x\})$,
(7) $\underline{apr}''_{C_i}(X) = X \Leftrightarrow \overline{apr}''_{C_i}(X) = X$.

Proof. For any $i \in \{1,2,3,4,\cap,\cup\}$, C_i is a covering of U. Then, by Proposition 2.3, we can obtain the proposition directly. □

Proposition 2.5. ([43]) *Let C be a covering of U. Then the following statements are equivalent:*
(1) $\overline{apr}''_C(\overline{apr}''_C(X)) = \overline{apr}''_C(X)$ *for all* $X \subseteq U$,
(2) $\overline{apr}''_C(-\overline{apr}''_C(X)) = -\overline{apr}''_C(X)$ *for all* $X \subseteq U$,
(3) $\forall x \in U$ *and* $K \in C$, *either* $K \subseteq st(x,C)$ *or* $K \cap st(x,C) \neq \emptyset$,
(4) $\{st(x,C)|x \in U\}$ *is a partition of* U.

Let the covering C in Proposition 2.5 be $C_i(i \in \{1,2,3,4,\cap,\cup\})$, then we obtain

Corollary 2.6. *Let C be a covering of U. Then, for any $i \in \{1,2,3,4,\cap,\cup\}$, the following statements are equivalent:*
(1) $\overline{apr}''_{C_i}(\overline{apr}''_{C_i}(X)) = \overline{apr}''_{C_i}(X)$ *for all* $X \subseteq U$,
(2) $\overline{apr}''_{C_i}(-\overline{apr}''_{C_i}(X)) = -\overline{apr}''_{C_i}(X)$ *for all* $X \subseteq U$,
(3) $\forall x \in U$ *and* $K \in C_i$, *either* $K \subseteq st(x,C_i)$ *or* $K \cap st(x,C_i) \neq \emptyset$,
(4) $\{st(x,C_i)|x \in U\}$ *is a partition of* U.

2.2. Basic Concepts in Topology

In this subsection, we introduce some basic concepts of topological spaces. For the other basic topological concepts, we refer to [11].

Definition 2.7. ([11]) Let U be a non-empty set. A topology on U is a collection τ of subsets of U having the following properties:
(1) $\emptyset$ and U are in τ,
(2) the union of the elements of any subcollection of τ is in τ,
(3) the intersection of the elements of any finite subcollection of τ is in τ.
Then (U,τ) is called a topological space, each element in τ is called an open set, and the complement of an open set is called a closed set. A family $\beta \subseteq \tau$ is called a base for (U,τ), if every open set of (U,τ) can be represented as a union of subfamily of β. A family $\gamma \subseteq \tau$ is called a subbase for (U,τ) if the family of all finite intersections is a base for (U,τ).

Definition 2.8. ([11]) Let (U,τ) be a topological space and $X \in \mathcal{P}(U)$. Then the topological interior and closure of X are, respectively, defined by
$int_\tau(X) = \cup\{G|G$ is an open set and $G \subseteq X\}$,
$cl_\tau(X) = \cap\{K|K$ is a closed set and $X \subseteq K\}$.
int_τ and cl_τ are, respectively, called the topological interior operator and the topological closure operator of τ.

It can be shown that $cl_\tau(X)$ is a closed set and $int_\tau(X)$ is an open set in (U,τ). X is an open set in (U,τ) if and only if $int_\tau(X) = X$, and X is a closed set in (U,τ) if and only if $cl_\tau(X) = X$.

The topological closure operator can be also defined by Kuratowski closure axioms.

Definition 2.9. ([5, 11]) Let U be a non-empty set and $cl : \mathcal{P}(U) \to \mathcal{P}(U)$. For any $X, Y \subseteq U$,
(1) $cl(\varnothing) = \varnothing$,
(2) $X \subseteq cl(X)$,
(3) $cl(X \cup Y) = cl(X) \cup cl(Y)$,
(4) $cl(cl(X)) = cl(X)$,
(5) $cl(X) = \cup_{x\in X} cl(\{x\})$.
If cl satisfies (1)–(3), then cl is called a closure operator, and (U,cl) is called a closure space [5]. If cl satisfies (1)–(4), that is, cl satisfies Kuratowski closure axiom, then cl is called a topological closure operator [11]. If a closure operator cl satisfies (5), then cl is called a quasi-discrete closure operator [5].

In fact, in a closure space (U,cl), it is easy to prove that $\tau(cl) = \{-X|cl(X) = X\}$ is a topology. Similarly, the topological interior operator can be defined by corresponding axioms.

Definition 2.10. ([11]) Let (U, τ) be a topological space. If $A \subseteq U$ is open in U if and only if A is closed in U, then (U, τ) is called a pseudo-discrete space. If the intersection of arbitrarily many open sets in U is still open, then τ is called an Alexandrov topology, and (U, τ) is said to be an Alexandrov space.

(U, τ) is said to be connected if the only subsets of U that are both open and closed are empty set and U itself. (U, τ) is said to be locally connected at x if for every neighborhood O of x, there is a connected neighborhood V of x contained in U. If (U, τ) is locally connected at each of its points, it is said simply to be locally connected.

(U, τ) is said to have a countable basis at x if there is a countable collection $\mathfrak{B}$ of neighborhoods of x such that each neighborhood of x contains at least one of the elements of $\mathfrak{B}$. If (U, τ) has a countable basis at each of its points, it is said to satisfy the first countability axiom, or to be first-countable. If (U, τ) has a countable basis for its topology, it is said to satisfy the second countability axiom, or to be second-countable. A space (U, τ) on which every open covering contains a countable subcovering is called a Lindelöf space. A space (U, τ) having a countable dense subset is often said to be separable.

(U, τ) is said to be regular if for each pair consisting of a point x and a closed set B disjoint from x, there exist disjoint open sets containing x and B, respectively. (U, τ) is said to be normal if for each pair A, B of disjoint closed sets of U, there exist disjoint open sets containing A and B, respectively.

3. Topological Structures of the Type of Granular Based Covering Rough Sets

In this section, we investigate topological properties of the covering approximation operators, and explore properties of the topologies induced by the covering approximation operators.

Proposition 3.1. *Let C be a covering of U. Then*
(1) $\overline{apr}''_C$ is a quasi-discrete closure operator,
(2) $\tau(\overline{apr}''_C) = \{X \subseteq U | \underline{apr}''_C(X) = X\} = \{-X | \overline{apr}''_C(X) = X\}$ is a topology.

Proof. (1) By Proposition 2.3(1-3,6) and Definition 2.9, we get that $\overline{apr}''_C$ is a quasi-discrete closure operator.
(2) According to Proposition 2.3(1-3), we obtain that $\tau(\overline{apr}''_C)$ is a topology. □

Corollary 3.2. *Let C be a covering of U. Then, for any $i \in \{1, 2, 3, 4, \cap, \cup\}$, we have:*
(1) $\overline{apr}''_{C_i}$ is a quasi-discrete closure operator,
(2) $\tau(\overline{apr}''_{C_i}) = \{X \subseteq U | \underline{apr}''_{C_i}(X) = X\} = \{-X | \overline{apr}''_{C_i}(X) = X\}$ is a topology.

Proof. (1) According to Corollary 2.4(1-3,6) and Definition 2.9, we can obtain that $\overline{apr}''_C$ is a quasi-discrete closure operator.
(2) It follows from Corollary 2.4(1-3) that $\tau(\overline{apr}''_{C_i})$ is a topology. □

Proposition 3.3. *Let C be a covering of U. Then the following are equivalent:*
(1) $\forall x \in U$ and $K \in C$, either $K \subseteq Friends(x)$ or $K \cap Friends(x) \neq \emptyset$,
(2) $\underline{apr}''_C$ is a topological interior operator,
(3) $\overline{apr}''_C$ is a topological closure operator.

Proof. We can obtain the proposition by Propositions 2.3(1-3) and 2.5. □

Corollary 3.4. *Let C be a covering of U. Then, for any $i \in \{1, 2, 3, 4, \cap, \cup\}$, the following are equivalent:*
(1) $\forall x \in U$ and $K \in C_i$, either $K \subseteq st(x, C_i)$ or $K \cap st(x, C_i) \neq \emptyset$,
(2) $\underline{apr}''_{C_i}$ is a topological interior operator,
(3) $\overline{apr}''_{C_i}$ is a topological closure operator.

Proof. According to Corollaries 2.4(1-3) and 2.6, it is easy to get the corollary. □

Proposition 3.3 (and Corollary 3.4) presents a necessary and sufficiency conditions for $\overline{apr}''_C$ ($\overline{apr}''_{C_i}, i \in \{1,2,3,4,\cap,\cup\}$) and $\underline{apr}''_C$ ($\underline{apr}''_{C_i}$) to be, respectively, a topological closure operator and a topological interior operator. In order to present more properties of the topological spaces $(U, \tau(\overline{apr}''_C))$ and $(U, \tau(\overline{apr}''_{C_i}))$, we define binary relations.

Definition 3.5. Let C be a covering of U. Define a binary relation R_C by: for any $x, y \in U$, xR_Cy if and only if there exist $K_1, K_2, \cdots, K_n \in C$ such that $x \in K_1, y \in K_n$, and $K_j \cap K_{j+1} \neq \emptyset$ $(j = 1, 2, \cdots, n-1)$. Define a binary relation R_{C_i} ($i \in \{1,2,3,4,\cap,\cup\}$) as: for any $x, y \in U$, $xR_{C_i}y$ if and only if there exist $K_1, K_2, \cdots, K_n \in C_i$ such that $x \in K_1, y \in K_n$, and $K_j \cap K_{j+1} \neq \emptyset$ $(j = 1, 2, \cdots, n-1)$.

Proposition 3.6. *Let C be a covering of U. Then*
(1) R_C is reflexive,
(2) R_C is symmetric,
(3) R_C is transitive.

Proof. (1) For any $x \in U$, there exists a $K \in C$ such that $x \in K$. Then, by Definition 2.8, xR_Cx. Hence R_C is reflexive.

(2) For any $x, y \in U$, if xR_Cy, then there exist $K_1, K_2, \cdots, K_n \in C$ such that $x \in K_1, y \in K_n$, and $K_j \cap K_{j+1} \neq \emptyset$ $(j = 1, 2, \cdots, n-1)$. Then we have yR_Cx. Therefore, R_C is symmetric.

(3) For any $x, y, z \in U$, if xR_Cy and yR_Cz, then there exist $K_1, K_2, \cdots, K_n \in C$ such that $x \in K_1, y \in K_n$, and $K_j \cap K_{j+1} \neq \emptyset$ $(j = 1, 2, \cdots, n-1)$, and we can find $G_1, G_2, \cdots, G_m \in C$ such that $y \in G_1, z \in G_m$, and $G_j \cap G_{j+1} \neq \emptyset$ $(j = 1, 2, \cdots, m-1)$. Thus $y \in K_n \cap G_1 \neq \emptyset$. Hence there exist $K_1, K_2, \cdots, K_n, G_1, G_2, \cdots, G_m \in C$ such that $x \in K_1, z \in G_m$, and $K_j \cap K_{j+1} \neq \emptyset$ $(j = 1, 2, \cdots, n-1), K_n \cap G_1 \neq \emptyset, G_j \cap G_{j+1} \neq \emptyset$ $(j = 1, 2, \cdots, m-1)$. Consequently, xR_Cz. □

Corollary 3.7. *Let C be a covering of U. Then, for any $i \in \{1,2,3,4,\cap,\cup\}$,*
(1) R_{C_i} is reflexive,
(2) R_{C_i} is symmetric,
(3) R_{C_i} is transitive.

Proof. Since C_i is a covering of U, by Proposition 3.6, we get this corollary. □

By Proposition 3.6, R_C is an equivalence relationship, i.e., $U/R_C \triangleq \{[x]_{R_C} | x \in U\}$ is a partition of U, where $[x]_{R_C}$ is the equivalence class of x. From Corollary 3.4, R_{C_i} is an equivalence relationship, i.e., $U/R_{C_i} \triangleq \{[x]_{R_{C_i}} | x \in U\}$ is a partition of U, where $[x]_{R_{C_i}}$ is the equivalence class of x.

Proposition 3.8. *Let C be a covering of U, then for any $x \in U$,*
(1) $\underline{apr}''_C([x]_{R_C}) = [x]_{R_C}$,
(2) $[x]_{R_C}$ is an open and closed subset of $(U, \tau(\overline{apr}''_C))$.

Proof. (1) According to Proposition 2.3(2), we obtain $\underline{apr}''_C([x]_{R_C}) \subseteq [x]_{R_C}$. For any $y \in [x]_{R_C}$, we have xR_Cy. Then, there exist $K_1, K_2, \cdots, K_n \in C$ such that $x \in K_1, y \in K_n$, and $K_j \cap K_{j+1} \neq \emptyset$ $(j = 1, 2, \cdots, n-1)$. For any $K \in C$ with $y \in K$, we have $K \subseteq [x]_{R_C}$. In fact, since $y \in K$ and $y \in K_n$, we obtain $K \cap K_n \neq \emptyset$. Then, for any $z \in K$, there exist $K_1, K_2, \cdots, K_n, K \in C$ such that $x \in K_1, z \in K$, and $K_j \cap K_{j+1} \neq \emptyset (j = 1, 2, \cdots, n-1), K_n \cap K \neq \emptyset$. It follows that $z \in [x]_{R_C}$. Hence $K \subseteq [x]_{R_C}$. It implies that $y \in \underline{apr}''_C([x]_{R_C})$. Therefore, $[x]_{R_C} \subseteq \underline{apr}''_C([x]_{R_C})$.

(2) Due to (1), $[x]_{R_C}$ is an open set. Then, by Proposition 2.3(6), we deduce that $[x]_{R_C}$ is a closed set. □

Corollary 3.9. *Let C be a covering of U. Then, for any $i \in \{1,2,3,4,\cap,\cup\}$, $x \in U$,*
(1) $\underline{apr}''_{C_i}([x]_{R_{C_i}}) = [x]_{R_{C_i}}$,
(2) $[x]_{R_{C_i}}$ is an open and closed subset of $(U, \tau(\overline{apr}''_{C_i}))$.

Proof. Thanks to Corollary 2.4(2), we can get the proof of the corollary referring to the proof of Proposition 3.8. □

Proposition 3.10. *Let C be a covering of U, then*
(1) for any $X \in \tau(\overline{apr}''_C)$ and $x \in X$, $[x]_{R_C} \subseteq X$,
(2) $\{[x]_{R_C}\}$ is an open neighborhood base of $x \in U$.

Proof. (1) Since $X \in \tau(\overline{apr}''_C)$, we have $\underline{apr}''_C(X) = X$. For any $y \in [x]_{R_C}$, there exist $K_1, K_2, \cdots, K_n \in C$ such that $x \in K_1, y \in K_n$, and $K_j \cap K_{j+1} \neq \emptyset (j = 1, 2, \cdots, n-1)$. Since $x \in K_1$ and $x \in X = \underline{apr}''_C(X)$, by Definition 2.1, $K_1 \subseteq X$. Let $z \in K_1 \cap K_2$, then $z \in K_1 \subseteq X = \underline{apr}''_C(X)$. Hence, by Definition 2.1, $K_2 \subseteq X$. In the same way, we have $K_n \subseteq X$. Then $y \in X$. Therefore, we conclude that $[x]_{R_C} \subseteq X$.

(2) According to (1) and Proposition 3.8(2), it is easy to see that $\{[x]_{R_C}\}$ is an open neighborhood base of x. □

Corollary 3.11. *Let C be a covering of U. Then, for any $i \in \{1, 2, 3, 4, \cap, \cup\}$,*
(1) for any $X \in \tau(\overline{apr}''_{C_i})$ and $x \in X$, $[x]_{R_{C_i}} \subseteq X$,
(2) $\{[x]_{R_{C_i}}\}$ is an open neighborhood base of $x \in U$.

Proof. It is similar to the proof of Proposition 3.10. □

Proposition 3.12. *Let C be a covering of U, then for any $x \in U$,*
(1) $cl_{\tau(\overline{apr}''_C)}(\{x\}) = [x]_{R_C}$,
(2) $[x]_{R_C}$ is a connected component that contains x.
(3) $(U, \tau(\overline{apr}''_C))$ is a locally connected space.

Proof. (1) By Proposition 2.3(7), $X \subseteq U$ is an open set of $(U, \tau(\overline{apr}''_C))$, if and only if X is a closed set. Then, according to Propositions 3.8 and 3.10, we have

$$\begin{aligned} cl_{\tau(\overline{apr}''_C)}(\{x\}) &= \cap\{B | x \in B, \text{ and } B \text{ is closed}\} \\ &= \cap\{B | x \in B, \text{ and } B \text{ is open}\} = [x]_{R_C}. \end{aligned}$$

(2) Suppose that C_x is a connected component containing x. Then C_x is closed. It implies that C_x is open. By Proposition 3.10, we get that $[x]_{R_C} \subseteq C_x$. Thus $[x]_{R_C} = C_x$. Otherwise, $[x]_{R_C} \neq C_x$. Hence $[x]_{R_C}$ is an open and closed proper subset of C_x, which contradicts the fact that C_x is a connected component. Therefore, $[x]_{R_C}$ is a connected component that contains x.

(3) For any $x \in U$ and $A \in \tau(\overline{apr}''_C)$ with $x \in A$, we obtain $[x]_{R_C} \subseteq A$. Due to (2), $[x]_{R_C}$ is a connected set. Then $(U, \tau(\underline{apr}''_C))$ is a locally connected space. □

Corollary 3.13. *Let C be a covering of U, then for any $x \in U$, $i \in \{1, 2, 3, 4, \cap, \cup\}$,*
(1) $cl_{\tau(\overline{apr}''_{C_i})}(\{x\}) = [x]_{R_{C_i}}$,
(2) $[x]_{R_{C_i}}$ is a connected component that contains x.
(3) $(U, \tau(\overline{apr}''_{C_i}))$ is a locally connected space.

Proof. By Corollaries 2.4 and 3.2, we can obtain the proof using the same method as in the proof of Proposition 3.12. □

Theorem 3.14. *Let C be a covering of U, then $(U, \tau(\overline{apr}''_C))$ is connected if and only if xR_Cy for all $x, y \in U$.*

Proof. "⇐". Suppose $U = X \cup Y$ and $X \cap Y = \emptyset$. Let $x \in X$ and $y \in Y$. By the assumption, we have xR_Cy. So there exist $K_1, K_2, \cdots, K_n \in C$ such that $x \in K_1, y \in K_n$, and $K_j \cap K_{j+1} \neq \emptyset$ $(j = 1, 2, \cdots, n-1)$. Then there at least exists a $j_0 (1 \leq j_0 \leq n)$ such that $K_{j_0} \cap X \neq \emptyset$ and $K_{j_0} \cap Y \neq \emptyset$. Otherwise, for any $j \in \{1, 2, \cdots, n\}$, $K_j \subseteq X$ or $K_j \subseteq Y$. Since $x \in X \cap K_1 \neq \emptyset$, we have $K_1 \subseteq X$. By $K_1 \cap K_2 \neq \emptyset$, we have $K_2 \subseteq X$. In the same way, we get $K_n \subseteq X$. Then $y \in K_n \subseteq X$, which contradicts $y \in Y$. Let $u \in K_{j_0} \cap X$ and $v \in K_{j_0} \cap Y$. Then $u \in \overline{apr}''_C(Y)$ and $u \notin Y$, $v \in \overline{apr}''_C(X)$ and $v \notin X$. Hence $\overline{apr}''_C(X) \neq X$ and $\overline{apr}''_C(Y) \neq Y$, which implies that X and Y are not closed. Thus U is not the union of two disjoint closed sets, that is, U is connected.

"⇒". Suppose that there exist $x, y \in U$ such that xR_Cy does not hold. Let $X = \{z \in U | xR_Cz\}$.

Then X is closed. In fact, for any $u \in \overline{apr}''_C(X)$, there exists a $K \in C$ such that $u \in K$ and $K \cap X \neq \emptyset$. Since $x \in X$, we get $xR_C u$, which implies that $u \in X$. Hence $\overline{apr}''_C(X) \subseteq X$. It follows that $\overline{apr}''_C(X) = X$, which implies that X is closed.

We are going to prove that $-X$ is closed. If not, $\overline{apr}''_C(-X) \not\subseteq -X$. Then there exists a $u \in U$ such that $u \in \overline{apr}''_C(-X)$ and $u \notin -X$. Hence there exists a $K \in C$ such that $u \in K$ and $K \cap (-X) \neq \emptyset$. Let $w \in K \cap (-X)$. It follows that $uR_C w$. Since $u \in X$, we have $w \in X$, which contradicts the fact $w \in -X$.

Since $x \in X$ and $y \in -X$, U is the union of two disjoint non-empty closed sets, that is, U is not connected. □

Corollary 3.15. *Let C be a covering of U. Then, for any $i \in \{1, 2, 3, 4, \cap, \cup\}$, $(U, \tau(\overline{apr}''_{C_i}))$ is connected if and only if $xR_{C_i}y$ for all $x, y \in U$.*

Proof. It is similar to the proof of Theorem 3.14. □

Proposition 3.16. *Let C be a covering of U, then*
(1) $\tau(\overline{apr}''_C)$ is pseudo-discrete,
(2) $\tau(\overline{apr}''_C)$ is an Alexandrov space.

Proof. (1) By Proposition 2.3(7), X is open if and only if X is closed. Then $\tau(\overline{apr}''_C)$ is pseudo-discrete.

(2) Since each open set in U is closed, the intersection of arbitrarily many open sets in U is still open. Hence $(U, \tau(\overline{apr}''_C)$ is an Alexandrov space. □

Corollary 3.17. *Let C be a covering of U. Then, for any $i \in \{1, 2, 3, 4, \cap, \cup\}$,*
(1) $\tau(\overline{apr}''_{C_i})$ is pseudo-discrete,
(2) $\tau(\overline{apr}''_C)$ is an Alexandrov space.

Proof. According to Corollary 2.4, it is easy to get the conclusion. □

By Proposition 3.16, for any $X \subseteq U$,
$X \subseteq U$ is definable about $\underline{apr}''_C$ and $\overline{apr}''_C$
$\Leftrightarrow \underline{apr}''_C(X) = X = \overline{apr}''_C(X)$
$\Leftrightarrow X$ is an open and closed set of $\tau(\overline{apr}''_C)$
$\Leftrightarrow X \in \tau(\overline{apr}''_C)$.
Hence the family of all definable subsets of U is $\tau(\overline{apr}''_C)$. On the other hand,
$(U, \tau(\overline{apr}''_C))$ is not connected
$\Leftrightarrow (U, \tau(\overline{apr}''_C))$ has non-empty open and closed proper subsets
$\Leftrightarrow (U, \tau(\overline{apr}''_C))$ has other definable sets besides Ø and U.
$(U, \tau(\overline{apr}''_C))$ is connected
$\Leftrightarrow (U, \tau(\overline{apr}''_C))$ do not have any non-empty open and closed proper subset
$\Leftrightarrow$ the definable sets about $\underline{apr}''_C$ and $\overline{apr}''_C$ are no other than Ø and U.
Then we can note that $(U, \tau(\overline{apr}''_C))$ is connected, if and only if the definable sets about $\underline{apr}''_C$ and $\overline{apr}''_C$ are no other than Ø and U. $(U, \tau(\overline{apr}''_C))$ is not connected, if and only if $(U, \tau(\overline{apr}''_C))$ has other definable sets besides Ø and U. Therefore, there exist relationships between the connectedness of topological spaces and the existence of definable sets in approximation spaces.

In the same way, $\tau(\overline{apr}''_{C_i})$ ($i \in \{1, 2, 3, 4, \cap, \cup\}$) is the family of all definable subsets about $\underline{apr}''_{C_i}$ and $\underline{apr}''_{C_i}$.

Proposition 3.18. *Let C be a covering of U. Then*
(1) $(U, \tau(\overline{apr}''_C))$ is a first countable space,
(2) $(U, \tau(\overline{apr}''_C))$ is a locally separable space.

Proof. (1) According to Proposition 3.10(2), we obtain that $\{[x]_{R_C}\}$ is an open neighborhood base of x. Hence $(U, \tau(\overline{apr}''_C))$ is first countable.

(2) By Proposition 3.12(1), $\{x\}$ is a dense subset of $[x]_{R_C}$, then $[x]_{R_C}$ is a separable subset. Hence, due to Proposition 3.10(1), each neighborhood of x has separable subset $[x]_{R_C}$. It implies that $(U, \tau(\overline{apr}''_C))$ is locally separable. □

Corollary 3.19. *Let C be a covering of U. Then, for any $i \in \{1, 2, 3, 4, \cap, \cup\}$,*

(1) $(U, \tau(\overline{apr}''_{C_i}))$ is a first countable space,

(2) $(U, \tau(\overline{apr}''_{C_i}))$ is a locally separable space.

Proof. According to Corollaries 3.11 and 3.13, we can obtain this corollary referring to the proof of Proposition 3.18. □

Proposition 3.20. *Let C be a covering of U. Then*

(1) $(U, \tau(\overline{apr}''_C))$ is a regular space,

(2) $(U, \tau(\overline{apr}''_C))$ is a normal space.

Proof. (1) For any $x \in U$ and closed set B with $x \notin B$, according to Proposition 3.16(1), we get that B is open. Thus there exist two disjoint open sets $X \backslash B$ and B such that $x \in X \backslash B$ and $B \subseteq B$. Then $(U, \tau(\overline{apr}''_C))$ is regular.

(2) For each pair A, B of disjoint closed sets of X, by Proposition 3.16(1), we have that A and B are open sets. Then there exist disjoint open sets A and B such that $A \subseteq A$ and $B \subseteq B$. It means that $(U, \tau(\overline{apr}''_C))$ is normal. □

Corollary 3.21. *Let C be a covering of U. Then, for any $i \in \{1, 2, 3, 4, \cap, \cup\}$,*

(1) $(U, \tau(\overline{apr}''_{C_i}))$ is a regular space,

(2) $(U, \tau(\overline{apr}''_{C_i}))$ is a normal space.

Proof. It is similar to the proof of Proposition 3.20. □

Proposition 3.22. *Let C be a covering of U. Then the following are equivalent:*

(1) U/R_C is countable,

(2) $(U, \tau(\overline{apr}''_C))$ is second-countable,

(3) $(U, \tau(\overline{apr}''_C))$ is separable,

(4) $(U, \tau(\overline{apr}''_C))$ is a Lindelöf space.

Proof. (1)⇒(2). By Proposition 3.10, $\{[x]_{R_C} | x \in U\}$ is a base of $(U, \tau(\overline{apr}''_C))$. Since $U/R_C = \{[x]_{R_C} | x \in U\}$ is countable, we obtain that $(U, \tau(\overline{apr}''_C))$ is second-countable.

(2)⇒(3). It is clear.

(3)⇒(4). Let C be an open covering of U, and D be a countable dense subset of U. For any $x \in D$, there exists a $K_x \in C$ such that $x \in K_x$. Let $C_0 = \{K_x | x \in D\}$. Then C_0 is countable. Now we prove that C_0 is a covering of U. For any $y \in U$, there exists a $K \in C$ such that $y \in K$. Hence $[y]_{R_C} \subseteq K$. Since D is a dense subset of U, we get $[y]_{R_C} \cap D \neq \emptyset$. Let $x \in [y]_{R_C} \cap D$. Then there exists a $K_x \in C_0$ such that $x \in K_x$. It follows that $[x]_{R_C} \subseteq K_x$. Since $x \in [y]_{R_C}$ and R_C is an equivalence relation, we obtain $[x]_{R_C} = [y]_{R_C}$. Then $y \in [y]_{R_C} = [x]_{R_C} \subseteq K_x$. Therefore, we can conclude that $(U, \tau(\overline{apr}''_C))$ is a Lindelöf space.

(4)⇒(1). $U/R_C = \{[x]_{R_C} | x \in U\}$ is an open covering of U. Since R_C is an equivalence relation, U/R_C is the only subcovering of U/R_C. Since $(U, \tau(\overline{apr}''_C))$ is a Lindelöf space, we obtain that U/R_C is countable. □

Corollary 3.23. *Let C be a covering of U. Then, for any $i \in \{1, 2, 3, 4, \cap, \cup\}$, the following are equivalent:*

(1) U/R_{C_i} is countable,

(2) $(U, \tau(\overline{apr}''_{C_i}))$ is second-countable,

(3) $(U, \tau(\overline{apr}''_{C_i}))$ is separable,

(4) $(U, \tau(\overline{apr}''_{C_i}))$ is a Lindelöf space.

Proof. It is similar to the proof of Theorem 3.14. □

Example 3.24. Let $U = \{a, b, c, d, e, f, g, h\}$, $C = \{\{a, b\}, \{b\}, \{b, c\}, \{a, b, c\}, \{d, e\}, \{e, f\}, \{g\}, \{g, h\}, \{h\}\}$. Then

$C_1 = \{\{a, b\}, \{b\}, \{b, c\}, \{d, e\}, \{e, f\}, \{g\}, \{h\}\}$,
$C_2 = \{\{a, b, c\}, \{d, e\}, \{e, f\}, \{g, h\}\}$,
$C_3 = \{\{a, b\}, \{b\}, \{b, c\}, \{d, e\}, \{e\}, \{e, f\}, \{g\}, \{h\}\}$,
$C_4 = \{\{a, b, c\}, \{d, e\}, \{d, e, f\}, \{e, f\}, \{g, h\}\}$,
$C_\cap = \{\{a, b\}, \{b, c\}, \{a, b, c\}, \{d, e\}, \{e, f\}, \{g\}, \{g, h\}, \{h\}\}$,
$C_\cup = \{\{a, b\}, \{b\}, \{b, c\}, \{d, e\}, \{e, f\}, \{g\}, \{h\}\}$.

Hence we have
$\tau(\overline{apr}''_C) = \tau(\overline{apr}''_{C_2}) = \tau(\overline{apr}''_{C_4}) = \tau(\overline{apr}''_{C_\cap}) = \{\emptyset, \{a, b, c\}, \{d, e, f\}, \{g, h\}, \{a, b, c, d, e, f\}, \{a, b, c, g, h\}, \{d, e, f, g, h\}, U\}$,
and
$\tau(\overline{apr}''_{C_3}) = \tau(\overline{apr}''_{C_\cup}) = \tau(\overline{apr}''_{C_1}) = \{\emptyset, \{a, b, c\}, \{d, e, f\}, \{g\}, \{h\}, \{a, b, c, d, e, f\}, \{a, b, c, g\}, \{a, b, c, h\}, \{d, e, f, g\}, \{d, e, f, h\}, \{g, h\}, \{a, b, c, d, e, f, g\}, \{a, b, c, d, e, f, h\}, U\}$.

We also obtain
$R_C = R_{C_2} = R_{C_4} = R_{C_\cap} = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c), (d, d), (d, e), (d, f), (e, d), (e, e), (e, f), (f, d), (f, e), (f, f), (g, g), (h, h)\}$,
and
$R_{C_1} = R_{C_3} = R_{C_\cup} = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c), (d, d), (d, e), (d, f), (e, d), (e, e), (e, f), (f, d), (f, e), (f, f), (g, g), (h, h)\}$.

Then $R_C = R_{C_2} = R_{C_4} = R_{C_\cap}$ and $R_{C_\cup} = R_{C_1} = R_{C_3}$ are equivalence relations, and
$U/R_C = U/R_{C_2} = U/R_{C_4} = U/R_{C_\cap} = \{\{a, b, c\}, \{d, e, f\}, \{g, h\}\}$,
$U/R_{C_1} = U/R_{C_3} = U/R_{C_\cup} = \{\{a, b, c\}, \{d, e, f\}, \{g\}, \{h\}\}$.

We also get that $(U, \tau(\overline{apr}''_C))$ and $(U, \tau(\overline{apr}''_{C_i}))(i \in \{1, 2, 3, 4, \cap, \cup\})$ are quasi-discrete, regular, normal and non-connected. $\tau(\overline{apr}''_C)$ is the family of all definable sets about $\underline{apr}''_C$ and $\overline{apr}''_C$, and $\tau(\overline{apr}''_{C_i})(i \in \{1, 2, 3, 4, \cap, \cup\})$ is the family of all definable sets about $\underline{apr}''_{C_i}$ and $\overline{apr}''_{C_i}$.

4. Relationships Among Topologies Induced by the Covering-Based Rough Sets

In [45], Zhao discussed a topology $\mathcal{T}$ with the covering C as a subbase. In this section, we present the relationships among the topologies $\tau(\overline{apr}''_C)$, $\tau(\overline{apr}''_{C_1})$, $\tau(\overline{apr}''_{C_2})$, $\tau(\overline{apr}''_{C_3})$, $\tau(\overline{apr}''_{C_4})$, $\tau(\overline{apr}''_{C_\cap})$, $\tau(\overline{apr}''_{C_\cup})$ and $\mathcal{T}$. The universe in this section is restricted to be finite.

Proposition 4.1. *Let C be a covering of U. Then*
$\tau(\overline{apr}''_C) = \tau(\overline{apr}''_{C_2}) = \tau(\overline{apr}''_{C_4}) = \tau(\overline{apr}''_{C_\cap}) \subseteq \tau(\overline{apr}''_{C_1}) = \tau(\overline{apr}''_{C_\cup}) \subseteq \tau(\overline{apr}''_{C_3}) \subseteq \mathcal{T}$.

Proof. (1) $\tau(\overline{apr}''_C) = \tau(\overline{apr}''_{C_2})$. For any $X \in \tau(\overline{apr}''_C)$, we have $\overline{apr}''_C(X) = X$. For any $x \in \overline{apr}''_{C_2}(X)$, there exists a $y \in U$ such that $K \in MD(C, y)$, $x \in K$ and $K \cap X \neq \emptyset$. Then $K \in C$. Hence $x \in K \subseteq \overline{apr}''_C(X) = X$. It follows that $\overline{apr}''_{C_2}(X) \subseteq X$. By Corollary 2.4, we get $X \subseteq \overline{apr}''_{C_2}(X)$. Therefore, $\overline{apr}''_{C_2}(X) = X$, which implies $X \in \tau(\overline{apr}''_{C_2})$. Then $\tau(\overline{apr}''_C) \subseteq \tau(\overline{apr}''_{C_2})$. Conversely, for any $X \in \tau(\overline{apr}''_{C_2})$, we have $\overline{apr}''_{C_2}(X) = X$. For any $x \in \overline{apr}''_C(X)$, there exists a $K \in C$ such that $x \in K$ and $K \cap X \neq \emptyset$. If $K \in MD(C, x)$, then $x \in \overline{apr}''_{C_2}(X) = X$. If $K \notin MD(C, x)$, then there exists a $G \in MD(C, x)$ such that $K \subseteq G$. It follows that $x \in G$ and $G \cap X \neq \emptyset$. Hence $x \in \overline{apr}''_{C_2}(X) = X$. Then we can conclude that $\overline{apr}''_C(X) \subseteq X$. It follows from Proposition 2.3 that $X \subseteq \overline{apr}''_C(X)$. Therefore, $\overline{apr}''_C(X) = X$, which means that $X \in \tau(\overline{apr}''_C)$. Then $\tau(\overline{apr}''_{C_2}) \subseteq \tau(\overline{apr}''_C)$.

(2) $\tau(\overline{apr}''_C) = \tau(\overline{apr}''_{C_4})$. For any $X \in \tau(\overline{apr}''_C)$, we have $\overline{apr}''_C(X) = X$. For any $x \in \overline{apr}''_{C_4}(X)$, there exists a $y \in U$ such that $x \in st(y, C)$, and $st(y, C) \cap X \neq \emptyset$. Then there exists $K_1, K_2 \in C$ such that $x \in K_1$, $y \in K_1$, $y \in K_2$, $K_2 \cap X \neq \emptyset$. Hence $K_2 \subseteq \overline{apr}''_C(X) = X$. It follows that $\{y\} \subseteq K_1 \cap K_2 \subseteq K_1 \cap X \neq \emptyset$. Then $K_1 \subseteq \overline{apr}''_C(X) = X$. Therefore, $x \in X$. It implies that $\overline{apr}''_{C_4}(X) \subseteq X$. Due to Corollary 2.4, we have $X \subseteq \overline{apr}''_{C_4}(X)$. Thus $\overline{apr}''_{C_4}(X) = X$, which follows that $X \in \tau(\overline{apr}''_{C_4})$. Hence $\tau(\overline{apr}''_C) \subseteq \tau(\overline{apr}''_{C_4})$.

Conversely, for any $X \in \tau(\overline{apr}''_{C_4})$, we have $\overline{apr}''_{C_4}(X) = X$. For any $x \in \overline{apr}''_C(X)$, there exists a $K \in C$ such that $x \in K$ and $K \cap X \neq \emptyset$. Then $K \subseteq st(x, C)$. It follows that $K \cap X \subseteq st(x, C) \cap X \neq \emptyset$. Hence

$x \in st(x, C) \subseteq \overline{apr}''_{C_4}(X) = X$, which implies that $\overline{apr}''_{C}(X) \subseteq X$. By Proposition 2.3, we get $X \subseteq \overline{apr}''_{C}(X)$. Therefore, $\overline{apr}''_{C}(X) = X$, which implies $X \in \tau(\overline{apr}''_{C})$. Then $\tau(\overline{apr}''_{C_2}) \subseteq \tau(\overline{apr}''_{C})$.

(3) $\tau(\overline{apr}''_{C}) = \tau(\overline{apr}''_{C_4})$. For any $X \in \tau(\overline{apr}''_{C})$, we have $\overline{apr}''_{C}(X) = X$. For any $x \in \overline{apr}''_{C_\cap}(X)$, there exists a $K \in C_\cap$ such that $x \in K$ and $K \cap X \neq \emptyset$. Since $C_\cap \subseteq C$, $K \in C$. Then $x \in \overline{apr}''_{C}(X) = X$, which implies that $\overline{apr}''_{C_\cap}(X) \subseteq X$. By Corollary 2.4, we have $X \subseteq \overline{apr}''_{C_\cap}(X)$. Thus $\overline{apr}''_{C_\cap}(X) = X$, which follows that $X \in \tau(\overline{apr}''_{C_\cap})$. Hence $\tau(\overline{apr}''_{C}) \subseteq \tau(\overline{apr}''_{C_\cap})$.

Conversely, For any $X \in \tau(\overline{apr}''_{C_\cap})$, we get $\overline{apr}''_{C_\cap}(X) = X$. For any $x \in \overline{apr}''_{C}(X)$, there exists a $K \in C$ such that $x \in K$ and $K \cap X \neq \emptyset$. If $K \in C_\cap$, then $x \in \overline{apr}''_{C_\cap}(X) = X$. If $K \notin C_\cap$, by the definition of $C_\cap$, there at least exists a $G \in C_\cap$ such that $K \subseteq G$. Then $x \in G$ and $G \cap X \neq \emptyset$, which follows that $x \in \overline{apr}''_{C_\cap}(X) = X$. Hence we can conclude that $\overline{apr}''_{C}(X) \subseteq X$. According to Proposition 2.3, we get $X \subseteq \overline{apr}''_{C}(X)$. Therefore, $\overline{apr}''_{C}(X) = X$, which implies that $X \in \tau(\overline{apr}''_{C})$. Therefore, $\tau(\overline{apr}''_{C_\cap}) \subseteq \tau(\overline{apr}''_{C})$.

(4) $\tau(\overline{apr}''_{C}) \subseteq \tau(\overline{apr}''_{C_\cup})$. For any $X \in \tau(\overline{apr}''_{C})$, we have $\overline{apr}''_{C}(X) = X$. For any $x \in \overline{apr}''_{C_\cup}(X)$, there exists a $K \in C_\cup$ such that $x \in K$ and $K \cap X \neq \emptyset$. Since $C_\cup \subseteq C$, $K \in C$. Then $x \in \overline{apr}''_{C}(X) = X$, which implies that $\overline{apr}''_{C_\cup}(X) \subseteq X$. By Corollary 2.4, we obtain $X \subseteq \overline{apr}''_{C_\cap}(X)$. So $\overline{apr}''_{C_\cup}(X) = X$, which follows that $X \in \tau(\overline{apr}''_{C_\cup})$. Hence, $\tau(\overline{apr}''_{C}) \subseteq \tau(\overline{apr}''_{C_\cup})$.

(5) $\tau(\overline{apr}''_{C_\cup}) = \tau(\overline{apr}''_{C_1})$. Firstly, we prove $C_\cup = C_1$. For any $K \in C$, if $K \notin C_1$, then $K \notin Md(C, x)$ for all $x \in U$. Hence, for any $x \in K$, there exists a $K_x \in C$ such that $K_x \subseteq K$ and $K_x \neq K$. Then $K = \cup_{x \in K} K_x$. It follows that $K \notin C_\cup$. Thus $C_\cup \subseteq C_1$. Conversely, for any $K \in C$, if $K \notin C_\cup$, then there exists $\mathcal{F} \subseteq C - \{K\}$ such that $K = \cup \mathcal{F}$. Then, for any $x \in K$, there exists $K_x \in \mathcal{F}$ such that $x \in K_x \subseteq K$, which follows that $K \notin Md(C, x)$. Therefore, $K \notin C_1$. Hence $C_1 \subseteq C_\cup$. Secondly, we have $(\underline{apr}''_{C_\cup}, \overline{apr}''_{C_\cup}) = (\underline{apr}''_{C_1}, \overline{apr}''_{C_1})$ and $\tau(\overline{apr}''_{C_\cup}) = \tau(\overline{apr}''_{C_1})$.

(6) $\tau(\overline{apr}''_{C_1}) \subseteq \tau(\overline{apr}''_{C_3})$. For any $X \in \tau(\overline{apr}''_{C_1})$, we obtain $\overline{apr}''_{C_1}(X) = X$. For any $x \in \overline{apr}''_{C_3}(X)$, there exists a $y \in U$ such that $x \in (y)_C$ and $(y)_C \cap X \neq \emptyset$. Then there exists a $K \in Md(C, y)$ such that $y \in K$. Hence $(y)_C \subseteq K$, which follows that $K \cap X \neq \emptyset$. Thus $y \in \overline{apr}''_{C_1}(X) = X$, which implies that $\overline{apr}''_{C_3}(X) \subseteq X$. Thank to Corollary 2.4, we get $X \subseteq \overline{apr}''_{C_3}(X)$. So $\overline{apr}''_{C_3}(X) = X$, which follows that $X \in \tau(\overline{apr}''_{C_3})$. Hence $\tau(\overline{apr}''_{C_1}) \subseteq \tau(\overline{apr}''_{C_3})$.

(7) $\tau(\overline{apr}''_{C_3}) \subseteq \mathcal{T}$. For any $x \in U$, we obtain $(x)_C \in \mathcal{T}$. Then, for any $X \in \tau(\overline{apr}''_{C_3})$, we have $X = \overline{apr}''_{C_3}(X) = \cup\{(x)_C | (x)_C \cap X \neq \emptyset\} \in \mathcal{T}$. Thus $\tau(\overline{apr}''_{C_3}) \subseteq \mathcal{T}$. □

Remark 4.2. Generally, $\tau(\overline{apr}''_{C_1}) \subseteq \tau(\overline{apr}''_{C_\cap})$, $\tau(\overline{apr}''_{C_3}) \subseteq \tau(\overline{apr}''_{C_\cup})$ and $\mathcal{T} \subseteq \tau(\overline{apr}''_{C_3})$ do not hold.

Example 4.3. (1) Let $U = \{a, b, c, d\}$, $C = \{\{a, b\}, \{b, c\}, \{d\}, \{a, b, c, d\}\}$. Then

$C_\cup = \{\{a, b\}, \{b, c\}, \{d\}\}$.

We obtain $\tau(\overline{apr}''_{C_\cap}) = \tau(\overline{apr}''_{C}) = \{\emptyset, U\}$ and $\tau(\overline{apr}''_{C_1}) = \tau(\overline{apr}''_{C_\cup}) = \{\emptyset, \{a, b, c\}, \{d\}, U\}$. Then we have $\tau(\overline{apr}''_{C_1}) \nsubseteq \tau(\overline{apr}''_{C_\cap})$.

(2) Let $U = \{a, b, c, d\}$, $C = \{\{a, b\}, \{b, c\}, \{c, d\}, \{d\}\}$. Hence

$C_1 = \{\{a, b\}, \{b, c\}, \{c, d\}, \{d\}\}$,

$C_3 = \{\{a, b\}, \{b\}, \{c\}, \{d\}\}$.

Therefore, $\tau(\overline{apr}''_{C_\cup}) = \tau(\overline{apr}''_{C_1}) = \{\emptyset, U\}$ and $\tau(\overline{apr}''_{C_3}) = \{\emptyset, \{a, b\}, \{c\}, \{d\}, \{a, b, c\}, \{a, b, d\}, \{c, d\}, U\}$. Thus we have $\tau(\overline{apr}''_{C_3}) \nsubseteq \tau(\overline{apr}''_{C_1})$.

(3) Let $U = \{a, b, c\}$, $C = \{\{a, b, c\}, \{b, c\}\}$. Then $C_3 = \{\{a, b, c\}, \{b, c\}\}$. Hence we get $\tau(\overline{apr}''_{C_3}) = \{\emptyset, U\}$ and $\mathcal{T} = \{\emptyset, \{b, c\}, U\}$. It implies that $\mathcal{T} \nsubseteq \tau(\overline{apr}''_{C_3})$.

5. Conclusion

In this paper, topological structures of a type of granular based covering approximation operators which have been discussed. Then topological properties of seven pairs of covering approximation operators. By defining binary relations from covering, we have presented connectedness, countability, separation property and Lindelöf property of the topological space induced by the approximation operators. We have obtained that the topological spaces are locally connected, first countable, pseudo-discrete, Alexandrov, regular, normal and so on. We have also described relationships between the connectedness of topological spaces and the existence of definable sets in rough sets to show an application of the discussion of topological structures of the covering approximation operators.

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