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Coefficient Estimates for Some Subclasses of m-Fold Symmetric Bi-Univalent Functions

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Abstract. In this work, we introduce and investigate a subclass $\mathcal{H}_{\Sigma_m}^{h,p}(\tau,\gamma)$ of analytic and bi-univalent functions when both $f(z)$ and $f^{-1}(z)$ are m-fold symmetric in the open unit disk $\mathbb{U}$. Moreover, we find upper bounds for the initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions belonging to this subclass $\mathcal{H}_{\Sigma_m}^{h,p}(\tau,\gamma)$. The results presented in this paper would generalize and improve those that were given in several recent works.

1. Introduction, Definitions and Preliminaries

Let $\mathcal{A}$ be a class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

The subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{U}$ is denoted by $\mathcal{S}$. Thus $\mathcal{S}$ is the class of all normalized univalent functions in $\mathbb{U}$.

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The *Koebe One-Quarter Theorem* [8] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$. So every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}\big(f(z)\big) = z \qquad (z \in \mathbb{U})$$

and

$$f\big(f^{-1}(w)\big) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geqq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \tag{2}$$

If both f and f^{-1} are univalent in $\mathbb{U}$, then we say that the function f is bi-univalent in $\mathbb{U}$. We denote by Σ the class of bi-univalent functions in $\mathbb{U}$, which are given by (1).

Lewin [18] (see also [4]) investigated the class Σ of bi-univalent functions and showed that $|a_2| < 1.51$ for the Taylor-Maclaurin coefficient $|a_2|$ of functions belonging to Σ. Subsequently, Brannan *et al.* [3] conjectured that $|a_2| \leqq \sqrt{2}$. Netanyahu [20], on the other hand, showed that

$$\max_{f\in\Sigma} |a_2| = \frac{4}{3}.$$

Many recent works, which are devoted to the study of the bi-univalent function class Σ, have derived non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. For a brief history and interesting examples of functions in the class Σ, one may refer to a pioneering paper by Srivastava *et al.* [29]. In fact, this widely-cited work by Srivastava *et al.* [29] actually revived the study of analytic and bi-univalent functions in recent years and it has led to a flood of papers on the subject by (for example) Srivastava *et al.* [24–26, 28, 30, 33, 34] and other authors (see, among others, [5–7, 9–12, 15, 16, 19, 21, 32]). The coefficient estimate problem, that is, finding upper bounds of the Taylor-Maclaurin coefficients $|a_n|$ $(n \in \mathbb{N} \setminus \{2,3\})$ for each $f \in \Sigma$ is still an open problem, $\mathbb{N}$ being the set of positive integers. There seems to be no direct way to get bounds for coefficients $|a_n|$ for $n > 3$. However, in special cases, there are several papers in which the Faber polynomial methods were used for determining upper bounds for higher-order coefficients (see, for example, [1, 2, 13, 14, 31, 35, 36]).

For each function $f \in \mathcal{S}$, the function $h(z)$ given by

$$h(z) = \sqrt[m]{f(z^m)} \qquad (z \in \mathbb{U};\ m \in \mathbb{N})$$

is univalent and maps the unit disk $\mathbb{U}$ into a region with m-fold symmetry. A function is called m-fold symmetric (see [26, 27, 30]) if the function f has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1}z^{mk+1} \qquad (z \in \mathbb{U}; m \in \mathbb{N}). \tag{3}$$

We denote by $\mathcal{S}_m$ the class of m-fold symmetric univalent functions in $\mathbb{U}$, which are normalized by the series expansion (3). In fact, the functions in the class $\mathcal{S}$ are one-fold symmetric, that is,

$$\mathcal{S}_1 = \mathcal{S}.$$

Analogous to the concept of m-fold symmetric univalent functions, we now introduce the concept of m-fold symmetric bi-univalent functions. Each function $f \in \Sigma$ generates an m-fold symmetric bi-univalent function for each integer $m \in \mathbb{N}$. The normalized form of f is given as in (3). Furthermore, the series expansion for f^{-1}, which was recently proven by Srivastava *et al.* [30], is given as follows:

$$\begin{aligned} g(w) &= w - a_{m+1}w^{m+1} + [(m+1)a_{m+1}^2 - a_{2m+1}]w^{2m+1} \\ &\quad - \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right]w^{3m+1} + \cdots, \end{aligned} \tag{4}$$

where $g = f^{-1}$. We denote by Σ_m the class of m-fold symmetric bi-univalent functions in $\mathbb{U}$. In the special case when $m = 1$, the formula (4) for the class Σ_m coincides with the formula (2) for the class Σ. Some examples of m-fold symmetric bi-univalent functions are given below:

$$\left(\frac{z^m}{1-z^m}\right)^{\frac{1}{m}} \qquad \text{and} \qquad [-\log(1-z^m)]^{\frac{1}{m}}$$

with the corresponding inverse functions given by

$$\left(\frac{w^m}{1-w^m}\right)^{\frac{1}{m}} \qquad \text{and} \qquad \left(\frac{e^{w^m}-1}{e^{w^m}}\right)^{\frac{1}{m}},$$

respectively.

Quite recently, Srivastava *et al.* [26] introduced two new general subclasses $\mathcal{H}_{\Sigma_m}(\tau, \gamma, \alpha)$ and $\mathcal{H}_{\Sigma_m}(\tau, \gamma, \beta)$ of the m-fold symmetric bi-univalent function class Σ_m consisting of analytic and m-fold symmetric bi-univalent functions in $\mathbb{U}$ and derived the coefficient bounds for $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in each of these new subclasses.

Definition 1. (see [26]) Let $0 < \alpha \leqq 1$, $0 \leqq \gamma \leqq 1$ and $\tau \in \mathbb{C} \setminus \{0\}$. A function $f(z)$ given by (3) is said to be in the class $\mathcal{H}_{\Sigma_m}(\tau, \gamma, \alpha)$ if the following conditions are satisfied:

$$f \in \Sigma_m \qquad \text{and} \qquad \left|\arg\left(1 + \frac{1}{\tau}\,[f'(z) + \gamma z f''(z) - 1]\right)\right| < \frac{\alpha\pi}{2} \qquad (z \in \mathbb{U})$$

and

$$\left|\arg\left(1 + \frac{1}{\tau}\,[g'(w) + \gamma w g''(w) - 1]\right)\right| < \frac{\alpha\pi}{2} \qquad (w \in \mathbb{U}),$$

where the function g is given by (4).

Theorem 1. (see [26]) *Let the function $f(z)$ given by (3) be in the class $\mathcal{H}_{\Sigma_m}(\tau, \gamma, \alpha)$. Then*

$$|a_{m+1}| \leqq \frac{2\alpha|\tau|}{\sqrt{|\tau\alpha(m+1)(2m+1)(1+2\gamma m) + (1-\alpha)(m+1)^2(1+\gamma m)^2|}},$$

and

$$|a_{2m+1}| \leqq \frac{2\alpha^2|\tau|^2}{(m+1)(1+\gamma m)^2} + \frac{2\alpha|\tau|}{(1+2m)(1+2\gamma m)}.$$

Definition 2. (see [26]) Let $0 \leqq \beta < 1$, $0 \leqq \gamma \leqq 1$ and $\tau \in \mathbb{C} \setminus \{0\}$. A function $f(z)$ given by (3) is said to be in the class $\mathcal{H}_{\Sigma_m}(\tau, \gamma, \beta)$ if the following conditions are satisfied:

$$f \in \Sigma_m \qquad \text{and} \qquad \Re\left(1 + \frac{1}{\tau}[f'(z) + \gamma z f''(z) - 1]\right) > \beta \qquad (z \in \mathbb{U}),$$

and

$$\Re\left(1 + \frac{1}{\tau}[g'(w) + \gamma w g''(w) - 1]\right) > \beta \qquad (w \in \mathbb{U}),$$

where the function g is given by (4).

Theorem 2. (see [26]) *Let the function $f(z)$ given by* (3) *be in the class $\mathcal{H}_{\Sigma_m}(\tau,\gamma,\beta)$. Then*

$$|a_{m+1}| \leqq \sqrt{\frac{4(1-\beta)|\tau|}{(m+1)(2m+1)(1+2\gamma m)}}$$

and

$$|a_{2m+1}| \leqq \frac{2(1-\beta)^2|\tau|^2}{(m+1)(1+\gamma m)^2} + \frac{2(1-\beta)|\tau|}{(1+2m)(1+2\gamma m)}.$$

The main objective of this paper is to present an elegant formula for computing the coefficients of the inverse functions for the class Σ_m of m-fold symmetric functions by means of the residue calculus. As an application, we introduce a new subclass of bi-univalent functions in which both f and f^{-1} are m-fold symmetric analytic functions and obtain upper bounds for the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in this new subclass. Our results for the bi-univalent function class $\mathcal{H}_{\Sigma_m}^{h,p}(\tau,\gamma)$, which we shall introduce in Section 2, would generalize and improve some recent works by Srivastava *et al.* [26, 29, 30] and by Frasin [9].

2. The Subclass $\mathcal{H}_{\Sigma_m}^{h,p}(\tau,\gamma)$ and Its Associated Coefficient Estimates

In this section, the following general subclass $\mathcal{H}_{\Sigma_m}^{h,p}(\tau,\gamma)$ is introduced and investigated.

Definition 3. Assume that the functions $h : \mathbb{U} \to \mathbb{C}$ and $p : \mathbb{U} \to \mathbb{C}$, analytic in $\mathbb{U}$, are given by

$$h(z) = 1 + h_m z^m + h_{2m} z^{2m} + h_{3m} z^{3m} + \cdots$$

and

$$p(w) = 1 + p_m w^m + p_{2m} w^{2m} + p_{3m} w^{3m} + \cdots,$$

such that

$$\min\{\Re\big(h(z)\big) \qquad \text{and} \qquad \Re\big(p(z)\big)\} > 0 \qquad (z \in \mathbb{U}).$$

Let $0 \leqq \gamma \leqq 1$ and $\tau \in \mathbb{C} \setminus \{0\}$. We say that a function f given by (3) is in the class $\mathcal{H}_{\Sigma_m}^{h,p}(\tau,\gamma)$ if the following conditions are satisfied:

$$f \in \Sigma_m \qquad \text{and} \qquad \left(1 + \frac{1}{\tau}\left[f'(z) + \gamma z f''(z) - 1\right]\right) \in h(\mathbb{U}) \qquad (z \in \mathbb{U}) \tag{5}$$

and

$$\left(1 + \frac{1}{\tau}\left[g'(w) + \gamma w g''(w) - 1\right]\right) \in p(\mathbb{U}) \qquad (w \in \mathbb{U}), \tag{6}$$

where the function g is defined by (4).

Remark 1. There are many choices of the functions h and p which would provide interesting subclasses of the general class $\mathcal{H}_{\Sigma_m}^{h,p}(\tau,\gamma)$. For example, if we set

$$h(z) = p(z) = \left(\frac{1+z^m}{1-z^m}\right)\alpha = 1 + 2\alpha z^m + 2\alpha^2 z^{2m} + \cdots,$$

it can easily be verified that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 3. Thus, if we have $f \in \mathcal{H}_{\Sigma_m}^{h,p}(\tau,\gamma)$, then

$$f \in \Sigma_m \qquad \text{and} \qquad \left|\arg\left(1+\frac{1}{\tau}[f'(z)+\gamma z f''(z)-1]\right)\right| < \frac{\alpha\pi}{2} \qquad (0<\alpha \leqq 1;\ z\in\mathbb{U})$$

and

$$\left|\arg\left(1+\frac{1}{\tau}[g'(w)+\gamma w g''(w)-1]\right)\right| < \frac{\alpha\pi}{2} \qquad (0<\alpha \leqq 1;\ w\in\mathbb{U}),$$

where the function g is given by (4). On the other hand, if we take

$$h(z)=p(z)=\frac{1+(1-2\beta)z^m}{1-z^m}=1+2(1-\beta)z^m+2(1-\beta)z^{2m}+\cdots,$$

then the conditions of Definition 3 are satisfied for both functions $h(z)$ and $p(z)$. Thus, if $f \in \mathcal{H}_{\Sigma_m}^{h,p}(\tau,\gamma)$, then

$$f \in \Sigma_m \qquad \text{and} \qquad \Re\left(1+\frac{1}{\tau}\left[f'(z)+\gamma z f''(z)-1\right]\right) > \beta \qquad (0 \leqq \beta < 1;\ z\in\mathbb{U})$$

and

$$\Re\left(1+\frac{1}{\tau}\left[g'(w)+\gamma w g''(w)-1\right]\right) > \beta, \qquad (0 \leqq \beta < 1;\ w\in\mathbb{U}),$$

where the function g is defined by (4).

We are now ready to express the bounds for the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for the subclass $\mathcal{H}_{\Sigma_m}^{h,p}(\tau,\gamma)$ of the normalized bi-univalent function class Σ.

Theorem 3. *Let the function $f(z)$ given by (3) be in the class $\mathcal{H}_{\Sigma_m}^{h,p}(\tau,\gamma)$. Then*

$$|a_{m+1}| \leqq \min\left\{\sqrt{\frac{|\tau|^2\left(|h^{(m)}(0)|^2+|p^{(m)}(0)|^2\right)}{2[(m+1)!(1+\gamma m)]^2}},\ \sqrt{\frac{|\tau|\left(|h^{(2m)}(0)|+|p^{(2m)}(0)|\right)}{(2m+1)!(m+1)(1+2\gamma m)}}\right\} \tag{7}$$

and

$$|a_{2m+1}| \leqq \min\left\{\frac{|\tau|\left(|h^{(2m)}(0)|+|p^{(2m)}(0)|\right)}{2(2m+1)!(1+2\gamma m)}+\frac{|\tau|^2\left(|h^{(m)}(0)|^2+|p^{(m)}(0)|^2\right)}{4m!(m+1)!(1+\gamma m)^2},\right.$$
$$\left.\frac{|\tau||h^{(2m)}(0)|}{(2m+1)!(1+2\gamma m)}\right\}. \tag{8}$$

Proof. The main idea in the proof of Theorem 3 is to get the desired bounds for the coefficient $|a_{m+1}|$ and $|a_{2m+1}|$. Indeed, by considering the relations (5) and (6), we have

$$1+\frac{1}{\tau}\left[f'(z)+\gamma z f''(z)-1\right]=h(z) \qquad (0\leqq\gamma\leqq 1;\ \tau\in\mathbb{C}\setminus\{0\};\ z\in\mathbb{U}) \tag{9}$$

and

$$1+\frac{1}{\tau}\left[g'(w)+\gamma w g''(w)-1\right]=p(w) \qquad (0\leqq\gamma\leqq 1;\ \tau\in\mathbb{C}\setminus\{0\};\ w\in\mathbb{U}), \tag{10}$$

where each of the functions h and p satisfies the conditions of Definition 3. In light of the following Taylor-Maclaurin series expansions for the functions h and p, we get

$$h(z) = 1 + h_m z^m + h_{2m} z^{2m} + h_{3m} z^{3m} + \cdots \tag{11}$$

and

$$p(w) = 1 + p_m w^m + p_{2m} w^{2m} + p_{3m} w^{3m} + \cdots . \tag{12}$$

Substituting from the relations (11) and (12) into (9) and (10), respectively, we get

$$\frac{(m+1)(1+\gamma m)}{\tau}\, a_{m+1} = h_m, \tag{13}$$

$$\frac{(2m+1)(1+2\gamma m)}{\tau}\, a_{2m+1} = h_{2m}, \tag{14}$$

$$-\frac{(m+1)(1+\gamma m)}{\tau}\, a_{m+1} = p_m \tag{15}$$

and

$$\frac{(2m+1)(1+2\gamma m)}{\tau}\, \left[(m+1)a_{m+1}^2 - a_{2m+1}\right] = p_{2m}. \tag{16}$$

Comparing the coefficients (13) and (15), we obtain

$$h_m = -p_m \tag{17}$$

and

$$\frac{2[(m+1)(1+\gamma m)]^2}{\tau^2}\, a_{m+1}^2 = h_m^2 + p_m^2. \tag{18}$$

Now, if we add (14) and (16), we get the following relation:

$$\frac{(m+1)(2m+1)(1+2\gamma m)}{\tau}\, a_{m+1}^2 = h_{2m} + p_{2m}. \tag{19}$$

Therefore, from (18) and (19), we have

$$a_{m+1}^2 = \frac{\tau^2\left(h_m^2 + p_m^2\right)}{2[(m+1)(1+\gamma m)]^2} \tag{20}$$

and

$$a_{m+1}^2 = \frac{\tau\left(h_{2m} + p_{2m}\right)}{(m+1)(2m+1)(1+2\gamma m)}, \tag{21}$$

respectively. Therefore, we find from the equations (20) and (21) that

$$|a_{m+1}|^2 \leqq \frac{|\tau|^2\left[|h^{(m)}(0)|^2 + |p^{(m)}(0)|^2\right]}{2[(m+1)!(1+\gamma m)]^2}$$

and

$$|a_{m+1}|^2 \leqq \frac{|\tau|\left[|h^{(2m)}(0)| + |p^{(2m)}(0)|\right]}{(2m+1)!(m+1)(1+2\gamma m)},$$

respectively. We have thus derived the desired bound on the coefficient $|a_{m+1}|$ as asserted in (7).

The proof is completed by finding the bound on the coefficient $|a_{2m+1}|$. Upon subtracting (16) from (14), we get

$$\frac{(2m+1)(1+2\gamma m)}{\tau}\left[2a_{2m+1}-(m+1)a_{m+1}^2\right]=h_{2m}-p_{2m}. \tag{22}$$

Putting the value of a_{m+1}^2 from (20) into (22), it follows that

$$a_{2m+1}=\frac{\tau^2\left(h_m^2+p_m^2\right)}{4(m+1)(1+\gamma m)^2}+\frac{\tau(h_{2m}-p_{2m})}{2(2m+1)(1+2\gamma m)},$$

Therefore, we conclude the following bound:

$$|a_{2m+1}|\leqq\frac{|\tau|^2\left[|h^{(m)}(0)|^2+|p^{(m)}(0)|^2\right]}{4(m!)^2(m+1)(1+\gamma m)^2}+\frac{|\tau|\left[\left|h^{(2m)}(0)\right|+\left|p^{(2m)}(0)\right|\right]}{2(2m+1)!(1+2\gamma m)}. \tag{23}$$

By substituting the value of a_{m+1}^2 from (21) into (22), we obtain

$$a_{2m+1}=\frac{\tau\left(h_{2m}-p_{2m}\right)}{2(2m+1)(1+2\gamma m)}+\frac{\tau[h_{2m}+p_{2m}]}{2(2m+1)(1+2\gamma m)}=\frac{\tau h_{2m}}{(1+2m)(1+2\gamma m)},$$

which readily yields

$$|a_{2m+1}|\leqq\frac{|\tau|\left|h^{(2m)}(0)\right|}{(2m+1)!(1+2\gamma m)}. \tag{24}$$

Finally, from (23) and (24), we get the desired estimate on the coefficient $|a_{2m+1}|$ as asserted in (8). The proof of Theorem 3 is thus completed. □

3. Corollaries and Consequences

If we put

$$h(z)=p(z)=\left(\frac{1+z^m}{1-z^m}\right)\alpha=1+2\alpha z^m+2\alpha^2z^{2m}+\cdots,$$

in Theorem 3, then Corollary 1 can be obtained.

Corollary 1. *Let the function $f(z)$ given by (3) be in the class $\mathcal{H}_{\Sigma_m}(\tau,\gamma,\alpha)$. Then*

$$|a_{m+1}|\leqq\min\left\{\frac{2\alpha|\tau|}{(m+1)(1+\gamma m)},\ \sqrt{\frac{4\alpha^2|\tau|}{(m+1)(2m+1)(1+2\gamma m)}}\right\}$$

and

$$|a_{2m+1}|\leqq\frac{2\alpha^2|\tau|}{(2m+1)(1+2\gamma m)}.$$

Remark 2. For the coefficient $|a_{2m+1}|$, it is easily seen that

$$\frac{2\alpha^2|\tau|}{(2m+1)(1+2\gamma m)}\leqq\frac{2\alpha^2|\tau|^2}{(m+1)(1+\gamma m)^2}+\frac{2\alpha|\tau|}{(2m+1)(1+2\gamma m)}.$$

Therefore, clearly, Corollary 1 provides an improvement over Theorem 1.

If we put $\tau = 1$ and $\gamma = 0$ in Corollary 1, then the class $\mathcal{H}_{\Sigma_m}(\tau, \gamma, \alpha)$ reduces to the class $\mathcal{H}^{\alpha}_{\Sigma_m}$ which was introduced and studied by Srivastava *et al.* [30]. We thus deduce the following corollary which is an improvement of a known result due to Srivastava *et al.* [30, Theorem 2] (see also Remark 3 below).

Corollary 2. *Let the function $f(z)$ given by* (3) *be in the class $\mathcal{H}^{\alpha}_{\Sigma_m}$. Then*

$$|a_{m+1}| \leqq \frac{2\alpha}{\sqrt{(m+1)(2m+1)}}$$

and

$$|a_{2m+1}| \leqq \frac{2\alpha^2}{2m+1}.$$

Remark 3. The bounds on $|a_{m+1}|$ and $|a_{2m+1}|$, which areasserted by Corollary 2, are better than those given by Srivastava *et al.* [30, Theorem 2].

Remark 4. If we set $m = 1$ in Corollary 2, then the class $\mathcal{H}^{\alpha}_{\Sigma_m}$ reduces to the class $\mathcal{H}^{\alpha}_{\Sigma}$ introduced and studied by Srivastava *et al.* [29]. We thus have the following Corollary.

Corollary 3. *Let the function $f(z)$ given by* (1) *be in the class $\mathcal{H}^{\alpha}_{\Sigma}$ $(0 < \alpha \leqq 1)$. Then*

$$|a_2| \leqq \sqrt{\frac{2}{3}}\,\alpha \tag{25}$$

and

$$|a_3| \leqq \frac{2\alpha^2}{3}. \tag{26}$$

Remark 5. Corollary 3 provides an improvement over a result which was obtained by Srivastava *et al.* [29, Theorem 1].

By setting $m = 1$ in Corollary 1, the class $\mathcal{H}_{\Sigma_m}(\tau, \gamma, \alpha)$ reduces to the class $\mathcal{R}^{\alpha}_{\Sigma}(\tau, \gamma)$ and we are thus led to the following corollary.

Corollary 4. *Let the function $f(z)$ given by* (1) *be in the class $\mathcal{R}^{\alpha}_{\Sigma}(\tau, \gamma)$. Then*

$$|a_2| \leqq \min\left\{\frac{|\tau|\alpha}{1+\gamma}, \sqrt{\frac{2}{3}\left(\frac{|\tau|\alpha}{1+2\gamma}\right)}\right\}$$

and

$$|a_3| \leqq \frac{2}{3}\left(\frac{|\tau|\alpha^2}{1+2\gamma}\right).$$

If we let $\tau = 1$ in Corollary 4, then we have Corollary 5 below.

Corollary 5. *Let the function f given by* (1) *be in the class $\mathcal{H}_{\Sigma}(\alpha, \gamma)$. Then*

$$|a_2| \leqq \sqrt{\frac{2}{3}\left(\frac{\alpha}{1+2\gamma}\right)}$$

and

$$|a_3| \leqq \frac{2}{3}\left(\frac{\alpha^2}{1+2\gamma}\right).$$

Remark 6. It is easy to see that

$$\sqrt{\frac{2}{3}\left(\frac{\alpha}{1+2\gamma}\right)} \leqq \frac{2\alpha}{\sqrt{2(\alpha+2)+4\gamma(\alpha+\gamma+2-\alpha\gamma)}}$$

and

$$\frac{2}{3}\left(\frac{\alpha^2}{1+2\gamma}\right) \leqq \left(\frac{\alpha}{1+\gamma}\right)^2 + \frac{2}{3}\left(\frac{\alpha}{1+2\gamma}\right).$$

Thus, clearly, Corollary 5 provides a refinement of the estimates which were obtained by Frasin [9, Theorem 2.2].

By letting

$$h(z) = p(z) = \frac{1+(1-2\beta)z^m}{1-z^m} = 1 + 2(1-\beta)z^m + 2(1-\beta)z^{2m} + \cdots$$

in Theorem 3, we deduce the following corollary.

Corollary 6. *Let the function $f(z)$ given by* (3) *be in the class $\mathcal{H}_{\Sigma_m}(\tau,\gamma,\beta)$. Then*

$$|a_{m+1}| \leqq \min\left\{\frac{2(1-\beta)|\tau|}{(m+1)(1+\gamma m)},\ \sqrt{\frac{4(1-\beta)|\tau|}{(m+1)(2m+1)(1+2\gamma m)}}\right\}$$

and

$$|a_{2m+1}| \leqq \frac{2(1-\beta)|\tau|}{(2m+1)(1+2\gamma m)}.$$

Remark 7. It is easy to see, for the coefficient $|a_{2m+1}|$, that

$$\frac{2(1-\beta)|\tau|}{(2m+1)(1+2\gamma m)} \leqq \frac{2\left[(1-\beta)|\tau|\right]^2}{(m+1)(1+\gamma m)^2} + \frac{2(1-\beta)|\tau|}{(2m+1)(1+2\gamma m)}.$$

Thus, obviously, an improvement of Theorem 2 is provided by Corollary 6.

Remark 8. If we take $\tau = 1$ and $\gamma = 0$ in Corollary 6, then the class $\mathcal{H}_{\Sigma_m}(\tau,\gamma,\beta)$ reduces to the class $\mathcal{H}^{\beta}_{\Sigma_m}$ which was introduced and studied by Srivastava *et al.* [30]. We are thus led to Corollary 7 below.

Corollary 7. *Let the function $f(z)$ given by* (3) *be in the class $\mathcal{H}^{\beta}_{\Sigma_m}$. Then*

$$|a_{m+1}| \leqq \begin{cases} \sqrt{\dfrac{4(1-\beta)}{(m+1)(2m+1)}} & \left(0 \leqq \beta < \dfrac{m}{2m+1}\right) \\[2ex] \dfrac{2(1-\beta)}{m+1} & \left(\dfrac{m}{2m+1} \leqq \beta < 1\right) \end{cases}$$

and

$$|a_{2m+1}| \leqq \frac{2(1-\beta)}{2m+1}.$$

Remark 9. Corollary 7 provides a refinement of a result which was proven by Srivastava *et al.* [30, Theorem 3].

Remark 10. If we set $m = 1$ in Corollary 7, then the class $\mathcal{H}^{\beta}_{\Sigma_m}$ reduces to the class $\mathcal{H}^{\beta}_{\Sigma}$ which was introduced and studied by Srivastava *et al.* [29]. In this special case, we get the following Corollary.

Corollary 8. *Let the function $f(z)$ given by* (1) *be in the class $\mathcal{H}^{\beta}_{\Sigma}$ $(0 \leqq \beta < 1)$. Then*

$$|a_2| \leqq \begin{cases} \sqrt{\dfrac{2(1-\beta)}{3}} & \left(0 \leqq \beta \leqq \frac{1}{3}\right) \\ 1-\beta & \left(\frac{1}{3} \leqq \beta < 1\right) \end{cases}$$

and

$$|a_3| \leqq \frac{2(1-\beta)}{3}.$$

Remark 11. The bounds on $|a_2|$ and $|a_3|$, which are asserted by Corollary 8, are better than those given by Srivastava *et al.* [29, Theorem 2].

By setting $m = 1$ in Corollary 6, the class $\mathcal{H}_{\Sigma_m}(\tau, \gamma, \beta)$ reduces to the class $\mathcal{R}^{\beta}_{\Sigma}(\tau, \gamma)$ and we thus obtain the following consequence.

Corollary 9. *Let the function $f(z)$ given by* (1) *be in the class $\mathcal{R}^{\beta}_{\Sigma}(\tau, \gamma)$. Then*

$$|a_2| \leqq \min\left\{\frac{|\tau|(1-\beta)}{1+\gamma}, \sqrt{\frac{2}{3}\left(\frac{|\tau|(1-\beta)}{1+2\gamma}\right)}\right\}$$

and

$$|a_3| \leqq \frac{2}{3}\left(\frac{|\tau|(1-\beta)}{1+2\gamma)}\right).$$

If we take $\tau = 1$ in Corollary 9, then we have Corollary 10 below.

Corollary 10. *Let the function f given by* (1) *be in the class $\mathcal{H}_{\Sigma}(\beta, \gamma)$. Then*

$$|a_2| \leqq \min\left\{\frac{1-\beta}{1+\gamma}, \sqrt{\frac{2}{3}\left(\frac{1-\beta}{1+2\gamma}\right)}\right\}$$

and

$$|a_3| \leqq \frac{2}{3}\left(\frac{1-\beta}{1+2\gamma}\right).$$

Remark 12. Corollary 10 is an improvement of the following estimates which were obtained by Frasin [9, Theorem 3.2]. In fact, for the coefficient $|a_2|$, if

$$\gamma > \frac{3\delta - 2 + \sqrt{3\delta(3\delta-2)}}{2} \qquad \text{and} \qquad \frac{2}{3} < \delta < \frac{8}{9} \qquad (\delta = 1-\beta),$$

then

$$\frac{1-\beta}{1+\gamma} < \sqrt{\frac{2}{3}\left(\frac{1-\beta}{1+2\gamma}\right)}.$$

Also, for the coefficient $|a_3|$, we have

$$\frac{2}{3}\left(\frac{1-\beta}{1+2\gamma}\right) \leqq \left(\frac{1-\beta}{1+\gamma}\right)^2 + \frac{2}{3}\left(\frac{1-\beta}{1+2\gamma}\right).$$

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