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Finite Hurwitz-Lerch Functions

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Abstract. We define the finite Hurwitz-Lerch function $H_n^{(s)}(x,y) = \sum_{k=0}^{n-1} \frac{x^k}{(k+y)^s}$. Then we explore some properties of them and give some applications, including Coppo's formula, the sum of powers of consecutive integers, Mellin differential operator, and some others.

1. Introduction

For $n \in \mathbb{N}$, $x, y, s \in \mathbb{C}$ with $y \neq 0, -1, -2, \cdots, -(n-1)$ (when $\Re(s) < 0$, $y = 0$ is allowed), we define the finite Hurwitz-Lerch function as follows:

$$H_n^{(s)}(x,y) = \sum_{k=0}^{n-1} \frac{x^k}{(k+y)^s}.$$

It is strongly related to the Hurwitz-Lerch zeta function $\Phi(x,s,y)$.

$$\Phi(x,s,y) = \sum_{k=0}^{\infty} \frac{x^k}{(k+y)^s} = \lim_{n\to\infty} H_n^{(s)}(x,y).$$

And the elementary translation property of $\Phi(z,s,x)$ for a fixed positive integer m is

$$\Phi(z,s,x) = z^m \Phi(z,s,m+x) + H_m^{(s)}(z,x).$$

The Hurwitz-Lerch zeta function $\Phi(x,s,y)$ is a generalization of the Hurwitz zeta function $\zeta(s;x) = \Phi(1,s,x)$, polylogarithm function $Li_s(z) = z\Phi(z,s,1)$, and the Lipschitz-Lerch zeta function $\phi(\xi,s,x) = \Phi(e^{2\pi i\xi},s,x)$ (see, for details, [20, 22, 23, 29–33]). On the other hand, $H_n^{(s)}(1,z+1) = H_n^{(s)}(z)$ which is called the generalized harmonic function, and

$$H_n^{(s)}(1,1) = H_n^{(s)}(0) = H_n^{(s)} = \sum_{k=1}^{n} \frac{1}{k^s}.$$

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The generalized harmonic functions appear often in mathematics and physics. For example, they appear in the computation of Feynman integrals in massive higher order perturbative calculations in renormalizable quantum field theories (see [1, 5, 21]) and in the computation of multiple zeta values or Euler sums (see [6, 15, 26–28, 34]).

For $x, y, s \in \mathbb{C}$ with $y \neq 0, -1, -2, \cdots, -n$ (when $\Re(s) < 0$, $y = 0$ is allowed), let us define

$$S_n^{(s)}(x,y) = \sum_{k=0}^{n} \binom{n}{k} \frac{x^k}{(k+y)^s}. \tag{1}$$

The sequence b_n is the binomial transform of the sequence a_n if $b_n = \sum_{k=0}^{n} \binom{n}{k} a_k$. It satisfies the binomial inversion formula $a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} b_k$. Moreover, if $b_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k$, we call b_n are the dual sequence of a_n. By Eq. (1) we know that $S_n^{(s)}(x,y)$ is the binomial transform of $x^n/(n+y)^s$.

In fact, $H_n^{(s)}(x,y)$ is the dual sequence of $-S_{n-1}^{(s)}(-x,y)$. Moreover they have the following relation:

$$\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(k+x)^m} = \frac{n!}{x(x+1)\cdots(x+n)} P_{m-1}(H_{n+1}^{(1)}(1,x), \ldots, H_{n+1}^{(m-1)}(1,x)), \tag{2}$$

where $P_m(x_1, \ldots, x_m)$ is the modified Bell polynomial which is defined by [7, 14]

$$\exp\left(\sum_{k=1}^{\infty} \frac{x_k}{k} z^k \right) = \sum_{m=0}^{\infty} P_m(x_1, \ldots, x_m) z^m.$$

The above equation was first derived by Coppo in 2003 (ref. [11, Page 17]). In 2006, Coffey [10] reported this formula again. Connon [11] reviewed it in 2008 and called this formula "Coppo's Formula".

In 2008, Chu and Yan [9] proved the following formula:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{x+k}{k}^{-1} \sum_{0 \le j_1 \le j_2 \le \cdots \le j_\ell \le k} \prod_{r=1}^{\ell} \frac{1}{x+j_r} = \frac{x}{(x+n)^{\ell+1}}. \tag{3}$$

Diaz-Barrero [16] proposed the special case $(\ell, x) = (2, 0)$ of this formula as a monthly problem in 2005. And then Diaz-Barrero et al [17] in 2007 proved for the special cases $\ell = 3, 4$ of this formula. In the next year Chu and Yan [9] gave a proof for the general case of this formula. In 2012, Chu [8, Theorem 3.1] gave another proof. However, this formula is the binomial inversion formula of Eq. (2). We will explain them in Section 3.

Another interesing thing is that the number $H_{n+1}^{(-m)}(1,0) = \sum_{j=1}^{n} j^m$ is the sum of powers of consecutive integers. The classical sums of powers of consecutive integers formula is usually called Bernoulli's formula:

$$1^m + 2^m + 3^m + \cdots + (n-1)^m = \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_k n^{m+1-k} \tag{4}$$

for $m \ge 0$, $n \ge 1$, where B_n is the Bernoulli number defined by $t/(e^t - 1) = \sum_{n=0}^{\infty} B_n t^n / n!$. d'Ocagne [25] in 1887 gave another formula using Stirling numbers of the second kind:

$$1^m + 2^m + \cdots + n^m = \sum_{k=1}^{n} k! \binom{n+1}{k+1} S(m,k),$$

where $S(m,k)$ is the Stirling number of the second kind defined by

$$S(n,k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n.$$

In 2012, Boyadzhiev stated this identity again in [3]. One of the generalizations of the Stirling numbers of the second kind is defined by [4, 25]:

$$S_c(\alpha, n) = \frac{1}{n!}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(c+k)^{\alpha}. \tag{5}$$

It can be seen that $S_c(\alpha, n) = \frac{(-1)^n}{n!}S_n^{(-\alpha)}(-1, c)$. Therefore we give a more general formula of sums of powers of consecutive integers:

$$\sum_{j=0}^{k}(c+j)^{\alpha} = \sum_{j=0}^{k}(-1)^j\binom{k+1}{j+1}S_j^{(-\alpha)}(-1, c).$$

The special case $c = 0$ leads to d'Ocagne's result. We explore this formula in Section 4 and also give more applications in Mellin differential operator $\Theta_c = (z \cdot d/dz + c)$, and the Riemann zeta values, etc.

2. Basic properties of $H_n^{(s)}(x, y)$ and $S_n^{(s)}(x, y)$

We list some well-known basic properties of $\Phi(z, s, x)$ and $S_n^{(s)}(x, y)$: If either $|z| < 1$ and $\Re(s) > 0$ or $z = 1$ and $\Re(s) > 1$ the integral representation of $\Phi(z, s, x)$ is (see [19, 30])

$$\Phi(z, s, x) = \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-(x-1)t}}{e^t - z}\,dt. \tag{6}$$

Connon found the integral representation of $S_n^{(s)}(x, y)$ as follows [12, Eq. (5)].

$$S_n^{(s)}(x, y) = \frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}e^{-yt}(1 + xe^{-t})^n\,dt$$

and the ordinary generating function (ogf) [12, Eq. (13)].

$$\sum_{n=0}^{\infty} z^n S_n^{(s)}(x, y) = \frac{1}{1-z}\Phi\left(\frac{zx}{1-z}, s, y\right) \quad |z| < 1. \tag{7}$$

And Connon also got the following identity [12, Eq. (11)]:

$$\sum_{n=0}^{\infty}\frac{S_n^{(s)}(-x, y)}{n+1} = s\Phi(x, s+1, y) - \log x \cdot \Phi(x, s, y).$$

First, we establish the binomial transform relationship between $H_n^{(s)}(x, y)$ and $S_n^{(s)}(x, y)$.

Proposition 2.1. *For $n \in \mathbb{N}$, we have*

$$H_n^{(s)}(x, y) = \sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}S_{k-1}^{(s)}(-x, y), \quad \text{and} \tag{8}$$

$$S_{n-1}^{(s)}(x, y) = \sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}H_k^{(s)}(-x, y). \tag{9}$$

Proof. Since $S_n^{(s)}(x, y)$ is the dual sequence of $(-x)^n/(n+y)^s$, we have

$$H_n^{(s)}(x, y) = \sum_{k=0}^{n-1}\sum_{r=0}^{k}\binom{k}{r}(-1)^r S_r^{(s)}(-x, y) = \sum_{r=0}^{n-1}(-1)^r S_r^{(s)}(-x, y)\sum_{k=r}^{n-1}\binom{k}{r}.$$

By using the well-known identity $\sum_{k=r}^{n-1}\binom{k}{r}=\binom{n}{r+1}$, we get

$$H_n^{(s)}(x,y)=\sum_{r=0}^{n-1}\binom{n}{r+1}(-1)^r S_r^{(s)}(-x,y).$$

Eq. (9) is just the binomial inversion formula of Eq. (8). □

The following identities are another forms of the above identities:

Corollary 2.2. *For $n\in\mathbb{N}$, we have*

$$\frac{S_n^{(s)}(-x,y)}{n+1}=\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k H_{k+1}^{(s)}(x,y)}{k+1},\qquad \frac{H_{n+1}^{(s)}(x,y)}{n+1}=\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k S_k^{(s)}(-x,y)}{k+1}.$$

Using the basic integral techniques, we get the integral representation of $H_n^{(s)}(x,y)$ as follows.

Proposition 2.3. *The integral representation of $H_n^{(s)}(x,y)$ is*

$$H_n^{(s)}(x,y)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-yt}\frac{1-x^ne^{-nt}}{1-xe^{-t}}\,dt. \tag{10}$$

We give some sum formulas for $H_n^{(s)}(x,y)$ and $S_n^{(s)}(x,y)$ in the sequel.

Proposition 2.4. *Let $x,y,z,s\in\mathbb{C}$ with $z\neq 1$, $n\in\mathbb{N}$, and $y\neq 0,-1,-2,-3,\cdots,-n$ (when $\Re(s)<0$, $y=0$ is allowed), then we have*

$$\sum_{k=1}^{n}H_k^{(s)}(x,y)z^k=\frac{z}{1-z}\left[H_n^{(s)}(xz,y)-z^nH_n^{(s)}(x,y)\right]. \tag{11}$$

Proof.

$$\begin{aligned}\sum_{k=1}^{n}H_k^{(s)}(x,y)z^k&=\sum_{k=2}^{n}\left[H_{k-1}^{(s)}(x,y)+\frac{x^{k-1}}{(k-1+y)^s}\right]z^k+\frac{z}{y^s}\\&=\sum_{k=1}^{n}H_k^{(s)}(x,y)z^{k+1}-H_n^{(s)}(x,y)z^{n+1}+\sum_{k=0}^{n-1}\frac{x^kz^{k+1}}{(k+y)^s}\\&=z\sum_{k=1}^{n}H_k^{(s)}(x,y)z^k-H_n^{(s)}(x,y)z^{n+1}+z\sum_{k=0}^{n-1}\frac{(xz)^k}{(k+y)^s}.\end{aligned}$$

□

For $|z|<1$, we let $n\to\infty$ in Eq. (11), we get the ordinary generating function of $H_n^{(s)}(x,y)$.

Proposition 2.5. *For $x,y,z,s\in\mathbb{C}$ with $|z|<1$ and $y\neq 0,-1,-2,-3,\cdots$ (when $\Re(s)<0$ $y=0$ is allowed), the ordinary generating function of $H_n^{(s)}(x,y)$ is*

$$\sum_{n=1}^{\infty}H_n^{(s)}(x,y)z^n=\frac{z}{1-z}\Phi(zx,s,y).$$

Let $z\to 1$ in Eq. (11) and apply L'Hospital's rule, then we get the following identitiy.

Proposition 2.6. *For any $n \in \mathbb{N}$, we have*

$$\sum_{k=1}^{n} H_k^{(s)}(x,y) = (y+n)H_n^{(s)}(x,y) - H_n^{(s-1)}(x,y).$$

Using $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$, we obtain

Lemma 2.7. *For a positive integer n, we have*

$$S_n^{(s)}(x,y) = S_{n-1}^{(s)}(x,y) + xS_{n-1}^{(s)}(x,y+1).$$

Proposition 2.8. *For a positive integer n, we have*

$$(z-1)\sum_{k=0}^{n} S_k^{(s)}(x,y)z^k + xz\sum_{k=0}^{n-1} S_k^{(s)}(x,y+1)z^k = S_n^{(s)}(x,y)z^{n+1} - S_0^{(s)}(x,y).$$

Proof. Using the above lemma we have

$$\begin{aligned}
\sum_{k=0}^{n} S_k^{(s)}(x,y)z^k &= S_0^{(s)}(x,y) + \sum_{k=1}^{n}\left[S_{k-1}^{(s)}(x,y) + xS_{k-1}^{(s)}(x,y+1)\right]z^k \\
&= S_0^{(s)}(x,y) + \sum_{k=0}^{n-1} S_k^{(s)}(x,y)z^{k+1} + x\sum_{k=0}^{n-1} S_k^{(s)}(x,y+1)z^{k+1}.
\end{aligned}$$

Then we conclude this proposition. □

Let $z = 1$ in the above proposition, we have the following formula.

Corollary 2.9. *For a positive integer n, we have*

$$x\sum_{k=0}^{n-1} S_k^{(s)}(x,y+1) = S_n^{(s)}(x,y) - S_0^{(s)}(x,y).$$

3. Coppo's formula

Here we list some preliminaries about symmetric functions. For more details, we refer the reader to [24]. Let h_m and p_m be the m-th complete homogeneous, and power-sum symmetric polynomials, in infinitely many variables $x_1, x_2, \ldots$, respectively. They have associated generating functions

$$\begin{aligned}
H(t) &:= \sum_{j=0}^{\infty} h_j t^j = \prod_{i=1}^{\infty} \frac{1}{1-tx_i}, \\
P(t) &:= \sum_{j=1}^{\infty} p_j t^{j-1} = \sum_{i=1}^{\infty} \frac{x_i}{1-tx_i} = \frac{H'(t)}{H(t)}.
\end{aligned}$$

Let the modified Bell polynomials $P_m(x_1, x_2, \ldots, x_m)$ be defined by [7, 14]

$$\exp\left(\sum_{k=1}^{\infty} \frac{x_k}{k} z^k\right) = \sum_{m=0}^{\infty} P_m(x_1, x_2, \ldots, x_m) z^m.$$

The general explicit expression for P_m is

$$P_m(x_1, \ldots, x_m) = \sum_{k_1+2k_2+\cdots+mk_m=m} \frac{1}{k_1! \cdots k_m!}\left(\frac{x_1}{1}\right)^{k_1} \cdots \left(\frac{x_m}{m}\right)^{k_m}.$$

It is well-known that for j being a nonnegative integer,

$$h_j = P_j(p_1, p_2, \ldots, p_j). \tag{12}$$

Let $x_k = (k + x)^{-1}$, for all $1 \le k \le n$ and $x_k = 0$ for $k > n$. Then

$$h_m = \sum_{1 \le k_1 \le \cdots \le k_m \le n} \frac{1}{(k_1 + x)(k_2 + x) \cdots (k_m + x)}, \quad \text{and} \quad p_m = H_n^{(m)}(1, x + 1),$$

where we write $h_0 = 1$. Now Eq. (12) gives us the evaluations of h_m:

$$\sum_{1 \le k_1 \le \cdots \le k_m \le n} \frac{1}{(k_1 + x)(k_2 + x) \cdots (k_m + x)} = P_m(H_n^{(1)}(1, x + 1), \ldots, H_n^{(m)}(1, x + 1)). \tag{13}$$

The right-hand side of Eq. (13) has a integral representation as stated in [7, 13]. Here we state it as a lemma.

Lemma 3.1. *For $n \in \mathbb{N}$, $m \in \mathbb{N}_0$, $x \in \mathbb{R}$ with $0 < 1 + x$, we have*

$$P_m\left(H_n^{(1)}(1, x + 1), \ldots, H_n^{(m)}(1, x + 1)\right) = \frac{1}{B(n, 1 + x)} \int_0^\infty e^{-(1+x)y}(1 - e^{-y})^{n-1} \frac{y^m}{m!} \, dy,$$

where $B(x, y)$ is the Euler beta function and $P_m(x_1, \ldots, x_m)$ is the modified Bell polynomial.

Proposition 3.2 (Coppo's formula). *For $m \in \mathbb{N}_0$, $S_n^{(m)}(-1, x)$ can be represented as*

$$\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(x + k)^m} = S_n^{(m)}(-1, x) = B(n + 1, x) P_{m-1}(H_{n+1}^{(1)}(1, x), \ldots, H_{n+1}^{(m-1)}(1, x)). \tag{14}$$

Proof.

$$\begin{aligned} S_n^{(m)}(-1, x) &= \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(k + x)^m} = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(m - 1)!} \int_0^\infty e^{-(k+x)t} t^{m-1} \, dt \\ &= \int_0^\infty \frac{t^{m-1} e^{-xt}}{(m - 1)!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} e^{-kt} \, dt = \int_0^\infty e^{-xt}(1 - e^{-t})^n \frac{t^{m-1}}{(m - 1)!} \, dt. \end{aligned}$$

Using the above lemma, we complete the proof. □

From Eq. (14) we know that $B(n + 1, z) P_{m-1}(H_{n+1}^{(1)}(1, x, \cdots, H_{n+1}^{(-1)}(1, x)))$ is the dual sequence of $1/(x + n^m)$. Hence, the dual form of Eq. (14) is

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k B(k + 1, x) P_{m-1}(H_{k+1}^{(1)}, \cdots, H_{k+1}^{(m-1)}(1, x)) = \frac{1}{(x + n)^m}. \tag{15}$$

Since

$$B(k + 1, x) = \frac{\Gamma(k + 1)\Gamma(x)}{\Gamma(k + 1 + x)} = x^{-1} \binom{x + k}{k}^{-1}$$

and by Eq. (13) we know that Eq. (15) is just Eq. (3) with $\ell + 1 = m$.

4. Sums of powers and other applications

The Mellin differential operator Θ_c is defined by $\Theta_c = (zD + c)$, where $D = d/dz$ and $c \in \mathbb{R}$. The Mellin differential operator of order $r \in \mathbb{N}$ is defined iteratively by

$$\Theta_c^1 = \Theta_c, \quad \Theta_c^r = \Theta(\Theta_c^{r-1}).$$

For convenience we set $\Theta_c^0 = I$ as the identity map.

Proposition 4.1. *[4, Lemma 9] Let $A(z) = \sum_{n=0}^{\infty} a_n z^n$. Then the Mellin derivative of order $r \in \mathbb{N}$ has the representation*

$$\sum_{n=0}^{\infty}(n+c)^r a_n z^n = \Theta_c^r A(z) = \sum_{k=0}^{\infty} S_c(r,k) z^k A^{(k)}(z). \tag{16}$$

The basic examples are

$$\Phi(z, s-1, x) = \Theta_x \Phi(z, s, x),$$

and Eq. (7.46) of [18]

$$\sum_{n=0}^{\infty} n^r z^n = \Theta_0^r\left(\frac{1}{1-z}\right) = \sum_{k=0}^{\infty} S(r,k)\frac{k! z^k}{(1-z)^{k+1}}.$$

Since

$$S_n^{(m)}(-1,x) = \sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{(k+x)^m} = n!(-1)^n S_x(-m,n), \tag{17}$$

we expand the positive power r of Eq. (16) to the negative power as follows.

Proposition 4.2. *For $c > 0$, we have*

$$\sum_{n=0}^{\infty}\frac{a_n z^n}{(n+c)^m} = \sum_{k=0}^{\infty} S_c(-m,k) z^k A^{(k)}(z).$$

Proof. Since $A^{(k)}(z) = \sum_{n=k}^{\infty}\frac{n!}{(n-k)!}a_n z^{n-k}$, we have

$$\begin{aligned}\sum_{k=0}^{\infty} S_c(-m,k) z^k A^{(k)}(z) &= \sum_{k=0}^{\infty} S_c(-m,k)\sum_{n=k}^{\infty}\frac{n!}{(n-k)!}a_n z^n \\ &= \sum_{n=0}^{\infty} a_n z^n \sum_{k=0}^{n}\binom{n}{k} k! S_c(-m,k).\end{aligned}$$

The desired result is followed by the fact that the sequence $n!(-1)^n S_c(-m,n)$ is the dual sequence of $1/(n+c)^m$ (See Eq. (5)). Thus we complete the proof. □

As final applications we apply our results to obtain the expressions of the Dirichlet beta values $\beta(m)$, the Dirichlet eta values $\eta(m)$, and the Riemann zeta values $\zeta(m+1)$.

First we note that $H_n^{(s)}(1, 1/2) = H_n^{(s)}(-1/2) = \sum_{k=1}^{n} 2^s/(2k-1)^s = 2^s O_n^{(s)}$. Therefore we calculate the values of $\beta(m)$ as follows [14, Eq. (10)]. We apply Eq. (7) and Eq. (14),

$$\begin{aligned}\beta(m) &= \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^m} = 2^{-m}\Phi(-1, m, 1/2) = \sum_{n=0}^{\infty}\frac{S_n^{(m)}(-1, 1/2)}{2^{m+n+1}} \\ &= \sum_{n=0}^{\infty}\frac{B(n+1, 1/2)P_{m-1}(H_{n+1}^{(1)}(1, 1/2), \dots, H_{n+1}^{(m-1)}(1, 1/2))}{2^{n+m+1}} \\ &= \sum_{n=1}^{\infty}\frac{2^{n-1}}{n\binom{2n}{n}}P_{m-1}(O_n^{(1)}, \dots, O_n^{(m-1)}).\end{aligned}$$

The last identity is obtained by the fact that $B(n+1,1/2) = \frac{2^{2n+2}}{(n+1)\binom{2n+2}{n+1}}$. Next we give the evaluation of the Dirichlet eta values $\eta(m)$ [2, Eq. (4.2)].

$$\begin{aligned}\eta(m) &= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^m} = \Phi(-1,m,1) = \sum_{n=0}^{\infty}\frac{S_n^{(m)}(-1,1)}{2^{n+1}}\\ &= \sum_{n=0}^{\infty}\frac{B(n+1,1)P_{m-1}(H_{n+1}^{(1)},\dots,H_{n+1}^{(m-1)})}{2^{n+1}} = \sum_{n=1}^{\infty}\frac{P_{m-1}(H_n^{(1)},\dots,H_n^{(m-1)})}{n\cdot 2^n}.\end{aligned}$$

Since $\eta(m) = (1-\frac{1}{2^{m-1}})\zeta(m)$, for $m \geq 1$ we have

$$\zeta(m+1) = \frac{2^m}{2^m-1}\sum_{n=1}^{\infty}\frac{P_m(H_n^{(1)},\dots,H_n^{(m)})}{n\cdot 2^n}.$$

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