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Some Explicit Formulas for the Generalized Frobenius-Euler Polynomials of Higher Order

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Abstract. Our aim is to derive some explicit formulas for the generalized Bernoulli and Euler polynomials in terms of Whitney and translated Whitney numbers of the second kind. Also we derive some explicit formulas for the generalized Euler polynomials and Genocchi-like polynomials in terms of generalized Whitney polynomials of the second kind. We provide an algorithm for computing the generalized Frobenius-Euler polynomials of higher order.

1. Introduction

For $v \in \mathbb{C}$ and for m a positive integer, the generalized Pochhamer symbol $(v|m)_n$ is defined by

$$(v|m)_n := v(v-m)\cdots(v-(n-1)m) \quad \text{with} \quad (v|m)_0 = 1, \tag{1}$$

the Pochhamer symbol is $(v)_n := (v|1)_n$.

Also, the extended binomial polynomial $\binom{x}{n}$, for a variable x, is given by $\binom{x}{n} := \frac{x(x-1)\cdots(x-n+1)}{n!}$.

Definition 1.1. *Let α be a complex number, the generalized Bernoulli polynomials $B_n^{(m)}(x,\alpha)$ and the generalized Euler polynomials $E_n^{(m)}(x,\alpha)$ of parameter m are given by the following generating functions, for $|z| < 2\pi$*

$$\left(\frac{z}{e^{mz}-1}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} B_n^{(m)}(x,\alpha)\frac{z^n}{n!}, \quad \text{and} \quad \left(\frac{2}{e^{mz}+1}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} E_n^{(m)}(x,\alpha)\frac{z^n}{n!}. \tag{2}$$

For $m = 1$, we get the generalized Bernoulli polynomials $B_n(x,\alpha)$ and the generalized Euler polynomials $E_n(x,\alpha)$, see for instance [12, 16] and references therein.

For $x = 0$, we get the generalized Bernoulli numbers $B_n^{(m)}(\alpha)$ and the generalized Euler numbers $E_n^{(m)}(\alpha)$.

As a second extension to the Frobenius-like polynomials we propose

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Definition 1.2. *Let $\lambda \in \mathbb{C}$ with $\lambda \neq 1$, the generalized Frobenius-Euler polynomials of order α, $H_n^{(m)}(x,\alpha|\lambda)$, $n \geq 0$, $\alpha \in \mathbb{C}$, are defined by the generating function, for $|z| < 2\pi$*

$$\left(\frac{1-\lambda}{e^{mz}-\lambda}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} H_n^{(m)}(x,\alpha|\lambda)\frac{z^n}{n!}. \tag{3}$$

When $x = 0$, we deal with the generalized Frobenius-Euler numbers of order α, $H_n^{(m)}(0,\alpha|\lambda) = H_n^{(m)}(\alpha|\lambda)$.

We have the following relation $H_n^{(m)}(x,\alpha|\lambda) = \sum_{k=0}^{n}\binom{n}{k}H_n^{(m)}(\alpha|\lambda)x^{n-k}$.

For $m = 1$, we have the Frobenius-Euler polynomials of order α, $H_n(x,\alpha|\lambda)$, see for instance [16, 17] and references therein. We have $E_n^{(m)}(x,\alpha) = H_n^{(m)}(x,\alpha|-1)$.

In the special case $\alpha = m = 1$, we get the Frobenius-Euler polynomials given by $H_n^{(1)}(x,1|\lambda) = H_n(x,\lambda)$. For more information we refer to [7, 12, 13].

We will also consider the case where α be a variable at the end of each of the following sections.

Recently, Boutiche et *al*. [16] introduce the generalized Stirling polynomials of the second kind as

$$S_n^k(x) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(x+j)^n, \tag{4}$$

they establish the following explicit formulas for the generalized Bernoulli polynomials and the generalized Euler polynomials in terms of the generalized Stirling polynomials of the second kind $S_n^k(x)$ respectively,

$$B_n(x,\alpha) = \sum_{k=0}^{n}(-1)^k\binom{n+k}{k}^{-1}\binom{n+\alpha}{n-k}\binom{\alpha+k-1}{k}S_n^k(x), \quad \text{and} \quad E_n(x,\alpha) = \sum_{k=0}^{n}\frac{(-1)^k}{2^k}(\alpha)_k\, S_n^k(x).$$

Also they proved an explicit formula for the Frobenius-Euler polynomials of order α in terms of the weighted Stirling numbers of the second kind.

We propose some explicit formulas of the generalized Frobenius-Euler polynomials of order α in terms of the generalized Whitney numbers of the second kind and of the generalized translated Whitney numbers of the second kind.

We start by some classical definitions concerning Whitney numbers.

Whitney numbers $(w_m(n,k))_{k=0}^n$ of the first kind are given by $(x-1|m)_n = \sum_{k=0}^n w_m(n,k)x^k$, they satisfy the following recursion formula [14],

$$w_m(n,k) = w_m(n-1,k-1) + (m-1-mn)w_m(n-1,k) \quad (1 \leq k \leq n).$$

Whitney numbers $(W_m(n,k))_{k=0}^n$ of the second kind are given by $x^n = \sum_{k=0}^n W_m(n,k)(x-1|m)_k$, they satisfy the following recursion formula [14],

$$W_m(n,k) = W_m(n-1,k-1) + (mk+1)W_m(n-1,k) \quad (1 \leq k \leq n).$$

The exponential generating functions for $w_m(n,k)$ and $W_m(n,k)$ are given respectively by, see [3],

$$\sum_{n=0}^{\infty} w_{m,r}(n,k)\frac{z^n}{n!} = (1+mz)^{(-1/m)}\frac{\ln^k(1+mz)}{m^k k!} \quad \text{and} \quad \sum_{n=0}^{\infty} W_m(n,k)\frac{z^n}{n!} = e^z\frac{(e^{mz}-1)^k}{k!m^k}.$$

For any nonnegative integer r, the r-Whitney numbers were introduced by Mezö [14] as a new class of numbers generalizing the Whitney and r- Stirling numbers. According to [9, 14], the nth power of $mx + r$

can be expressed in terms of the falling factorial as follows $(mx+r)^n = \sum_{k=0}^{n} m^k W_{m,r}(n,k)(x)_k$, where the coefficients $W_{m,r}(n,k)$ are called r-Whitney numbers of the second kind. Their generating function is

$$\sum_{n=0}^{\infty} W_{m,r}(n,k)\frac{z^n}{n!} = e^{rz}\frac{(e^{mz}-1)^k}{k!m^k}.$$

We note that the r-Whitney numbers of the second kind may be reduced to the r-Stirling numbers of the second kind by setting $m=1$, i.e., $W_{1,0}(n,k) = S(n,k)$ and $W_{1,r}(n,k) = S_r(n+r,k+r)$.

We introduce now the generalized Whitney polynomials $W_{m,n}^k(x)$ and numbers $W_{m,n}^k$ of the second kind:

$$W_{m,n}^k(x) = \frac{1}{k!m^k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(mj+x+1)^n, \quad \text{we denote } W_{m,n}^k := W_{m,n}^k(0). \tag{5}$$

The corresponding exponential generating function is given by

$$\sum_{n=k}^{\infty} W_{m,n}^k(x)\frac{z^n}{n!} = \frac{1}{k!m^k}e^{(x+1)z}(e^{mz}-1)^k, \tag{6}$$

it satisfies the following recurrence relation,

$$W_{m,n+1}^k(x) = W_{m,n}^{(k-1)}(x) + (x+mk+1)W_{m,n}^k(x) \qquad (1 \le k \le n). \tag{7}$$

As a consequence, from Identity (6), one can deduce the following results,

$$W_{m,n}^k(x) = \sum_{j=0}^{n}\binom{n}{j}W_m(j,k)x^{n-j}, \quad \text{where} \quad W_{m,n}^k(0) = W_m(n,k), \tag{8}$$

$$W_{m,n}^k(r-1) = W_{m,r}(n,k), \ W_{m,n}^k(mr-1) = m^{n-k}S_r(n+r,k+r), \quad \text{and} \quad W_{m,n}^k(x) = m^{n-k}S_n^k\left(\frac{x+1}{m}\right). \tag{9}$$

2. Explicit formulas for the generalized Bernoulli polynomials of parameter m.

We begin by establishing an explicit formula of the generalized Bernoulli polynomials of parameter m in terms of the generalized Whitney polynomials $W_{m,n}^k(x)$ of the second kind. According to (1), we have the following result.

Theorem 2.1. *The following identity holds true*

$$B_n^{(m)}(x,\alpha) = \sum_{k=0}^{n}(-1)^k\binom{n+k}{k}^{-1}\binom{n+\alpha}{n-k}\binom{\alpha+k-1}{k}m^k W_{m,n+k}^k(x-1). \tag{10}$$

Proof. We proceed as in proof of the explicit formula of Srivastava and Todorov [18],
$B_n^{(m)}(x,\alpha) = \sum_{k=0}^{n}(-1)^k\binom{\alpha+k-1}{k}\sum_{p=0}^{n-k}\binom{n}{p+k}\frac{(p+k)!}{(p+2k)}(x-1)^{n-k-p}k!m^k W_{m,p+2k}^k\sum_{l=0}^{p}\binom{\alpha+k+l-1}{l}$.
Since $\sum_{l=0}^{n-k}\binom{\alpha+k+l-1}{l} = \binom{n+\alpha}{n-k}$, after some rearrangement, we obtain

$$B_n^{(m)}(x,\alpha) = \sum_{k=0}^{n}(-1)^k\frac{\binom{\alpha+k-1}{k}\binom{n+\alpha}{n-k}}{\binom{n+k}{k}}m^k\sum_{j=0}^{n+k}\binom{n+k}{j}(x-1)^{n+k-j}W_{m,j}^k.$$

Employing (8), the result is obtained. □

Using (9), for $m = 1$ we get, see [15], $B_n^{(\alpha,1)}(x) = \sum_{k=0}^{n}(-1)^k\binom{n+k}{k}^{-1}\binom{n+\alpha}{n-k}\binom{\alpha+k-1}{k}S_{n+k}^{k}(x).$

Setting $x = 0$ in (10), we obtain the explicit formula for the generalized Bernoulli numbers $B_n^{(m)}(\alpha)$ given in [6]. Setting $\alpha = -k$ (k being a nonnegative integer) in (10) and using (9) we get

$$B_n^{(m)}(x,-k) = \sum_{j=k}^{n}(-1)^j\binom{n+j}{j}^{-1}\binom{n-k}{j-k}\binom{-k+j-1}{j}m^n S_{n+j}^{j}(\frac{x}{m}).$$

For $x = r$ in (10), we obtain $B_n^{(m)}(r,\alpha) = \sum_{k=0}^{n}(-1)^k\binom{n+k}{k}^{-1}\binom{n+\alpha}{n-k}\binom{\alpha+k-1}{k}m^k W_{m,r}(n+k,k).$

By setting $x = r$ and $\alpha = 1$ in Theorem 10, we get a combinatorial identity with the Catalan numbers C_n:

$$B_n^{(m)}(r,1) = \sum_{k=0}^{n}(-1)^k\binom{n+k}{k}^{-1}\binom{n+1}{k+1}m^k W_{m,r}(n+k,k) = \frac{1}{C_n}\sum_{k=0}^{n}\frac{(-1)^k}{k+1}\binom{2n}{n-k}m^k W_{m,r}(n+k,k),$$

For $m = 1$, we get the result obtained in [6, Remark 2.5].

Setting $x = \alpha$ in the generating function (2), we get the generating function $B_n^{(m)}(\alpha,\alpha)$, and get explicitly $B_n^{(m)}(\alpha,\alpha) = \sum_{k=0}^{n}(-1)^k\binom{n+k}{k}^{-1}\binom{n+\alpha}{n-k}\binom{\alpha+k-1}{k}m^k W_{m,n+k}^{k}(\alpha-1).$

For $m = 2$, we get the following interesting result.

Theorem 2.2. *We have the following identity* $\left(\frac{z/2}{\sinh z}\right)^x = \sum_{n=0}^{\infty} B_n^{(2)}(x,x)\frac{z^n}{n!}.$

For $x = 1$, we generate the coefficient of the hyperbolic cosecant number in terms of the generalized Bernoulli number.

Corollary 2.3. *The following identity holds true* $z \operatorname{csc} h\, z = 2\sum_{n=0}^{\infty} B_n^{(2)}(1,1)\frac{z^n}{n!}.$

3. Explicit formulas for the generalized Euler polynomials

Here we give an explicit formula for the generalized Euler polynomials in terms of generalized Whitney polynomials of the second kind.

Theorem 3.1. *The following relationship holds true*

$$E_n^{(m)}(x,\alpha) = \sum_{k=0}^{n}\frac{(-1)^k}{2^k}(\alpha)_k\, m^k W_{m,n}^{k}(x-1). \tag{11}$$

Proof. We observe that $\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{(-1)^k}{2^k}(\alpha)_k\, m^k W_{m,n}^{k}(x)\right)\frac{z^n}{n!} = \sum_{k=0}^{\infty}\frac{(-1)^k}{2^k}(\alpha)_k\, m^k\left(\sum_{n=k}^{\infty}W_{m,n}^{k}(x)\frac{z^n}{n!}\right)$ $= \sum_{k=0}^{\infty}\frac{(-1)^k}{2^k}(\alpha)_k\, m^k e^{(x+1)z}\frac{(e^{mz}-1)^k}{m^k k!} = e^{(x+1)z}\sum_{k=0}^{\infty}\binom{\alpha+k-1}{k}\left(\frac{1-e^{mz}}{2}\right)^k = e^{(x+1)z}\left(1-\frac{1-e^{mz}}{2}\right)^{-\alpha}$, this concludes the proof. □

Using Identity (9), we have the explicit formula $E_n^{(m)}(x,\alpha) = m^n\sum_{k=0}^{n}\frac{(-1)^k}{2^k}(\alpha)_k\, S_n^k\left(\frac{x}{m}\right).$

In particular, for $m = 1$, we get Relation (3.3) given in [6], and setting $x = r$ for a fixed m in (11), we get

$$E_n^{(m)}(r,\alpha) = \sum_{k=0}^{n}\frac{(-1)^k}{2^k}(\alpha)_k\, m^k W_{m,r}(n,k). \tag{12}$$

For $m = 1$ in (12), we have $\quad E_n(r,\alpha) = \displaystyle\sum_{k=0}^{n} \frac{(-1)^k}{2^k} (\alpha)_k\, S_r(n+r, k+r).$

For $x = \alpha$ in the generating function of Euler polynomials, we get the generating function of $E_n^{(m)}(\alpha,\alpha)$, and get explicitly $\quad E_n^{(m)}(\alpha,\alpha) = \displaystyle\sum_{k=0}^{n} \frac{(-1)^k}{2^k} (\alpha)_k\, m^k W_{m,n}^k(\alpha-1).$

For $m = 2$, we get the following result $\quad \left(\dfrac{1}{\cosh z}\right)^x = \displaystyle\sum_{n=0}^{\infty} E_n^{(2)}(x,x)\frac{z^n}{n!}.$

For $x = 1$, we generate the coefficient of the hyperbolic secant number in terms of the generalized Bernoulli number.

Corollary 3.2. *We have the following identity* $\quad \operatorname{sech} z = \displaystyle\sum_{n=0}^{\infty} E_n^{(2)}(1,1)\frac{z^n}{n!}.$

4. Explicit formulas for the generalized Frobenius-Euler Polynomials

Now, we deal with the generalized Frobenius-Euler polynomials $H_n^{(m)}(x,\alpha|\lambda)$ of order $\alpha \in \mathbb{C}$, with $\lambda \neq 1$. We express it in terms of the generalized Whitney polynomials of the second kind as follows

Theorem 4.1. *The following relationship holds true*

$$H_n^{(m)}(x,\alpha|\lambda) = \sum_{k=0}^{n} \frac{(\alpha)_k}{(\lambda-1)^k} m^k W_{m,n}^k(x-1). \tag{13}$$

Proof. From (6), we have $\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{(\alpha)_k}{(\lambda-1)^k} m^k W_{m,n}^k(x)\right)\frac{z^n}{n!} = \sum_{k=0}^{\infty}\frac{(\alpha)_k}{(\lambda-1)^k} m^k \left(\sum_{n=k}^{\infty} W_{m,n}^k(x)\frac{z^n}{n!}\right) = \sum_{k=0}^{\infty}\frac{(\alpha)_k}{(\lambda-1)^k} m^k e^{(x+1)z}\frac{(e^{mz}-1)^k}{m^k k!} = e^{(x+1)z}\sum_{k=0}^{\infty}(\alpha)_k\frac{1}{k!}\left(\frac{e^{mz}-1}{\lambda-1}\right)^k = e^{(x+1)z}\left(1-\frac{e^{mz}-1}{\lambda-1}\right)^{-\alpha} = \sum_{n=0}^{\infty} H_n^{(m)}(x+1,\alpha|\lambda)\frac{z^n}{n!}$, which gives the desired result. □

Setting $m = 1$, we have $\quad H_n(x,\alpha|\lambda) = \displaystyle\sum_{k=0}^{n}\frac{(\alpha)_k}{(\lambda-1)^k} S_n^k(x)$, which is due to [6, Eq. (7)].

For $x = 1$ and $\alpha = s$ (s being a positive integer) in (13), we have $\quad H_n^{(m)}(1,s|\lambda) = \displaystyle\sum_{k=0}^{n}\frac{(s)_k}{(\lambda-1)^k} m^k W_m(n,k).$

Substituting $\alpha = 1$ in (13), we have $\quad H_n^{(m)}(x,1|\lambda) = H_n^{(m)}(x,\lambda) = \displaystyle\sum_{k=0}^{n}\frac{k!}{(\lambda-1)^k} m^k W_{m,n}^k(x-1).$

For $x = r$ in (13), we have $\quad H_n^{(m)}(r,\alpha|\lambda) = \displaystyle\sum_{k=0}^{n}\frac{(\alpha)_k}{(\lambda-1)^k} m^k W_{m,r}(n,k).$

In particular, for $m = 1$, we have $\quad H_n(r,\alpha|\lambda) = \displaystyle\sum_{k=0}^{n}\frac{(\alpha)_k}{(\lambda-1)^k} S_r(n+r,k+r)$, which is given in [16, Remark 3].

Setting $\lambda = -1$, we obtain Formula (11) for the generalized Euler polynomials $E_n^{(m)}(x,\alpha)$,

In particular, for $m = 1$, we have $\quad E_n(x,\alpha) = \displaystyle\sum_{k=0}^{n}\frac{(-1)^k}{2^k}(\alpha)_k S_n^k(x)$, which is given in [6, Eq. 3.3].

Setting $x = \alpha$ in the generating function (3), we get the generating function $H_n^{(m)}(\alpha,\alpha|\lambda)$, and get explicitly $H_n^{(m)}(\alpha,\alpha|\lambda) = \displaystyle\sum_{k=0}^{n}\frac{(\alpha)_k}{(\lambda-1)^k} m^k W_{m,n}^k(\alpha-1).$

According to (13) and (5), we have, for $m > 0$.

Proposition 4.2. *We have* $H_n^{(m)}(x,\alpha|\lambda) = \sum_{i=0}^{n}\binom{n}{i}m^{n-i}\left(\sum_{0\le j\le k\le n}\frac{(\alpha)_k}{(\lambda-1)^k}\frac{(-1)^{k-j}}{k!}\binom{k}{j}j^{n-i}\right)x^i.$

Proof. In fact, $H_n^{(m)}(x,\alpha|\lambda) = \sum_{k=0}^{n}\frac{(\alpha)_k}{(\lambda-1)^k}\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(mj+x)^n = \sum_{\substack{0\le j\le k\le n\\ 0\le i\le n}}\frac{(\alpha)_k}{(\lambda-1)^k}\frac{(-1)^{k-j}}{k!}\binom{k}{j}\binom{n}{i}(mj)^{n-i}x^i =$
$\sum_{i=0}^{n}\binom{n}{i}m^{n-i}\left(\sum_{0\le j\le k\le n}\frac{(\alpha)_k}{(\lambda-1)^k}\frac{(-1)^{k-j}}{k!}\binom{k}{j}j^{n-i}\right)x^i$ □

5. Generalized Frobenius-Genocchi polynomials of higher order

Here, we give an explicit formula for Frobenius-Genocchi polynomials introduced by Yaşar and Őzarslan [19], by the mean of the following generating function $\frac{(1-\lambda)z}{e^z-\lambda}e^{xz} = \sum_{n=0}^{\infty}G_n(x|\lambda)\frac{z^n}{n!}$.

It is natural that we define the generalization of the Frobenius-Genocchi polynomials by means of the following generating function

$$\left(\frac{(1-\lambda)z}{e^{mz}-\lambda}\right)^{\alpha}e^{xz} = \sum_{n=0}^{\infty}G_n^{(m)}(x,\alpha|\lambda)\frac{z^n}{n!}. \tag{14}$$

When $x=0$, $G_n^{(m)}(0,\alpha|\lambda) = G_n^{(m)}(\alpha|\lambda)$ denote the generalized Frobenius-Genocchi numbers of order α. For a unified extension of Euler and Genocchi polynomials, see [2]

The generalized Genocchi polynomials $G_n^{(m)}(x,\alpha)$, given by $G_n^{(m)}(x,\alpha) = G_n^{(m)}(x,\alpha|-1)$ are defined by the following generating function

$$\left(\frac{2z}{e^{mz}+1}\right)^{\alpha}e^{xz} = \sum_{n=0}^{\infty}G_n^{(m)}(x,\alpha)\frac{z^n}{n!}. \tag{15}$$

By setting $\alpha = m = 1$ in (15), we obtain the classical Genocchi polynomials $G_n(x) = G_n^{(1)}(x,1)$.
According to (3), (14) and (13), we have the corollary

Corollary 5.1. *The following explicit formula holds true*

$$G_n^{(m)}(x,l|\lambda) = \frac{n!}{(n-l)!}H_{n-l}^{(m)}(x,l|\lambda) = \frac{n!}{(n-l)!}\sum_{k=0}^{n-l}\frac{(l)_k}{(\lambda-1)^k}m^kW_{m,n-l}^k(x-1). \tag{16}$$

By setting $m=1$, Corollary 5.1 is reduced to the [6, Eq. 12].
By substituting $x=1$ in (16), we obtain the following explicit formula in terms of the Whitney numbers of the second kind $G_n^{(m)}(1,l|\lambda) = \frac{n!}{(n-l)!}\sum_{k=0}^{n-l}\frac{(l)_k}{(\lambda-1)^k}m^kW_m(n-l,k)$.

If we set $x=r$, we obtain $G_n^{(m)}(r,l|\lambda) = \frac{n!}{(n-l)!}\sum_{k=0}^{n-\alpha}\frac{(l)_k}{(\lambda-1)^k}m^kW_{m,r}(n-l,k)$.

By setting $\lambda=-1$ in (16), we obtain an explicit formula for the generalized Genocchi polynomials $G_n^{(m)}(x,l)$ of order l as follows, $G_n^{(m)}(x,l) = \frac{n!}{(n-l)!}\sum_{k=0}^{n-l}\frac{(-1)^k}{2^k}(l)_km^kW_{m,n-l}^k(x-1)$.

Setting $x=l$ in the generating function (14), we have $\left(\frac{(1-\lambda)z}{e^{(m-1)z}-\lambda e^{-z}}\right)^l = \sum_{n=0}^{\infty}G_n^{(m)}(l|\lambda,l)\frac{z^n}{n!}$, which have the explicit formula $G_n^{(m)}(l,l|\lambda) = \frac{n!}{(n-l)!}\sum_{k=0}^{n-\alpha}\frac{(l)_k}{(\lambda-1)^k}m^kW_{m,n-l}^k(l-1)$.

6. What about translated Whitney numbers

We introduce the generalized translated Whitney polynomials of the second kind by

$$\mathcal{W}_n^{k,m}(x)=\frac{1}{k!m^k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(x+mj)^n\ .s$$

The exponential generating function of $\mathcal{W}_n^{k,m}(x)$ is given by

$$\sum_{n=k}^{\infty}\mathcal{W}_n^{k,m}(x)\frac{z^n}{n!}=\frac{1}{k!m^k}e^{xz}(e^{mz}-1)^k, \tag{17}$$

and they satisfy the following recurrence relation $\mathcal{W}_{n+1}^{k,m}(x)=\mathcal{W}_n^{k-1,m}(x)+(x+mk)\mathcal{W}_n^{k,m}(x)$, $(1\le k\le n)$.

From (17), we have $\mathcal{W}_n^{k,m}(x+1)=W_n^{k,m}(x) \quad (1\le k\le n)$.

For $m=1$, we recover the classical generalized Stirling numbers of the second kind $S_n^k(x)$.

For $x=0$, we have the translated Whitney numbers of the second kind $\mathcal{W}^m(n,k)$, see [1].

For $x=r$, we have the translated r-Whitney numbers of the second kind $\mathcal{W}_r^m(n+r,k+r)$, see also [1].

This section will be presented without proofs (they are similar to precedent ones).

6.1. Generalized Bernoulli polynomials

Employing the generalized translated Whitney polynomials $\mathcal{W}_n^{k,m}(x)$ of the second kind, we derive the following result concerning the generalized Bernoulli polynomials of parameter m,

Theorem 6.1. *The following relationship holds true* $B_n^{(m)}(x,\alpha)=\sum_{k=0}^{n}(-1)^k\binom{n+k}{k}^{-1}\binom{n+\alpha}{n-k}\binom{\alpha+k-1}{k}m^k\mathcal{W}_{n+k}^{k,m}(x)$.

Setting $x=\alpha$, we get $B_n^{(m)}(\alpha,\alpha)=\sum_{k=0}^{n}(-1)^k\binom{n+k}{k}^{-1}\binom{n+\alpha}{n-k}\binom{\alpha+k-1}{k}m^k\mathcal{W}_{n+k}^{k,m}(\alpha)$.

For $m=1$, we get the explicit formula for the generalized Bernoulli polynomials $B_n(x,\alpha)$, see [6, Eq. 2.1].

Setting $x=0$ in Theorem 6.1, we get an explicit formula for the generalized Bernoulli numbers $B_n^{(\alpha,m)}$.

For $x=r$, $\alpha=-k$ (k in $\mathbb{N}$), we get $B_n^{(m)}(x,-k)=\sum_{j=0}^{n}(-1)^j\binom{n+j}{j}^{-1}\binom{n-k}{j-k}\binom{j-k-1}{j}\mathcal{W}_r^m(n+j,j)$.

And setting $x=r$ and $\alpha=1$, we get $B_n^{(m)}(r,1)=\sum_{k=0}^{n}(-1)^k\binom{n+k}{k}^{-1}\binom{n+1}{k+1}m^k\mathcal{W}_r^m(n+k,k)$.

6.2. Generalized Euler polynomials

An explicit formula for the generalized Euler polynomials in terms of the translated Whitney numbers:

Theorem 6.2. *The following relationship holds true* $E_n^{(m)}(x,\alpha)=\sum_{k=0}^{n}\frac{(-1)^k}{2^k}(\alpha)_k\,m^k\mathcal{W}_n^{k,m}(x)$.

In particular, for $m=1$, we have an explicit formula given in [6, Eq. (3.3)].

For $x=0$, we get an explicit formula for the generalized Euler numbers $E_n^{(m)}(0,\alpha)=\sum_{k=0}^{n}\frac{(-1)^k}{2^k}(\alpha)_k\,m^k\mathcal{W}^m(n,k)$.

Setting $x=r$, we get $E_n^{(m)}(r,\alpha)=\sum_{k=0}^{n}\frac{(-1)^k}{2^k}(\alpha)_k\,m^k\mathcal{W}_r^m(n+r,k+r)$.

Setting $x=\alpha$, we have $E_n^{(m)}(\alpha,\alpha)=\sum_{k=0}^{n}\frac{(-1)^k}{2^k}(\alpha)_k\,m^k\mathcal{W}_n^{k,m}(\alpha)$.

6.3. Generalized Frobenius-Euler polynomials

In the previous section, we proposed an explicit formula for the generalized Frobenius-Euler polynomials $H_n^{(m)}(x,\alpha|\lambda)$: what about the generalized Frobenius-Euler numbers $H_n^{(m)}(\alpha|\lambda)$? So let's setting $H_n^{(m)}(x,\alpha|\lambda)$ in Formula (13), we obtain an explicit formula for the generalized Frobenius-Euler polynomials $H_n^{(m)}(x,\alpha|\lambda)$ in terms of the generalized translated Whitney numbers.

An explicit formula for the generalized Frobenius-Euler polynomials $H_n^{(m)}(x,\alpha|\lambda)$ is given by the following theorem.

Theorem 6.3. *The following relationship holds true*

$$H_n^{(m)}(x,\alpha|\lambda) = \sum_{k=0}^{n} \frac{(\alpha)_k}{(\lambda-1)^k} m^k \mathcal{W}_n^{k,m}(x). \tag{18}$$

For $x = \alpha$, we have $H_n^{(m)}(\alpha,\alpha|\lambda) = \sum_{k=0}^{n} \frac{(\alpha)_k}{(\lambda-1)^k} m^k \mathcal{W}_n^{k,m}(\alpha)$.

For $m = 1$, we obtain $H_n^{(1)}(x,\alpha|\lambda) = \sum_{k=0}^{n} \frac{(\alpha)_k}{(\lambda-1)^k} S_n^k(x)$, which is done by Boutiche et *al*. [6, Eq. (7)].

Setting $x = 0$ and $\alpha = s$ (s being a positive integer) in (18), we have an explicit formula for the generalized Frobenius-Euler numbers $H_n^{(m)}(s|\lambda)$, $H_n^{(m)}(s|\lambda) = \sum_{k=0}^{n} \frac{(s)_k}{(\lambda-1)^k} m^k \mathcal{W}^m(n,k)$.

By substituting $\alpha = 1$, we have $H_n^{(m)}(x,1|\lambda) = \sum_{k=0}^{n} \frac{k!}{(\lambda-1)^k} m^k \mathcal{W}_n^{k,m}(x)$.

For $x = r$, we have $H_n^{(m)}(r,\alpha|\lambda) = \sum_{k=0}^{n} \frac{(\alpha)_k}{(\lambda-1)^k} m^k \mathcal{W}_r^m(n+r,k+r)$.

In particular, for $m = 1$, we obtain $H_n^{(1)}(r,\alpha|\lambda) = \sum_{k=0}^{n} \frac{(\alpha)_k}{(\lambda-1)^k} \mathcal{S}_r(n+r,k+r)$, which is in [16, Remark 3].

Setting $\lambda = -1$, we obtain the following explicit formula for the generalized Euler polynomials $E_n^{(m)}(x,\alpha)$, $E_n^{(m)}(x,\alpha) = \sum_{k=0}^{n} \frac{(-1)^k}{2^k} (\alpha)_k m^k \mathcal{W}_n^{k,m}(x)$.

In particular, for $m = 1$, we obtain the following explicit formula for the Euler polynomials $E_n^{(1)}(x,\alpha) = \sum_{k=0}^{n} \frac{(-1)^k}{2^k} (\alpha)_k S_n^k(x)$, which was given by Boutiche et *al*. [6, Eq. (3.3)].

Corollary 6.4. *The explicit formula for the generalized Frobenius-Genocchi polynomials of order l, $l \in \mathbb{N}$ is* $G_n^{(m)}(x,l|\lambda) = \frac{n!}{(n-l)!} H_{n-l}^{(m)}(x,l|\lambda) = \frac{n!}{(n-l)!} \sum_{k=0}^{n-l} \frac{(l)_k}{(\lambda-1)^k} m^k \mathcal{W}_{n-l}^{k,m}(x)$.

By setting $x = 0$, we obtain the following explicit formula for the generalized Frobenius-Genocchi numbers of order α, $G_n^{(m)}(l|\lambda) = \frac{n!}{(n-l)!} \sum_{k=0}^{n-l} \frac{(l)_k}{(\lambda-1)^k} m^k \mathcal{W}^m(n-l,k)$.

For $m = 1$, we obtain the explicit formula $G_n^{(1)}(l|\lambda) = \frac{n!}{(n-l)!} \sum_{k=0}^{n-l} \frac{(l)_k}{(\lambda-1)^k} S(n-l,k)$, see [16, Eq. (12)].

By setting $x = r$, we obtain $\quad G_n^{(m)}(r, l|\lambda) = \dfrac{n!}{(n-l)!} \displaystyle\sum_{k=0}^{n-l} \dfrac{(l)_k}{(\lambda-1)^k} m^k \mathcal{W}_r^m(n-l+r, k+r).$

In particular, for $m = 1$, we obtain the explicit formula $\quad G_n^{(1)}(r, l|\lambda) = \dfrac{n!}{(n-l)!} \displaystyle\sum_{k=0}^{n-l} \dfrac{(l)_k}{(\lambda-1)^k} \mathcal{S}_r(n-l+r, k+r).$

By setting $\lambda = -1$, we have an explicit formula for the generalized Genocchi polynomials $G_n^{(l,m)}(x)$ of order l, $\quad G_n^{(m)}(x, l) = \dfrac{n!}{(n-l)!} \displaystyle\sum_{k=0}^{n-l} \dfrac{(-1)^k}{2^k} (l)_k m^k \mathcal{W}_{n-l}^{k,m}(x).$

For $m = 1$, we get an explicit formula for the Genocchi polynomials $G_n^{(1)}(x, l)$ of order l, see [16, Eq. (12)].

For $x = 0$, we get an explicit formula for the generalized Genocchi numbers $G_n^{(m)}(l)$ of order l, $\quad G_n^{(m)}(l) = \dfrac{n!}{(n-l)!} \displaystyle\sum_{k=0}^{n-l} \dfrac{(-1)^k}{2^k} (l)_k m^k \mathcal{S}^m(n-l, k).$

7. Recurrence relations for the generalized Frobenius-Euler polynomials of higher order

In this section, we propose an algorithm based on a three-term recurrence relation for calculating the generalized Frobenius-Euler polynomials $H_n^{(m)}(x, \alpha|\lambda)$ of order α.

First, by setting $x = 0$, we obtain the following explicit formula for the generalized Frobenius-Euler numbers $H_n^{(m)}(\alpha|\lambda) = \displaystyle\sum_{k=0}^{n} \dfrac{(\alpha)_k}{(\lambda-1)^k} m^k \mathcal{W}^m(n, k).$

By means of the Stirling transform, we obtain $\dfrac{(\alpha)_n}{(\lambda-1)^n} m^n = \displaystyle\sum_{k=0}^{n} w^m(n, k) H_k^{(m)}(\alpha|\lambda)$.

Now, we introduce the sequence $(A_{n,l}^{\lambda,m}(\alpha))$ with two indices:

$$A_{n,l} = A_{n,l}^{\lambda,m}(\alpha) = \frac{(\lambda-1)^l}{m^l (\alpha)_l} \sum_{k=0}^{l} w^m(l, k) H_{n+k}^{(m)}(\alpha|\lambda), \; l, n \in \mathbb{N} \tag{19}$$

with $A_{0,l} = 1$, $A_{n,0} = H_n^{(m)}(\lambda, \alpha)$, where $w^m(n, k)$ are the translated Whitney numbers of the first kind, given by $(x|m)_n = \sum_{k=0}^{n} w^m(n, k) x^k$, and satisfying the recurrence relation (see for details [1]), $w^m(n, k) = w^m(n-1, k-1) + m(n-1) w^m(n-1, k)$.

Theorem 7.1. *The sequence $A_{n,l}^{\lambda,m}(\alpha)$ satisfies the following three term recurrence relation*

$$A_{n+1,l} = \frac{m(l+\alpha)}{\lambda-1} A_{n,l+1} - ml A_{n,l}, \quad n \geq 0, \text{ with } A_{0,l} = 1. \tag{20}$$

Proof. From (7) and (19), we have $\quad A_{n,l+1} = \frac{(\lambda-1)^{l+1}}{m^{l+1}(\alpha)_{l+1}} \sum_{k=0}^{l+1} (w^m(l, k-1) + ml w^m(l, k)) H_{n+k}^{(m)}(\lambda, \alpha) = \frac{(\lambda-1)^{l+1}}{m^{l+1}(\alpha)_{l+1}} \sum_{k=1}^{l+1} w^m(l, k-1) H_{n+k}^{(m)}(\lambda, \alpha) + \frac{(\lambda-1)^{l+1}}{m^{l+1}(\alpha)_{l+1}} ml \sum_{k=0}^{l+1} w^m(l, k) H_{n+k}^{(m)}(\lambda, \alpha) = \frac{(\lambda-1)^{l+1}}{m^{l+1}(\alpha)_{l+1}} \sum_{k=0}^{l} w^m(l, k) H_{n+1+k}^{(m)}(\lambda, \alpha) + \frac{(\lambda-1)^{m+1}}{m^{l+1}(\alpha)_{l+1}} ml \sum_{k=0}^{l} w^m(l, k) H_{n+k}^{(m)}(\lambda, \alpha) = \frac{\lambda-1}{m(\alpha+l)} A_{n+1,l} + l \frac{(\lambda-1)}{\alpha+l} A_{n,l}$, which completes the proof. $\square$

Finally, we consider the polynomials $A_{n,l}^{(\lambda,m)}(x, \alpha)$ defined by $A_{0,l}^{\lambda,m}(x, \alpha) = 1$, $A_{n,0}^{\lambda,m}(x, \alpha) = H_n^{(m)}(x, \alpha|\lambda)$ and

$$A_{n,l}(x) = A_{n,l}^{\lambda,m}(x, \alpha) = \sum_{k=0}^{n} \binom{n}{k} A_{k,l}^{\lambda,m}(\alpha) x^{n-k}. \tag{21}$$

Theorem 7.2. *The polynomials $A_{n,l}^{(\lambda,m)}(x,\alpha)$ satisfy the following three-term recurrence relation:*

$$A_{n+1,l}(x) = (x - ml)A_{n,l}(x) + \frac{m(\alpha + l)}{\lambda - 1}A_{n,l+1}(x),$$

with an initial sequence given by $A_{0,l}(x) = 1$.

Proof. From (7) and (21), we have $A_{n,l+1}(x) = \sum_{k=0}^{n}\binom{n}{k}A_{k,l+1}^{\lambda,m}(\alpha)x^{n-k} = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{\lambda-1}{m(\alpha+l)}A_{k+1,l}^{\lambda,m}(\alpha) + l\frac{(\lambda-1)}{\alpha+l}A_{k,l}^{\lambda,m}(\alpha)\right)x^{n-k} = \sum_{k=1}^{n+1}\binom{n}{k-1}\frac{\lambda-1}{m(\alpha+l)}A_{k,l}^{\lambda,m}(\alpha)x^{n-k+1} + l\frac{(\lambda-1)}{\alpha+l}\sum_{k=0}^{n}\binom{n}{k}A_{k,l}^{\lambda,m}(\alpha)x^{n-k}$.

Using Pascal rule, we get $A_{n,l+1}(x) = \sum_{k=1}^{n+1}\binom{n+1}{k}\frac{\lambda-1}{m(\alpha+l)}A_{k,l}^{\lambda,m}(\alpha)x^{n-k+1} - \sum_{k=1}^{n+1}\binom{n}{k}\frac{\lambda-1}{m(\alpha+l)}A_{k,l}^{\lambda,m}(\alpha)x^{n-k+1} + l\frac{(\lambda-1)}{\alpha+l}A_{n,l}(x) = \sum_{k=0}^{n+1}\binom{n+1}{k}\frac{\lambda-1}{m(\alpha+l)}A_{k,l}^{\lambda,m}(\alpha)x^{n-k+1} - \sum_{k=0}^{n}\binom{n}{k}\frac{\lambda-1}{m(\alpha+l)}A_{k,l}^{\lambda,m}(\alpha)x^{n-k+1} + l\frac{(\lambda-1)}{\alpha+l}A_{n,l}(x) = \frac{\lambda-1}{m(\alpha+l)}A_{n+1,l}(x) - x\frac{\lambda-1}{m(\alpha+l)}A_{n,l}(x) + l\frac{(\lambda-1)}{\alpha+l}A_{n,l}(x) = \frac{\lambda-1}{m(\alpha+l)}\left(A_{n+1,l}(x) + (ml - x)A_{n,l}(x)\right)$. Then $A_{n+1,l}(x) = (x - ml)A_{n,l}(x) + \dfrac{m(\alpha + l)}{\lambda - 1}A_{n,l+1}(x)$. □

For $\alpha = x$, we have $A_{n+1,l}(x) = (x - ml)A_{n,l}(x) + \dfrac{m(x + l)}{\lambda - 1}A_{n,l+1}(x)$.

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References

[1] H. Belbachir and I. E. Bousbaa, Translated Whitney and *r*-Whitney Numbers: A Combinatorial Approach, J. Inte. Seq. 16 (2013).
[2] H. Belbachir, S. Hadj Brahim, M. Rachidi, *On another approach for a family of Appell polynomials*, Filomat,Vol 32, No 12 (2018).
[3] M. Benoumhani, On Whitney numbers of Dowling lattices, Disc. Math. 159 (1996), 13-33.
[4] M. Benoumhani, On some numbers Related to Whitney numbers of Dowling lattices, Adv. in Appl. Math.19 (1997), 106-116.
[5] A. Z. Broder, The *r*-Stirling numbers, Disc. Math. 49 (1984), 241-259.
[6] M. A. Boutiche, M. Rahmani and H. M. Srivastava, Explicit Formulas Associated with Some Families of Generalized Bernoulli and Euler Polynomials, Mediterr. J. Math. (2017)
[7] L. Carlitz, Eulerian numbers and polynomials of higher order, Duke Math. J. 27 (1960), 401–423.
[8] L. Carlitz, Eulerian numbers and polynomials, Math. Mag. 32 (1958/1959), 247–260.
[9] G. S. Cheon, J. H. Jung, *r*-Whitney numbers of Dowling Lattices. Disc. Math. 312 (2012), 2337-2348.
[10] T. A. Dowling, A class of geometric lattices based on finite groups, J. Combin. Theory, Ser B 14 (1973), 61-86.
[11] Y. He, S. Araci, H. M. Srivastava and M. AçikgÖz, Some new identities for the Apostol-Bernoulli polynomials and the Apostol-Genocchi polynomials, Appl. Math. Comput. 262 (2015), 31–41.
[12] D. S. Kim and T. Kim, Higher-order Frobenius-Euler and poly-Bernoulli mixed-type polynomials, Adv. Difference Equ. 2013 (2013), Article ID 251, 1–13.
[13] B. Kurt and Y. Simsek, On the generalized Apostol-type Frobenius-Euler polynomials, Adv. Difference Equ. 2013 (2013), Article ID 1, 1–9.
[14] I. Mezö, A new Formula for the Bernoulli polynomials, Results. Math. 58 (2010), 329-335.
[15] H. M. Srivastava, Generating relations and other results associated with some families of the extended Hurwitz-Lerch Zeta functions, SpringerPlus 2 (2013), Article ID 2: 67, 1–14.
[16] H. M. Srivastava, M. A. Boutiche and M. Rahmani, Some explicit formulas for the Frobenius- Euler polynomials of higher order, Appl. Math. Inf. Sci. 11, No. 2, (2017) 621-626.
[17] H. M. Srivastava, M. A. Boutiche and M. Rahmani, A class of Frobenius-type Eulerian polynomials. Rocky Mountain Journal of Mathematics 48.3 (2018): 1003-1013.
[18] H. M., Srivastava, P. G., Todorov, An explicit formula for the generalized Bernoulli polynomials. J. Math. Anal. Appl 130(2),(1988) 509-513.
[19] B. Yılmaz Yaşar and M. A. Özarslan, Frobenius-Euler and Frobenius-Genocchi polynomials and their differential equations, New Trends Math. Sci. 3 (2015), 172–180.