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On a Topology Between $\mathcal{T}_\alpha$ and $\mathcal{T}_{\gamma\alpha}$

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Abstract. Using the topology $\mathcal{T}_\gamma$ in a topological space $(X,\mathcal{T})$, a new class of generalized open sets called Γ-preopen sets, is introduced and studied. This class generates a new topology $\mathcal{T}_g$ which is larger than $\mathcal{T}_\alpha$ and smaller than $\mathcal{T}_{\gamma\alpha}$. By means of the corresponding interior and closure operators, among other results, necessary and sufficient conditions are given for $\mathcal{T}_g$ to coincide with $\mathcal{T}_\alpha$, $\mathcal{T}_\gamma$ or $\mathcal{T}_{\gamma\alpha}$.

1. Introduction

In the past few decades there has been a considerable interest in the study of generalized open sets in topological spaces. Four of these concepts were simply defined using the closure operator "cl" and the interior operator"int". We denote a topological space by $(X,\mathcal{T})$ or simply by X when there is no possibility of confusion. The class of closed sets in $(X,\mathcal{T})$ will be denoted by $\mathrm{C}(\mathcal{T})$.

Definition 1. A subset A of a space X is called
(1) an *α-set* if $A \subset \mathrm{int}(\mathrm{cl}(\mathrm{int} A))$,
(2) *semi-open* if $A \subset \mathrm{cl}(\mathrm{int} A)$,
(3) *preopen* if $A \subset \mathrm{int}(\mathrm{cl} A)$,
(4) *semi-preopen* if $A \subset \mathrm{cl}(\mathrm{int}(\mathrm{cl} A))$.

The first three notions were defined by Njåstad [12], Levine [10] and Mashhour et al. [11]. The concept of preopen sets was introduced by Corson and Michael [8] who used the term "locally dense sets". The fourth concept was introduced by Abd El-Monsef et al. [1] under the name "β-open", and in [3] these sets were called semi-preopen sets. We denote the classes of these sets in a space $(X,\mathcal{T})$ by $\mathcal{T}_\alpha$, $\mathrm{SO}(\mathcal{T})$, $\mathrm{PO}(\mathcal{T})$ and $\mathrm{SPO}(\mathcal{T})$ respectively. All of them are larger than $\mathcal{T}$ and closed under forming arbitrary unions. It was shown in [12] that $\mathcal{T}_\alpha$ is a topology on X. The closure and the interior of a set A in $(X,\mathcal{T}_\alpha)$ will be denoted by $\mathrm{cl}_\alpha A$ and $\mathrm{int}_\alpha A$. In general, $\mathrm{SO}(\mathcal{T})$ need not be a topology on X, but the intersection of a semi-open set and an open set is semi-open. The same holds for $\mathrm{PO}(\mathcal{T})$ and $\mathrm{SPO}(\mathcal{T})$. The complement of a semi-open set is called *semi-closed*. Thus A is semi-closed if and only if $\mathrm{int}(\mathrm{cl} A) \subset A$. *Preclosed* and *semi-preclosed* sets are similarly defined.We denote these classes by $\mathrm{SC}(\mathcal{T})$, $\mathrm{PC}(\mathcal{T})$ and $\mathrm{SPC}(\mathcal{T})$ respectively. For a subset A of a space X the *semi-closure* (resp. *preclosure*, *semi-preclosure*) of A, denoted by $\mathrm{scl} A$ (resp. $\mathrm{pcl} A$, $\mathrm{spcl} A$), is the intersection of all semi-closed (resp. preclosed, semi-preclosed) subsets of X containing A.

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The *semi-interior* (resp. *preinterior, semi-preinterior*) of A, denoted by $\mathrm{sint}\,A$ (resp. $\mathrm{pint}\,A$, $\mathrm{spint}\,A$), is the union of all semi-open (resp. preopen, semi-preopen) subsets of X contained in A. Finally, the classes of regular open sets, dense sets and nowhere dense sets in $(X,\mathcal{T})$ will be denoted by $\mathrm{RO}(\mathcal{T})$, $\mathrm{D}(\mathcal{T})$ and $\mathrm{N}(\mathcal{T})$ respectively.

Let $\mathcal{A}$ be a class of sets in $(X,\mathcal{T})$ which is larger than $\mathcal{T}$ and closed under forming arbitrary unions. Then $\mathcal{T}(\mathcal{A}) = \{G \in \mathcal{A} \,|\, G \cap A \in \mathcal{A}$ whenever $A \in \mathcal{A}\}$ is a topology on X such that that $\mathcal{T} \subset \mathcal{T}(\mathcal{A}) \subset \mathcal{A}$. It was shown in [12] that $\mathcal{T}(\mathcal{A}) = \mathcal{T}_\alpha$ for $\mathcal{A} = \mathrm{SO}(\mathcal{T})$. The topology generated by $\mathrm{PO}(\mathcal{T})$ was studied in [4] and denoted by $\mathcal{T}_\gamma$. It was proved in [9] that $\mathcal{T}(\mathcal{A}) = \mathcal{T}_\gamma$ for $\mathcal{A} = \mathrm{SPO}(\mathcal{T})$. The closure and the interior of a set A in $(X,\mathcal{T}_\gamma)$ are denoted by $\mathrm{cl}_\gamma A$ and $\mathrm{int}_\gamma A$.

Now we are going to present our main results. In Section 2 we introduce a new class of generalized open sets by the condition $A \subset \mathrm{int}\,(\mathrm{cl}_\gamma A)$. This class of Γ-preopen sets, denoted by $\Gamma\mathrm{PO}(\mathcal{T})$, generates a new topology $\mathcal{T}_g = \mathcal{T}\,(\Gamma\mathrm{PO}(\mathcal{T}))$ which we study in Section 3. This is a topology between $\mathcal{T}_\alpha$ and $\mathcal{T}_{\gamma\alpha}$ and among other results we show that $\mathcal{T}_g = \Gamma\mathrm{PO}(\mathcal{T}) \cap \mathrm{SO}(\mathcal{T}_\gamma)$, $\mathcal{T}_{\alpha g} = \mathcal{T}_{g\alpha} = \mathcal{T}_g$, $\mathcal{T}_{g\gamma} = \mathcal{T}_{\gamma g} = \mathcal{T}_{\gamma\alpha}$ and $\mathcal{T}_{gg} = \mathcal{T}_g$. In Section 4 we study the topologies generated by the other classes of generalized open sets which are introduced by using various combinations of the closure and interior operators in $\mathcal{T}$ and $\mathcal{T}_\gamma$. Besides Γ-preopen sets, seven new classes are obtained and we show that three of them generate the same topology $\mathcal{T}_g$ as does the class $\Gamma\mathrm{PO}(\mathcal{T})$. As for the remaining classes, it turns out that they generate the topology $\mathcal{T}_{\gamma\alpha}$.

Now we recollect some results which will be needed in the sequel.

Proposition 1.1. ([2, 3]) *Let A be a subset of a space X. Then:*
(1) $\mathrm{cl}_\alpha A = A \cup \mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}\,A))$, $\mathrm{int}_\alpha A = A \cap \mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}\,A))$,
(2) $\mathrm{scl}\,A = A \cup \mathrm{int}\,(\mathrm{cl}\,A)$, $\mathrm{sint}\,A = A \cap \mathrm{cl}\,(\mathrm{int}\,A)$,
(3) $\mathrm{pcl}\,A = A \cup \mathrm{cl}\,(\mathrm{int}\,A)$, $\mathrm{pint}\,A = A \cap \mathrm{int}\,(\mathrm{cl}\,A)$,
(4) $\mathrm{spcl}\,A = A \cup \mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}\,A))$, $\mathrm{spint}\,A = A \cap \mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}\,A))$.

Proposition 1.2. ([3]) *Let A be a subset of a space X. Then:*
(1) $\mathrm{pint}\,(\mathrm{cl}\,A) = \mathrm{int}\,(\mathrm{cl}\,A) = \mathrm{int}\,(\mathrm{scl}\,A)$,
(2) $\mathrm{pcl}\,(\mathrm{int}\,A) = \mathrm{cl}\,(\mathrm{int}\,A) = \mathrm{cl}\,(\mathrm{sint}\,A)$,
(3) $\mathrm{int}\,(\mathrm{pcl}\,A) = \mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}\,A)) = \mathrm{scl}\,(\mathrm{int}\,A)$,
(4) $\mathrm{cl}\,(\mathrm{pint}\,A) = \mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}\,A)) = \mathrm{sint}\,(\mathrm{cl}\,A)$.

Proposition 1.3. ([2]) *Let A be a subset of a space X. Then:*
(1) $\mathrm{int}\,(\mathrm{cl}_\alpha A) = \mathrm{int}_\alpha \mathrm{cl}\,A = \mathrm{int}_\alpha \mathrm{cl}_\alpha A = \mathrm{int}\,(\mathrm{cl}\,A)$,
(2) $\mathrm{cl}_\alpha \mathrm{int}\,A = \mathrm{cl}\,(\mathrm{int}_\alpha A) = \mathrm{cl}_\alpha \mathrm{int}_\alpha A = \mathrm{cl}\,(\mathrm{int}\,A)$.

Proposition 1.4. *Let $(X,\mathcal{T})$ be a space. Then:*
(1) $\mathcal{T}_\alpha = \{U \backslash A \mid U \in \mathcal{T}, A \in \mathrm{N}(\mathcal{T})\}$ ([12]),
(2) $\mathcal{T}_\alpha = \mathrm{SO}(\mathcal{T}) \cap \mathrm{PO}(\mathcal{T})$ ([13]),
(3) $\mathcal{T}_{\alpha\alpha} = \mathcal{T}_\alpha$ ([12]).

Proposition 1.5. ([12]) *Let $\mathcal{T}$ and $\mathcal{U}$ be topologies on a set X such that $\mathcal{T} \subset \mathcal{U} \subset \mathcal{T}_\alpha$. Then $\mathcal{U}_\alpha = \mathcal{T}_\alpha$.*

Proposition 1.6. ([9]) *For a space $(X,\mathcal{T})$ and $x \in X$ the following are equivalent:*
(a) $\{x\} \in \mathrm{SPO}(\mathcal{T})$.
(b) $\{x\} \in \mathrm{PO}(\mathcal{T})$.
(c) $\{x\} \in \mathcal{T}_\gamma$.

Proposition 1.7. ([7]) *Let $(X,\mathcal{T})$ be a space. Then $\mathcal{T}_{\gamma\gamma} = \mathcal{T}_{\gamma\alpha}$ and $\mathcal{T}_{\alpha\gamma} = \mathcal{T}_\gamma$.*

Proposition 1.8. ([6]) *Let A be a subset of a space $(X,\mathcal{T})$ and $x \in \mathrm{int}\,(\mathrm{cl}\,A) \backslash \mathrm{cl}_\gamma A$. Then $\{x\} \in \mathrm{PO}(\mathcal{T}) \backslash \mathcal{T}$.*

Proposition 1.9. ([4]) *Let A be a subset of a space X. Then:*
(1) $int_\gamma(cl\,A) = int\,(cl\,A) = int_\gamma scl\,A = int_\gamma cl_\alpha A$,
(2) $cl_\gamma int\,A = cl\,(int\,A) = cl_\gamma sint\,A = cl_\gamma int_\alpha A$.
(3) $int_\alpha cl_\gamma A = int\,(cl_\gamma A)$,
(4) $cl_\alpha int_\gamma A = cl\,(int_\gamma A)$.

Proposition 1.10. ([4]) *Let A be a subset of a space X. Then:*
(1) $cl_\alpha A = cl_\gamma A \cup int\,(cl\,A)$,
(2) $int_\alpha A = int_\gamma A \cap cl\,(int\,A)$.

Since the operators "cl" and "cl_α" coincide on the class of semi-preopen sets, we have

Corollary 1.11. *Let A be a subset of a space X. Then:*
(1) $cl\,(int_\gamma A) = cl_\gamma int_\gamma A \cup int\,(cl\,(int_\gamma A))$,
(2) $int\,(cl_\gamma A) = int_\gamma cl_\gamma A \cap cl\,(int\,(cl_\gamma A))$.

Proposition 1.12. *Let A be a subset of a space X. Then:*
(1) $cl_\gamma int_\gamma A = cl_\gamma A \cap cl\,(int_\gamma A)$,
(2) $cl_\gamma int_\gamma cl_\gamma A = cl_\gamma A \cap cl\,(int_\gamma cl_\gamma A)$.

Proof. (1) Suppose that $x \in \mathrm{cl}_\gamma A \cap \mathrm{cl}\,(\mathrm{int}_\gamma A)$ and let $x \notin \mathrm{cl}_\gamma \mathrm{int}_\gamma A$. Then by 1.11(1) $x \in \mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}_\gamma A))$ and so $\{x\} \in \mathrm{PO}(\mathcal{T})$ by 1.8. Hence $\{x\} \in \mathcal{T}_\gamma$ by 1.6 and so $x \in \mathrm{int}_\gamma A$, a contradiction. Therefore $\mathrm{cl}_\gamma A \cap \mathrm{cl}\,(\mathrm{int}_\gamma A) \subset \mathrm{cl}_\gamma \mathrm{int}_\gamma A$, while the converse follows immediately.
The statement (2) follows easily from (1). □

Dually we have

Proposition 1.13. *Let A be a subset of a space X. Then:*
(1) $int_\gamma cl_\gamma A = int_\gamma A \cup int\,(cl_\gamma A)$,
(2) $int_\gamma cl_\gamma int_\gamma A = int_\gamma A \cup int\,(cl_\gamma int_\gamma A)$.

Proposition 1.14. ([6]) *Let A be a subset of a space $(X, \mathcal{T})$. Then $A \in \mathcal{T}_\gamma$ if and only if $A = G \cup H$ with $G \in \mathcal{T}_\alpha$ and $\{h\} \in \mathrm{PO}(\mathcal{T}) \setminus \mathcal{T}$ for every $h \in H$.*

Proposition 1.15. ([9]) *Let $\mathcal{T}$ and $\mathcal{U}$ be topologies on X. Then $\mathrm{SPO}(\mathcal{T}) = \mathrm{SPO}(\mathcal{U})$ if and only if $\mathcal{T}_\gamma = \mathcal{U}_\gamma$.*

Proposition 1.16. ([4]) *Let $(X, \mathcal{T})$ be a space. Then*
(1) $\mathrm{SO}(\mathcal{T}) \subset \mathrm{SO}(\mathcal{T}_\gamma)$,
(2) $\mathrm{PO}(\mathcal{T}) \supset \mathrm{PO}(\mathcal{T}_\gamma)$,
(3) $\mathrm{SPO}(\mathcal{T}) \supset \mathrm{SPO}(\mathcal{T}_\gamma)$.

We conclude this section with the following chart.

$\mathcal{A}$	$\mathcal{T}(\mathcal{A})$
$\mathrm{SO}(\mathcal{T})$	$\mathcal{T}_\alpha$
$\mathrm{PO}(\mathcal{T})$	$\mathcal{T}_\gamma$
$\mathrm{SPO}(\mathcal{T})$	$\mathcal{T}_\gamma$
$\mathrm{SO}(\mathcal{T}_\gamma)$	$\mathcal{T}_{\gamma\alpha}$
$\mathrm{PO}(\mathcal{T}_\gamma)$	$\mathcal{T}_{\gamma\alpha}$
$\mathrm{SPO}(\mathcal{T}_\gamma)$	$\mathcal{T}_{\gamma\alpha}$

2. On Γ-Preopen Sets

Now we consider a new class of generalized open sets.

Definition 2. A subset A of a space $(X,\mathcal{T})$ is called Γ-*preopen* if $A \subset \text{int}\,(\text{cl}_\gamma \text{A})$. The class of all Γ-preopen sets in $(X,\mathcal{T})$ will be denoted by $\Gamma\text{PO}(\mathcal{T})$.

By 1.2 we have that $\text{int}\,(\text{cl}\,(\text{int}\,A)) = \text{int}\,(\text{pcl}\,A) \subset \text{int}\,(\text{cl}_\gamma A) \subset \text{int}_\gamma\text{cl}_\gamma A \subset \text{pint}\,(\text{cl}\,A) = \text{int}\,(\text{cl}\,A)$ and therefore $\mathcal{T}_\alpha \subset \Gamma\text{PO}(\mathcal{T}) \subset \text{PO}(\mathcal{T}_\gamma) \subset \text{PO}(\mathcal{T})$.

On the other hand, $\text{D}(\mathcal{T}_\gamma) \subset \Gamma\text{PO}(\mathcal{T})$ is clear.

Proposition 2.1. *For a subset A of a space X the following are equivalent:*

(a) $A \in \Gamma\text{PO}(\mathcal{T})$.

(b) $A \in \text{PO}(\mathcal{T})$ and $cl\,A = cl_\gamma A$.

Proof. (a) ⇒ (b): Let A be Γ-preopen, that is $A \subset\text{int}\,(\text{cl}_\gamma \text{A})$. Since $\text{cl}_\gamma A$ is preclosed, we have that $\text{cl}\,A \subset \text{cl}\,(\text{int}\,(\text{cl}_\gamma A)) \subset \text{cl}_\gamma A$ and thus $\text{cl}\,A=\text{cl}_\gamma A$.

The converse is obvious. □

Proposition 2.2. *The union of any family of Γ-preopen sets is a Γ-preopen set. The intersection of an open and a Γ-preopen set ia a Γ-preopen set.*

Proof. The statements are proved by using the same method as in proving the corresponding results for the other classes of generalized open sets (see [3]). □

Since $\text{PO}(\mathcal{T}_\alpha) = \text{PO}(\mathcal{T})$ implies $\mathcal{T}_{\alpha\gamma} = \mathcal{T}_\gamma$ ([4]) and having in mind that the operators "int" and "int_α" coincide on the class of semi-preclosed sets, we have that $\text{int}_\alpha\text{cl}_{\alpha\gamma}A = \text{int}_\alpha\text{cl}_\gamma A = \text{int}\,(\text{cl}_\gamma A)$. On the other hand, by 1.7 and 1.3 we obtain $\text{int}_\gamma\text{cl}_{\gamma\gamma}A = \text{int}_\gamma\text{cl}_{\gamma\alpha}A = \text{int}_\gamma\text{cl}_\gamma A$. Therefore we have

Proposition 2.3. *Let $(X,\mathcal{T})$ be a space. Then $\Gamma\text{PO}(\mathcal{T}_\alpha) = \Gamma\text{PO}(\mathcal{T})$ and $\Gamma\text{PO}(\mathcal{T}_\gamma) = \text{PO}(\mathcal{T}_\gamma)$.*

Corollary 2.4. *If $A \in \Gamma\text{PO}(\mathcal{T})$ and $G \in\mathcal{T}_\alpha$, then $A \cap G \in \Gamma\text{PO}(\mathcal{T})$.*

Recall that a space $(X,\mathcal{T})$ is called *semi-T_D* if $\text{cl}\,\{x\}\backslash\{x\}$ is semi-closed for each $x \in X$. It was proved in [7] that a space $(X,\mathcal{T})$ is semi-T_D if and only if $\mathcal{T}_\gamma =\mathcal{T}_\alpha$. So we have

Proposition 2.5. *Let $(X,\mathcal{T})$ be semi-T_D. Then $\Gamma\text{PO}(\mathcal{T}) = \text{PO}(\mathcal{T})$.*

Definition 3. A subset A of a space X is called Γ-*preclosed* if $X\backslash A$ is Γ-preopen.

Thus A is Γ-preclosed if and only if $\text{cl}\,(\text{int}_\gamma A) \subset A$. The class of all Γ-preclosed sets in $(X,\mathcal{T})$ will be denoted by $\Gamma\text{PC}(\mathcal{T})$.

Dually to 2.1 we have

Proposition 2.6. *A subset A of a space X is Γ-preclosed if and only if $A \in \text{PC}(\mathcal{T})$ and $int\,A = int_\gamma A$.*

Definition 4. For a subset A of a space X the Γ-*preclosure* of A , denoted by $\text{gcl}\,A$, is the smallest Γ-preclosed set containing A . The Γ-*preinterior* of A , denoted by $\text{gint}\,A$, is the largest Γ-preopen set contained in A .

Proposition 2.7. *Let A be a subset of a space X. Then:*

(1) $gcl\,A = A \cup cl\,(int_\gamma A)$,

(2) $gint\,A = A \cap int\,(cl_\gamma A)$.

Proof. We shall prove only the first statement. Since $\mathrm{cl}(\mathrm{int}_\gamma(A \cup \mathrm{cl}(\mathrm{int}_\gamma A))) \subset \mathrm{cl}(\mathrm{int}_\gamma A \cup \mathrm{cl}(\mathrm{int}_\gamma A)) = \mathrm{cl}(\mathrm{int}_\gamma A) \subset A \cup \mathrm{cl}(\mathrm{int}_\gamma A)$, we have that $A \cup \mathrm{cl}(\mathrm{int}_\gamma A)$ is Γ-preclosed and so $\mathrm{gcl}\, A \subset A \cup \mathrm{cl}(\mathrm{int}_\gamma A)$. On the other hand, $\mathrm{gcl}\, A$ is Γ-preclosed and so $\mathrm{cl}(\mathrm{int}_\gamma A) \subset \mathrm{cl}(\mathrm{int}_\gamma \mathrm{gcl}\, A) \subset \mathrm{gcl}\, A$ which implies $A \cup \mathrm{cl}(\mathrm{int}_\gamma A) \subset \mathrm{gcl}\, A$. □

Corollary 2.8. *Let $(X, \mathcal{T})$ be a space. Then $gcl\, A = cl\, A$ for every $A \in \mathrm{SO}(\mathcal{T}_\gamma)$ and $gint\, A = int\, A$ for every $A \in \mathrm{SC}(\mathcal{T}_\gamma)$.*

Now we shall relate the operators of Γ-preclosure and Γ-preinterior to some other operators concerning generalized open sets.

Proposition 2.9. *Let A be a subset of a space X. Then:*
(1) $cl\,(gint\, A) = cl\,(int\,(cl_\gamma A)) = cl_\gamma gint\, A$,
(2) $int\,(gcl\, A) = int\,(cl\,(int_\gamma A)) = int_\gamma gcl\, A$.

Proof. We shall prove only (1). First we notice that $\mathrm{cl}(\mathrm{gint}\, A) = \mathrm{cl}_\gamma \mathrm{gint}\, A$ by 2.1. On the other hand, $\mathrm{cl}(\mathrm{gint}\, A) = \mathrm{cl}(A \cap \mathrm{int}(\mathrm{cl}_\gamma A)) \supset \mathrm{cl}\, A \cap \mathrm{int}(\mathrm{cl}_\gamma A) = \mathrm{int}(\mathrm{cl}_\gamma A)$ and thus $\mathrm{cl}(\mathrm{gint}\, A) \supset \mathrm{cl}(\mathrm{int}(\mathrm{cl}_\gamma A)) \supset \mathrm{cl}(A \cap \mathrm{int}(\mathrm{cl}_\gamma A)) = \mathrm{cl}(\mathrm{gint}\, A)$. □

Proposition 2.10. *Let A be a subset of a space X. Then:*
(1) $cl_\gamma gcl\, A = gcl\,(cl_\gamma A) = cl_\gamma A \cup cl\,(int_\gamma A)$,
(2) $int_\gamma gint\, A = gint\,(int_\gamma A) = int_\gamma A \cap int\,(cl_\gamma A)$.

Proof. Again we prove only (1). By 1.13(1) and the fact that $\mathrm{cl}_\gamma A$ is preclosed we have that $\mathrm{gcl}(\mathrm{cl}_\gamma A) = \mathrm{cl}_\gamma A \cup \mathrm{cl}(\mathrm{int}_\gamma \mathrm{cl}_\gamma A) = \mathrm{cl}_\gamma A \cup \mathrm{cl}(\mathrm{int}_\gamma A \cup \mathrm{int}(\mathrm{cl}_\gamma A)) = \mathrm{cl}_\gamma A \cup \mathrm{cl}(\mathrm{int}_\gamma A) \cup \mathrm{cl}(\mathrm{int}(\mathrm{cl}_\gamma A)) = \mathrm{cl}_\gamma A \cup \mathrm{cl}(\mathrm{int}_\gamma A) = \mathrm{cl}_\gamma(A \cup \mathrm{cl}(\mathrm{int}_\gamma A)) = \mathrm{cl}_\gamma \mathrm{gcl}\, A$. □

Proposition 2.11. *Let A be a subset of a space X. Then:*
(1) $sint\,(gcl\, A) = cl\,(int_\gamma A)$, $scl\,(gint\, A) = int\,(cl_\gamma A)$,
(2) $sint\,(gint\, A) = sint\, A \cap int\,(cl_\gamma A)$, $scl\,(gcl\, A) = scl\, A \cup cl\,(int_\gamma A)$,
(3) $gint\,(sint\, A) = int_\alpha A$, $gcl\,(scl\, A) = cl_\alpha A$,
(4) $pcl\,(gint\, A) = gint\, A \cup cl\,(int\, A)$, $pint\,(gcl\, A) = gcl\, A \cap int\,(cl\, A)$,
(5) $gint\,(pcl\, A) = pcl\, A \cap int\,(cl_\gamma A)$, $gcl\,(pint\, A) = pint\, A \cup cl\,(int_\gamma A)$.

Proof. For (1) we use 2.9, for (3) 1.9, and the other statements follow easily. □

3. Topology Generated by Γ-Preopen Sets

Let $\mathcal{T}_g = \{G \in \Gamma\mathrm{PO}(\mathcal{T}) \mid G \cap A \in \Gamma\mathrm{PO}(\mathcal{T})$ whenever $A \in \Gamma\mathrm{PO}(\mathcal{T})\}$. Clearly, $\mathcal{T}_g$ is a topology on X, and by 2.4 it is larger than $\mathcal{T}_\alpha$. The closure and the interior of a set A in $(X, \mathcal{T}_g)$ will be denoted by $\mathrm{cl}_g A$ and $\mathrm{int}_g A$ respectively.

Example 3.1. (1) Let $\mathcal{T}$ be a topology on a finite set X and $A \in \Gamma\mathrm{PO}(\mathcal{T})$. Then $A \subset \mathrm{int}(\mathrm{cl}\, A)$, $\mathrm{cl}\, A = \mathrm{cl}_\gamma A$ by 2.1 and so $\mathrm{int}(\mathrm{cl}\{y\}) = \emptyset$ for every $y \in \mathrm{cl}_\gamma A \backslash A$. Then $\{y\} \in \mathrm{C}(\mathcal{T}_\alpha)$ and hence $\mathrm{cl}_\gamma A \backslash A \in \mathrm{C}(\mathcal{T}_\alpha)$ since X is finite. Thus $A \cup (X \backslash \mathrm{cl}_\gamma A) \in \mathcal{T}_\alpha$ and hence $A = (A \cup (X \backslash \mathrm{cl}_\gamma A)) \cap \mathrm{int}(\mathrm{cl}\, A) \in \mathcal{T}_\alpha$. Therefore $\Gamma\mathrm{PO}(\mathcal{T}) = \mathcal{T}_\alpha$ and thus $\mathcal{T}_g = \mathcal{T}_\alpha$ whenever X is finite.

(2) Let X be an infinite set and $p \in X$. Then $\mathcal{T} = \{\emptyset\} \cup \{U \subset X \mid p \in U$ and $X \backslash U$ is finite$\}$ is a topology on X with $\mathrm{PO}(\mathcal{T}) = \{\emptyset\} \cup \{S \subset X \mid p \in S$ or S is infinite$\}$, $\mathcal{T}_\gamma = \mathcal{T} \cup \{\{p\}\}$ and $\mathrm{PO}(\mathcal{T}_\gamma) = \mathcal{T}_{\gamma\alpha} = \{\emptyset\} \cup \{S \subset X \mid p \in S\}$ (see [7]). Then $\Gamma\mathrm{PO}(\mathcal{T}) = \mathcal{T}_{\gamma\alpha}$ and so $\mathcal{T}_g = \mathcal{T}_{\gamma\alpha}$.

Proposition 3.2. *Let $(X, \mathcal{T})$ be a space. Then $\mathcal{T}_g \subset \mathrm{SO}(\mathcal{T}_\gamma)$.*

Proof. Let $A \in \mathcal{T}_g$ and suppose that $\text{int}_\gamma A = \emptyset$. Then $X \backslash A \in \text{D}(\mathcal{T}_\gamma) \subset \Gamma\text{PO}(\mathcal{T})$ and so $(X \backslash A) \cup \{a\} \in \Gamma\text{PO}(\mathcal{T})$ for every $a \in A$. Then $\{a\} = A \cap ((X \backslash A) \cup \{a\}) \in \Gamma\text{PO}(\mathcal{T})$ and so $\{a\} \in \mathcal{T}_\gamma$ by 1.6, a contradiction. Hence $\text{int}_\gamma A \neq \emptyset$ and put $G = A \backslash \text{cl}(\text{int}_\gamma A)$. Then $G \in \mathcal{T}_g$, $\text{int}_\gamma G = \emptyset$, thus $G = \emptyset$. Therefore $A \subset \text{cl}(\text{int}_\gamma A)$ which implies $A \in \text{SO}(\mathcal{T}_\gamma)$ by 1.12(1). □

From 1.4(2) we have

Corollary 3.3. *Let* $(X, \mathcal{T})$ *be a space. Then* $\mathcal{T}_g \subset \mathcal{T}_{\gamma\alpha}$.

And now the following diagram relates the topology $\mathcal{T}_g$ to $\mathcal{T}_\alpha$, $\mathcal{T}_\gamma$ and $\mathcal{T}_{\gamma\alpha}$.

$$\begin{array}{ccccc} & & \mathcal{T}_\gamma & & \\ & \nearrow & & \searrow & \\ \mathcal{T}_\alpha & & \not\uparrow\not\downarrow & & \mathcal{T}_{\gamma\alpha} \\ & \searrow & & \nearrow & \\ & & \mathcal{T}_g & & \end{array}$$

Proposition 3.4. *Let* $(X, \mathcal{T})$ *be a space. Then* $\mathcal{T}_g = \Gamma\text{PO}(\mathcal{T}) \cap \text{SO}(\mathcal{T}_\gamma)$.

Proof. It remains to show that $\Gamma\text{PO}(\mathcal{T}) \cap \text{SO}(\mathcal{T}_\gamma) \subset \mathcal{T}_g$. Suppose that $A \in \Gamma\text{PO}(\mathcal{T}) \cap \text{SO}(\mathcal{T}_\gamma)$ and let $B \in \Gamma\text{PO}(\mathcal{T})$. Then $\text{cl}_\gamma(A \cap B) \supset \text{cl}_\gamma(\text{int}_\gamma A \cap B) \supset \text{int}_\gamma A \cap \text{cl}_\gamma B$ and hence $\text{cl}_\gamma(A \cap B) \supset \text{cl}_\gamma(\text{int}_\gamma A \cap \text{cl}_\gamma B) \supset \text{cl}_\gamma(\text{int}_\gamma A \cap \text{int}(\text{cl}_\gamma B)) \supset \text{cl}_\gamma \text{int}_\gamma A \cap \text{int}(\text{cl}_\gamma B) = \text{cl}_\gamma A \cap \text{int}(\text{cl}_\gamma B)$ because $A \in \text{SO}(\mathcal{T}_\gamma)$. Thus $\text{int}(\text{cl}_\gamma(A \cap B)) \supset \text{int}(\text{cl}_\gamma A) \cap \text{int}(\text{cl}_\gamma B) \supset A \cap B$. That is $A \cap B \in \Gamma\text{PO}(\mathcal{T})$ and finally $A \in \mathcal{T}_g$. □

Proposition 3.5. *Let* A *be a subset of a space* X. *Then:*

(1) $\text{int}_g\text{cl}_g A = \text{int}(\text{cl}_g A)$,
(2) $\text{cl}_g\text{int}_g A = \text{cl}(\text{int}_g A)$,
(3) $\text{cl}_g\text{int}_g\text{cl}_g A = \text{cl}(\text{int}(\text{cl}_g A))$,
(4) $\text{int}_g\text{cl}_g\text{int}_g A = \text{int}(\text{cl}(\text{int}_g A))$.

Proof. (1) Since $\mathcal{T}_g \subset \mathcal{T}_{\gamma\alpha}$ and $\text{cl}_g A \in \text{C}(\mathcal{T}_{\gamma\alpha})$, by 2.6 we have that $\text{int}_g\text{cl}_g A \subset \text{int}_{\gamma\alpha}\text{cl}_g A = \text{int}_\gamma\text{cl}_g A = \text{int}(\text{cl}_g A) \subset \text{int}_g\text{cl}_g A$.

The rest is similarly proved. □

Proposition 3.6. *Let* A *be a subset of a space* X. *Then:*

(1) $\text{int}_\gamma\text{cl}_\gamma A \subset \text{int}(\text{cl}_g A)$,
(2) $\text{cl}_\gamma\text{int}_\gamma\text{cl}_\gamma A \subset \text{cl}(\text{int}(\text{cl}_g A))$,
(3) $\text{cl}(\text{int}_g A) \subset \text{cl}_\gamma\text{int}_\gamma A$,
(4) $\text{int}(\text{cl}(\text{int}_g A)) \subset \text{int}_\gamma\text{cl}_\gamma\text{int}_\gamma A$.

Proof. (1) From 1.3(1), 3.3 and 2.6 it follows that $\text{int}_\gamma\text{cl}_\gamma A = \text{int}_\gamma\text{cl}_{\gamma\alpha} A \subset \text{int}_\gamma\text{cl}_g A = \text{int}(\text{cl}_g A)$.

The rest is proved in a similar way. □

Corollary 3.7. *Let* $(X, \mathcal{T})$ *be a space. Then:*

(1) $\text{PO}(\mathcal{T}_\gamma) \subset \text{PO}(\mathcal{T}_g) \subset \text{PO}(\mathcal{T})$,
(2) $\text{SPO}(\mathcal{T}_\gamma) \subset \text{SPO}(\mathcal{T}_g) \subset \text{SPO}(\mathcal{T})$,
(3) $\text{SO}(\mathcal{T}) \subset \text{SO}(\mathcal{T}_g) \subset \text{SO}(\mathcal{T}_\gamma)$,
(4) $\mathcal{T}_\alpha \subset \mathcal{T}_{g\alpha} \subset \mathcal{T}_{\gamma\alpha}$,
(5) $\text{N}(\mathcal{T}) \subset \text{N}(\mathcal{T}_g) \subset \text{N}(\mathcal{T}_\gamma)$.

Now we shall look further into the various relations between $\mathcal{T}_g$, $\mathcal{T}_\alpha$, $\mathcal{T}_\gamma$ and $\mathcal{T}_{\gamma\alpha}$. Notice that $\mathcal{T}_{\alpha g} = \mathcal{T}_g$ follows easily from 2.3.

Proposition 3.8. *Let* $(X, \mathcal{T})$ *be a space. Then:*

(1) $\mathcal{T}_{\gamma g} = \mathcal{T}_{\gamma\alpha}$,
(2) $\mathcal{T}_{g\gamma} \supset \mathcal{T}_\gamma$.

Proof. The first statement follows immediately from 2.3 and 1.7. As for the second statement suppose that $A \in \mathcal{T}_\gamma$. Then by 1.14, $A = G \cup H$ with $G \in \mathcal{T}_\alpha$ and $\{h\} \in \mathrm{PO}(\mathcal{T}) \setminus \mathcal{T}$ for every $h \in H$. By 1.6 and 3.7(1), $\{h\} \in \mathcal{T}_\gamma \subset \mathrm{PO}(\mathcal{T}_\gamma) \subset \mathrm{PO}(\mathcal{T}_g)$ and so $\{h\} \in \mathcal{T}_{g\gamma}$ for every $h \in H$. Hence $A \in \mathcal{T}_{g\gamma}$. □

Proposition 3.9. *Let A be a subset of a space X such that $\mathrm{cl}_\gamma \mathrm{int}_\gamma A = X$. Then $A \in \mathcal{T}_g$.*

Proof. Let $\mathrm{cl}_\gamma \mathrm{int}_\gamma A = X$ and $B \in \Gamma\mathrm{PO}(\mathcal{T})$. Then $\mathrm{cl}_\gamma(A \cap B) \supset \mathrm{cl}_\gamma(\mathrm{int}_\gamma A \cap B) \supset \mathrm{int}_\gamma A \cap \mathrm{cl}_\gamma B \supset \mathrm{int}_\gamma A \cap \mathrm{int}\,(\mathrm{cl}_\gamma B)$ and hence $\mathrm{cl}_\gamma(A \cap B) \supset \mathrm{cl}_\gamma(\mathrm{int}_\gamma A \cap \mathrm{int}\,(\mathrm{cl}_\gamma B)) \supset \mathrm{cl}_\gamma \mathrm{int}_\gamma A \cap \mathrm{int}\,(\mathrm{cl}_\gamma B) = \mathrm{int}\,(\mathrm{cl}_\gamma B)$. Therefore $\mathrm{int}\,(\mathrm{cl}_\gamma(A \cap B)) \supset \mathrm{int}\,(\mathrm{cl}_\gamma B) \supset B \supset A \cap B$ and so $A \cap B \in \Gamma\mathrm{PO}(\mathcal{T})$. That is $A \in \mathcal{T}_g$. □

Corollary 3.10. *Let $(X, \mathcal{T})$ be a space. Then $\mathcal{T}_{g\alpha} = \mathcal{T}_g$.*

Proof. Let $G \in \mathcal{T}_{g\alpha}$. By 1.4(1), $G = U \backslash A$ with $U \in \mathcal{T}_g$ and $A \in \mathrm{N}(\mathcal{T}_g)$. By 3.7 we have that $A \in \mathrm{N}(\mathcal{T}_\gamma)$ and hence $A \in \mathrm{C}(\mathcal{T}_g)$ by 3.9. Thus $G \in \mathcal{T}_g$. □

Proposition 3.11. *Let $(X, \mathcal{T})$ be a space. Then $\mathrm{N}(\mathcal{T}_g) = \mathrm{N}(\mathcal{T}_\gamma)$.*

Proof. It remains to show that $\mathrm{N}(\mathcal{T}_\gamma) \subset \mathrm{N}(\mathcal{T}_g)$. Suppose that $\mathrm{int}_\gamma \mathrm{cl}_\gamma A = \emptyset$. Then $A = \mathrm{cl}_g A$ by 3.9 and so by 3.5, $\mathrm{int}_g \mathrm{cl}_g A = \mathrm{int}\,(\mathrm{cl}_g A) = \mathrm{int}\, A = \emptyset$. Thus $A \in \mathrm{N}(\mathcal{T}_g)$. □

It was shown in [9] that $\mathrm{SPO}(\mathcal{T}) = \mathrm{SPO}(\mathcal{U})$ implies $\mathrm{N}(\mathcal{T}) = \mathrm{N}(\mathcal{U})$. The converse holds under the condition $\mathcal{T} \subset \mathcal{U} \subset \mathcal{T}_\gamma$ which was proved in [6]. The next statement gives us a slight improvement.

Lemma 3.12. *Let $\mathcal{T}$ and $\mathcal{U}$ be topologies on X such that $\mathcal{U} \subset \mathcal{T}_\gamma$ and $\mathcal{T} \subset \mathcal{U}_\gamma$. Then $\mathrm{SPO}(\mathcal{T}) = \mathrm{SPO}(\mathcal{U})$ if and only if $\mathrm{N}(\mathcal{T}) = \mathrm{N}(\mathcal{U})$.*

Proof. Suppose $A \in \mathrm{SPO}(\mathcal{T})$ and let $\mathrm{N}(\mathcal{T}) = \mathrm{N}(\mathcal{U})$. Then $B = A \backslash \mathrm{cl}_{\mathcal{U}} \mathrm{int}_{\mathcal{U}} \mathrm{cl}_{\mathcal{U}} A \in \mathrm{N}(\mathcal{U}) = \mathrm{N}(\mathcal{T})$. On the other hand, $\mathcal{U} \subset \mathcal{T}_\gamma$ implies $\mathrm{cl}_{\mathcal{U}} \mathrm{int}_{\mathcal{U}} \mathrm{cl}_{\mathcal{U}} A \in \mathrm{C}(\mathcal{T}_\gamma)$ and so $B \in \mathrm{SPO}(\mathcal{T})$. Hence $B = \emptyset$ and thus $A \in \mathrm{SPO}(\mathcal{U})$. □

Proposition 3.13. *Let $(X, \mathcal{T})$ be a space. Then $\mathrm{SPO}(\mathcal{T}_g) = \mathrm{SPO}(\mathcal{T}_\gamma)$.*

Proof. We have $\mathcal{T}_g \subset \mathcal{T}_{\gamma\gamma} = \mathcal{T}_{\gamma\alpha}$ (3.3 and 1.7) and $\mathcal{T}_\gamma \subset \mathcal{T}_{g\gamma}$ (3.8) and the statement follows from 3.11 and 3.12. □

Now we have from 1.15 and 1.7

Corollary 3.14. *Let $(X, \mathcal{T})$ be a space. Then $\mathcal{T}_{g\gamma} = \mathcal{T}_{\gamma\alpha}$.*

The next statement follows immediately from 2.8.

Proposition 3.15. *Let $(X, \mathcal{T})$ be a space. Then $\mathrm{cl}_g A = \mathrm{cl}\, A$ for every $A \in \mathrm{SO}(\mathcal{T}_\gamma)$ and $\mathrm{int}_g A = \mathrm{int}\, A$ for every $A \in \mathrm{SC}(\mathcal{T}_\gamma)$.*

Lemma 3.16. *Let A be a subset of a space X. Then $\mathrm{int}\,(\mathrm{cl}_{\gamma\alpha} A) = \mathrm{int}\,(\mathrm{cl}_\gamma A)$.*

Proof. By applying 1.12 (2) we obtain $\mathrm{int}\,(\mathrm{cl}_{\gamma\alpha} A) = \mathrm{int}\,(A \cup \mathrm{cl}_\gamma \mathrm{int}_\gamma \mathrm{cl}_\gamma A) \supset \mathrm{int}\,(\mathrm{cl}_\gamma \mathrm{int}_\gamma \mathrm{cl}_\gamma A) = \mathrm{int}\,(\mathrm{cl}_\gamma A \cap \mathrm{cl}\,(\mathrm{int}_\gamma \mathrm{cl}_\gamma A)) = \mathrm{int}\,(\mathrm{cl}_\gamma A) \cap \mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}_\gamma \mathrm{cl}_\gamma A)) \supset \mathrm{int}\,(\mathrm{cl}_\gamma A) \cap \mathrm{int}_\gamma \mathrm{cl}_\gamma A = \mathrm{int}\,(\mathrm{cl}_\gamma A)$. The reverse inclusion is clear. □

Proposition 3.17. *Let $(X, \mathcal{T})$ be a space. Then $\Gamma\mathrm{PO}(\mathcal{T}_g) = \Gamma\mathrm{PO}(\mathcal{T})$.*

Proof. Applying 3.14, 3.15 and 3.16 we have that $\mathrm{int}_g \mathrm{cl}_{g\gamma} A = \mathrm{int}_g \mathrm{cl}_{\gamma\alpha} A = \mathrm{int}\,(\mathrm{cl}_{\gamma\alpha} A) = \mathrm{int}\,(\mathrm{cl}_\gamma A)$ and thus $\Gamma\mathrm{PO}(\mathcal{T}_g) = \Gamma\mathrm{PO}(\mathcal{T})$. □

Corollary 3.18. *Let $(X, \mathcal{T})$ be a space. Then $\mathcal{T}_{gg} = \mathcal{T}_g$.*

Now we are in a position to complete the chart from Section 1.

$\mathcal{A}$	$\mathcal{T}(\mathcal{A})$
$\mathrm{SO}(\mathcal{T}_g)$	$\mathcal{T}_g$
$\mathrm{PO}(\mathcal{T}_g)$	$\mathcal{T}_{\gamma\alpha}$
$\mathrm{SPO}(\mathcal{T}_g)$	$\mathcal{T}_{\gamma\alpha}$

We conclude this section with the conditions under which the topology $\mathcal{T}_g$ coincides with $\mathcal{T}_\alpha$, $\mathcal{T}_\gamma$ or $\mathcal{T}_{\gamma\alpha}$.

Proposition 3.19. *Let* $(X, \mathcal{T})$ *be a space. Then* $\mathrm{RO}(\mathcal{T}_g) = \mathrm{RO}(\mathcal{T})$.

Proof. Suppose $A \in \mathrm{RO}(\mathcal{T}_g)$, that is $A = \mathrm{int}_g \mathrm{cl}_g A$. Hence $A = \mathrm{int}\,(\mathrm{cl}_g A)$ by 3.5 and so $A \in \mathrm{RO}(\mathcal{T})$ since $\mathrm{cl}_g A \in \mathrm{PC}(\mathcal{T})$. The converse follows from 3.15 □

The next statement was proved in [5].

Lemma 3.20. *Let* $\mathcal{T}$ *and* $\mathcal{U}$ *be topologies on a set* X. *Then* $\mathcal{T}_\alpha = \mathcal{U}_\alpha$ *if and only if* $\mathrm{RO}(\mathcal{T}) = \mathrm{RO}(\mathcal{U})$ *and* $\mathrm{SPO}(\mathcal{T}) = \mathrm{SPO}(\mathcal{U})$.

Proposition 3.21. $\mathcal{T}_g = \mathcal{T}_\alpha$ *if and only if* $\mathcal{T}_{\gamma\alpha} = \mathcal{T}_\gamma$.

Proof. Suppose $\mathcal{T}_g = \mathcal{T}_\alpha$ and let $U \in \mathcal{T}_{\gamma\alpha}$. Then by 1.4, $U = G \backslash A$ with $G \in \mathcal{T}_\gamma$ and $A \in \mathrm{N}(\mathcal{T}_\gamma)$. By 3.11 and 3.10 we have that $A \in \mathrm{N}(\mathcal{T}_g) \subset \mathrm{C}(\mathcal{T}_{g\alpha}) = \mathrm{C}(\mathcal{T}_g) = \mathrm{C}(\mathcal{T}_\alpha)$ and so $U \in \mathcal{T}_\gamma$. Conversely, suppose that $\mathcal{T}_{\gamma\alpha} = \mathcal{T}_\gamma$. By 1.15 and 3.13 we have $\mathrm{SPO}(\mathcal{T}) = \mathrm{SPO}(\mathcal{T}_\gamma) = \mathrm{SPO}(\mathcal{T}_g)$. Hence $\mathcal{T}_\alpha = \mathcal{T}_{g\alpha}$ by 3.19 and 3.20, and finally, $\mathcal{T}_\alpha = \mathcal{T}_g$ by 3.10. □

Corollary 3.22. $\mathcal{T}_g = \mathcal{T}_\gamma$ *if and only if* $(X, \mathcal{T})$ *is semi-*T_D.

Proof. Suppose that $\mathcal{T}_g = \mathcal{T}_\gamma$. Then by 3.10 we have $\mathcal{T}_{\gamma\alpha} = \mathcal{T}_{g\alpha} = \mathcal{T}_g = \mathcal{T}_\gamma$ and thus $\mathcal{T}_g = \mathcal{T}_\alpha$ by 3.21. Therefore $\mathcal{T}_\gamma = \mathcal{T}_\alpha$, that is $(X, \mathcal{T})$ is semi-T_D. The converse follows from 2.5. □

It remains to find out when the topologies $\mathcal{T}_g$ and $\mathcal{T}_{\gamma\alpha}$ coincide. For that, let $B = \{x \in X \mid \{x\} \in \mathcal{T}_\gamma \setminus \mathcal{T}\}$ and $R(B) = \{x \in B \mid \{x\} \in \mathrm{RO}(\mathcal{T}_\gamma)\}$.

Proposition 3.23. *Let* $(X, \mathcal{T})$ *be a space. Then:*

(1) $\{x\} \in \mathcal{T}_g$ *for every* $x \in B \backslash R(B)$,

(2) $\{x\} \in \mathcal{T}_\gamma \backslash \mathcal{T}_g$ *for every* $x \in R(B)$.

Proof. Let $x \in B \backslash R(B)$. By 1.13(1) we have that $\mathrm{int}_\gamma \mathrm{cl}_\gamma \{x\} = \{x\} \cup \mathrm{int}\,(\mathrm{cl}_\gamma \{x\})$ and so $x \in \mathrm{int}\,(\mathrm{cl}_\gamma \{x\})$. Hence $\{x\} \in \Gamma\mathrm{PO}(\mathcal{T})$ which implies $\{x\} \in \mathcal{T}_g$.

(2) Let $x \in R(B)$ and suppose that $\mathrm{cl}_\gamma \{x\} = \mathrm{cl}\,\{x\}$. Then $\{x\} = \mathrm{int}_\gamma \mathrm{cl}_\gamma \{x\} = \mathrm{int}_\gamma \mathrm{cl}\,\{x\} = \mathrm{int}\,(\mathrm{cl}\,\{x\})$ and so $\{x\} \in \mathcal{T}$, a contradiction. Thus $\{x\} \notin \mathcal{T}_g$. □

Corollary 3.24. $\mathcal{T}_g = \mathcal{T}_{\gamma\alpha}$ *if and only if* $R(B) = \emptyset$.

Proof. Suppose that $R(B) = \emptyset$. Then $\mathcal{T}_\gamma \subset \mathcal{T}_g \subset \mathcal{T}_{\gamma\alpha}$ by 1.14 and 3.23. Hence $\mathcal{T}_{g\alpha} = \mathcal{T}_{\gamma\alpha}$ by 1.5 and so $\mathcal{T}_g = \mathcal{T}_{\gamma\alpha}$ by 3.10. The converse is clear. □

4. Topologies Generated by the Other Classes of Generalized Open Sets Related to $\mathcal{T}_\gamma$

Besides Γ-preopen sets, by using various combinations of operators in $\mathcal{T}$ and $\mathcal{T}_\gamma$ we can introduce several classes of generalized open sets. By 1.9 it is not difficult to see that only seven types of sets can give us classes that are possibly new. These seven types are as follows: $\mathrm{cl}\,(\mathrm{int}_\gamma A)$, $\mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}_\gamma A))$, $\mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}_\gamma A))$, $\mathrm{int}\,(\mathrm{cl}_\gamma \mathrm{int}_\gamma A)$, $\mathrm{cl}\,(\mathrm{int}_\gamma \mathrm{cl}_\gamma A)$, $\mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}_\gamma \mathrm{int}_\gamma A))$ and $\mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}_\gamma \mathrm{cl}_\gamma A))$.

(A) $A \subset \mathbf{cl}\,(\mathbf{int}_\gamma A)$

By 1.12(1), the class of sets satisfying this condition coincides with $\mathrm{SO}(\mathcal{T}_\gamma)$ and thus the generated topology is $\mathcal{T}_{\gamma\alpha}$.

(B) $A \subset \mathbf{cl}\,(\mathbf{int}\,(\mathbf{cl}_\gamma A))$

Definition 5. A subset A of a space X is called *semi-Γ-preopen* if $A \subset \mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}_\gamma A))$. The class of all semi-Γ-preopen sets in $(X, \mathcal{T})$ will be denoted by $\mathrm{S\Gamma PO}(\mathcal{T})$. It is clear that $\Gamma\mathrm{PO}(\mathcal{T}) \subset \mathrm{S\Gamma PO}(\mathcal{T})$ and

$$\mathrm{SO}(\mathcal{T}) \subset \mathrm{S\Gamma PO}(\mathcal{T}) \subset \mathrm{SPO}(\mathcal{T}_\gamma) \subset \mathrm{SPO}(\mathcal{T}) .$$

Besides, $\mathrm{S\Gamma PO}(\mathcal{T})$ is closed under forming arbitrary unions and the intersection of an open set and a semi-Γ-preopen set is semi-Γ-preopen. The next statement follows easily from 2.9(1).

Proposition 4.1. *For a subset A of a space X the following are equivalent:*
(a) $A \in \mathrm{S\Gamma PO}(\mathcal{T})$.
(b) $cl_\gamma A \in \mathrm{RC}(\mathcal{T})$.
(c) $A \in \mathrm{SPO}(\mathcal{T})$ *and* $cl_\gamma A = cl\,A$.
(d) There exists a Γ-preopen set U such that $U \subset A \subset cl\,U$.

Proposition 4.2. *Let $(X, \mathcal{T})$ be a space, $A \in \mathrm{SO}(\mathcal{T})$ and $B \in \Gamma\mathrm{PO}(\mathcal{T})$. Then $A \cap B \in \mathrm{S\Gamma PO}(\mathcal{T})$.*

Proof. $\mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}_\gamma(A \cap B))) \supset \mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}_\gamma(\mathrm{int}\,A \cap B))) \supset \mathrm{cl}\,(\mathrm{int}\,(\mathrm{int}\,A \cap \mathrm{cl}_\gamma B)) = \mathrm{cl}\,(\mathrm{int}\,A \cap \mathrm{int}\,(\mathrm{cl}_\gamma B)) \supset \mathrm{cl}\,(\mathrm{int}\,A) \cap \mathrm{int}\,(\mathrm{cl}_\gamma B) \supset A \cap B$ and thus $A \cap B \in \mathrm{S\Gamma PO}(\mathcal{T})$. □

Proposition 4.3. *Every semi-Γ-preopen set can be represented as the intersection of a semi-open set and a Γ-preopen set.*

Proof. Let $A \in \mathrm{S\Gamma PO}(\mathcal{T})$. Then by 4.1 $\mathrm{cl}_\gamma A \in \mathrm{RC}(\mathcal{T}) \subset \mathrm{SO}(\mathcal{T})$, $A \cup (X \backslash \mathrm{cl}_\gamma A) \in \mathrm{D}(\mathcal{T}_\gamma) \subset \Gamma\mathrm{PO}(\mathcal{T})$ and $A = \mathrm{cl}_\gamma A \cap (A \cup (X \backslash \mathrm{cl}_\gamma A))$. □

Denote by $\mathcal{T}_h$ the topology generated by $\mathrm{S\Gamma PO}(\mathcal{T})$, that is $\mathcal{T}_h = \{G \in \mathrm{S\Gamma PO}(\mathcal{T}) \mid G \cap A \in \mathrm{S\Gamma PO}(\mathcal{T})$ whenever $A \in \mathrm{S\Gamma PO}(\mathcal{T})\}$.

Proposition 4.4. *Let $(X, \mathcal{T})$ be a space. Then $\mathcal{T}_h \subset \mathrm{SO}(\mathcal{T}_\gamma)$.*

Proof. Let $A \in \mathcal{T}_h$ and suppose that $x \in A \backslash \mathrm{cl}\,(\mathrm{int}_\gamma A)$. Since $X \backslash (A \backslash \mathrm{cl}\,(\mathrm{int}_\gamma A)) \in \mathrm{D}(\mathcal{T}_\gamma) \subset \mathrm{S\Gamma PO}(\mathcal{T})$ and $A \backslash \mathrm{cl}\,(\mathrm{int}_\gamma A) \in \mathcal{T}_h$, we have that $(\{x\} \cup (X \backslash (A \backslash \mathrm{cl}\,(\mathrm{int}_\gamma A)))) \cap (A \backslash \mathrm{cl}\,(\mathrm{int}_\gamma A)) = \{x\} \in \mathrm{S\Gamma PO}(\mathcal{T})$. Therefore $\{x\} \in \mathrm{SPO}(\mathcal{T})$ and thus by 1.6, $\{x\} \in \mathcal{T}_\gamma$, a contradiction. Hence $A \subset \mathrm{cl}\,(\mathrm{int}_\gamma A)$ and so $A \in \mathrm{SO}(\mathcal{T}_\gamma)$. □

Proposition 4.5. *Let $(X, \mathcal{T})$ be a space. Then $\mathcal{T}_h \subset \Gamma\mathrm{PO}(\mathcal{T})$.*

Proof. Let $A \in \mathcal{T}_h$ and suppose that $x \in A \backslash \mathrm{int}\,(\mathrm{cl}_\gamma A)$. Then by 4.1(c) and 1.9(1), $x \notin \mathrm{int}_\gamma \mathrm{cl}_\gamma A = \mathrm{int}_\gamma \mathrm{cl}\,A = \mathrm{int}\,(\mathrm{cl}\,A)$ and so $x \notin \mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}_\gamma A))$. Hence $x \in X \backslash \mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}_\gamma A)) = \mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}_\gamma (X \backslash A))) = \mathrm{cl}\,(\mathrm{gint}\,(X \backslash A))$ by 2.9(1). Therefore $\{x\} \cup \mathrm{gint}\,(X \backslash A) \in \mathrm{S\Gamma PO}(\mathcal{T})$ by 4.1(d) and so $(\{x\} \cup \mathrm{gint}\,(X \backslash A)) \cap A = \{x\} \in \mathrm{S\Gamma PO}(\mathcal{T})$. Thus $\{x\} \in \mathcal{T}_\gamma$, a contradiction. Hence $A \subset \mathrm{int}\,(\mathrm{cl}_\gamma A)$, that is $A \in \Gamma\mathrm{PO}(\mathcal{T})$. □

Proposition 4.6. *Let $(X, \mathcal{T})$ be a space. Then $\mathcal{T}_h = \mathcal{T}_g$.*

Proof. By 4.4, 4.5 and 3.4 we have that $\mathcal{T}_h \subset \mathcal{T}_g$. To prove the converse, suppose that $A \in \mathcal{T}_g$ and let $B \in \mathrm{S\Gamma PO}(\mathcal{T})$. By 4.3 we have that $B = C \cap D$ with $C \in \mathrm{SO}(\mathcal{T})$ and $D \in \Gamma\mathrm{PO}(\mathcal{T})$. Then $A \cap D \in \Gamma\mathrm{PO}(\mathcal{T})$ and so $A \cap B = (A \cap D) \cap C \in \mathrm{S\Gamma PO}(\mathcal{T})$ by 4.2. Hence $A \in \mathcal{T}_h$. □

(C) $A \subset$ **int (cl$_\gamma$int$_\gamma$**A**)**

It follows easily that $A \subset \mathrm{int}\,(\mathrm{cl}_\gamma \mathrm{int}_\gamma A)$ if and only if $A \in \Gamma\mathrm{PO}(\mathcal{T}) \cap \mathrm{SO}(\mathcal{T}_\gamma)$, therefore this class coincides with $\mathcal{T}_g$ by 3.4.

(D) $A \subset$ **cl (int (cl$_\gamma$int$_\gamma$**A**))**

Noticing that $\mathrm{cl}_\gamma \mathrm{int}_\gamma A$ is preclosed, it follows that $A \subset \mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}_\gamma \mathrm{int}_\gamma A))$ if and only if $A \in \mathrm{S\Gamma PO}(\mathcal{T}) \cap \mathrm{SO}(\mathcal{T}_\gamma)$. Denote the topology generated by this class by $\mathcal{T}_i$, that is $\mathcal{T}_i = \{G \in \mathrm{S\Gamma PO}(\mathcal{T}) \cap \mathrm{SO}(\mathcal{T}_\gamma) \mid G \cap A \in \mathrm{S\Gamma PO}(\mathcal{T}) \cap \mathrm{SO}(\mathcal{T}_\gamma)$ whenever $A \in \mathrm{S\Gamma PO}(\mathcal{T}) \cap \mathrm{SO}(\mathcal{T}_\gamma)\}$.

Proposition 4.7. *Let* $(X, \mathcal{T})$ *be a space. Then* $\mathcal{T}_i \subset \Gamma\mathrm{PO}(\mathcal{T})$ *.*

Proof. Let $A \in \mathcal{T}_i$ and suppose that $x \in A \backslash \mathrm{int}\,(\mathrm{cl}_\gamma A) = A \backslash \mathrm{int}\,(\mathrm{cl}\, A)$. Then $(X \backslash \mathrm{cl}\, A) \cup \{x\} \in \mathrm{SO}(\mathcal{T}) \subset \mathrm{S\Gamma PO}(\mathcal{T}) \cap \mathrm{SO}(\mathcal{T}_\gamma)$ by 1.16, and so $((X \backslash \mathrm{cl}\, A) \cup \{x\}) \cap A = \{x\} \in \mathrm{SO}(\mathcal{T}_\gamma) \cap \Gamma\mathrm{PO}(\mathcal{T})$. Hence $\{x\} \in \mathcal{T}_\gamma$, a contradiction. Therefore $A \in \Gamma\mathrm{PO}(\mathcal{T})$. □

Proposition 4.8. *Let* $(X, \mathcal{T})$ *be a space. Then* $\mathcal{T}_i = \mathcal{T}_g$ *.*

Proof. $\mathcal{T}_i \subset \mathcal{T}_g$ follows from 4.7 and 3.4. To prove the converse, let $A \in \mathcal{T}_g$ and $B \in \mathrm{S\Gamma PO}(\mathcal{T}) \cap \mathrm{SO}(\mathcal{T}_\gamma)$. Then $A \cap B \in \mathrm{S\Gamma PO}(\mathcal{T})$ by 4.6. On the other hand, $A \in \mathcal{T}_{\gamma\alpha}$ by 3.3, thus $A \cap B \in \mathrm{SO}(\mathcal{T}_\gamma)$. Hence $A \in \mathcal{T}_i$. □

(E) $A \subset$ **cl (int$_\gamma$cl$_\gamma$**A**)**

By 1.12(2), the class of sets satisfying this condition coincides with $\mathrm{SPO}(\mathcal{T}_\gamma)$ and thus the generated topology is $\mathcal{T}_{\gamma\alpha}$.

(F) $A \subset$ **int (cl (int$_\gamma$**A**))**

It follows easily that the class of sets satisfying this condition coincides with $\mathrm{PO}(\mathcal{T}) \cap \mathrm{SO}(\mathcal{T}_\gamma)$. Denote the topology generated by this class by $\mathcal{T}_j$, that is $\mathcal{T}_j = \{G \in \mathrm{PO}(\mathcal{T}) \cap \mathrm{SO}(\mathcal{T}_\gamma) \mid G \cap A \in \mathrm{PO}(\mathcal{T}) \cap \mathrm{SO}(\mathcal{T}_\gamma)$ whenever $A \in \mathrm{PO}(\mathcal{T}) \cap \mathrm{SO}(\mathcal{T}_\gamma)\}$. The closure and the interior of a set A in $(X, \mathcal{T}_j)$ will be denoted by $\mathrm{cl}_j A$ and $\mathrm{int}_j A$. The next statement follows immediately.

Proposition 4.9. *Let* $(X, \mathcal{T})$ *be a space. Then* $\mathcal{T}_\gamma \subset \mathcal{T}_j \subset \mathrm{PO}(\mathcal{T}) \cap \mathrm{SO}(\mathcal{T}_\gamma)$ *.*

Proposition 4.10. *Let* A *be a subset of a space* X*. Then*

(1) $int_\gamma cl_j A = int_\gamma cl_\gamma A$,

(2) $cl_\gamma int_j A = cl_\gamma int_\gamma A$.

Proof. From 4.9 and 1.2(1) we have that $\mathrm{int}_\gamma \mathrm{cl}_\gamma A = \mathrm{int}_\gamma \mathrm{scl}_\gamma A \subset \mathrm{int}_\gamma \mathrm{cl}_j A \subset \mathrm{int}_\gamma \mathrm{cl}_\gamma A$. The second equality is similarly proved. □

Corollary 4.11. *Let* A *be a subset of a space* X*. Then*

(1) $int_j cl_j A \supset int_\gamma cl_\gamma A$,

(2) $cl_j int_j A \subset cl_\gamma int_\gamma A$,

(3) $int_j cl_j int_j A \supset int_\gamma cl_\gamma int_\gamma A$,

(4) $cl_j int_j cl_j A \subset cl_\gamma int_\gamma cl_\gamma A$.

Proposition 4.12. *Let* $(X, \mathcal{T})$ *be a space. Then* $\mathcal{T}_g \subset \mathcal{T}_j$ *.*

Proof. Suppose that $A \in \mathcal{T}_g$ and $B \in \mathrm{PO}(\mathcal{T}) \cap \mathrm{SO}(\mathcal{T}_\gamma)$, that is $A \subset \mathrm{int}\,(\mathrm{cl}_\gamma \mathrm{int}_\gamma A)$ and $B \subset \mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}_\gamma B))$. Now we have that $\mathrm{cl}\,(\mathrm{int}_\gamma (A \cap B)) = \mathrm{cl}\,(\mathrm{int}_\gamma A \cap \mathrm{int}_\gamma B) = \mathrm{cl}\,(\mathrm{cl}_\gamma (\mathrm{int}_\gamma A \cap \mathrm{int}_\gamma B)) \supset \mathrm{cl}\,(\mathrm{cl}_\gamma \mathrm{int}_\gamma A \cap \mathrm{int}_\gamma B) \supset \mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}_\gamma \mathrm{int}_\gamma A \cap \mathrm{int}_\gamma B)) \supset \mathrm{int}\,(\mathrm{cl}_\gamma \mathrm{int}_\gamma A) \cap \mathrm{cl}\,(\mathrm{int}_\gamma B)$ and hence $\mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}_\gamma (A \cap B))) \supset \mathrm{int}\,(\mathrm{cl}_\gamma \mathrm{int}_\gamma A) \cap \mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}_\gamma B)) \supset A \cap B$. Therefore $A \cap B \in \mathrm{PO}(\mathcal{T}) \cap \mathrm{SO}(\mathcal{T}_\gamma)$ and thus $A \in \mathcal{T}_j$. □

Corollary 4.13. *Let* $(X,\mathcal{T})$ *be a space. Then* $\mathcal{T}_{j\alpha} = \mathcal{T}_j$.

Proof. By 1.4(1) it remains to show that $N(\mathcal{T}_j) \subset C(\mathcal{T}_j)$ and suppose that $\mathrm{int}_j\mathrm{cl}_jA = \emptyset$. Then $A \in N(\mathcal{T}_\gamma)$ by 4.11(1) and so $A \in N(\mathcal{T}_g) \subset C(\mathcal{T}_g) \subset C(\mathcal{T}_j)$ by 3.11, 3.10 and 4.12. □

From 4.11(3) and 4.13 we have

Corollary 4.14. *Let* $(X,\mathcal{T})$ *be a space. Then* $\mathcal{T}_{\gamma\alpha} \subset \mathcal{T}_j$.

Our next step is to prove the converse. By 1.4(2) it remains to show that $\mathcal{T}_j \subset PO(\mathcal{T}_\gamma)$. First we establish a simple lemma.

Lemma 4.15. *Let* A *be a set in a space* $(X,\mathcal{T})$ *such that* $A \cap cl\,(int\,(cl_\gamma A)) \in PO(\mathcal{T})$. *Then* $A \cap cl\,(int\,(cl_\gamma A)) \subset int\,(cl_\gamma A)$.

Proof. Since $\mathrm{int}\,(\mathrm{cl}_\gamma A) \in RO(\mathcal{T})$ we have that $A \cap \mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}_\gamma A)) \subset \mathrm{int}\,(\mathrm{cl}\,(A \cap \mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}_\gamma A)))) \subset \mathrm{int}\,(\mathrm{cl}\,A \cap \mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}_\gamma A))) = \mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}_\gamma A))) = \mathrm{int}\,(\mathrm{cl}_\gamma A)$. □

Proposition 4.16. *Let* $(X,\mathcal{T})$ *be a space. Then* $\mathcal{T}_j \subset PO(\mathcal{T}_\gamma)$.

Proof. Let $A \in \mathcal{T}_j$. We consider two cases:

(1) $\mathrm{int}\,(\mathrm{cl}_\gamma A) = \emptyset$: Suppose that $x \in A\backslash\mathrm{int}_\gamma A$ and put $B = \{x\} \cup (X\backslash\mathrm{cl}_\gamma A)$. Then $B \in D(\mathcal{T})$ and so $B \in PO(\mathcal{T})$. On the other hand, applying 1.13(1) we have that $\mathrm{cl}_\gamma(X\backslash\mathrm{cl}_\gamma A) = X\backslash\mathrm{int}_\gamma\mathrm{cl}_\gamma A = X\backslash(\mathrm{int}_\gamma A \cup \mathrm{int}\,(\mathrm{cl}_\gamma A)) = X\backslash\mathrm{int}_\gamma A \ni x$ and thus $B \in SO(\mathcal{T}_\gamma)$. Hence $A \cap B = \{x\} \in PO(\mathcal{T}) \cap SO(\mathcal{T}_\gamma)$ and so $\{x\} \in \mathcal{T}_\gamma$, a contradiction. Therefore $A \in \mathcal{T}_\gamma$. Moreover, $\mathrm{int}_\gamma\mathrm{cl}_\gamma A = \mathrm{int}_\gamma A = A$ and thus $A \in RO(\mathcal{T}_\gamma)$.

(2) $\mathrm{int}\,(\mathrm{cl}_\gamma A) \neq \emptyset$: Then $A = (A \cap \mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}_\gamma A))) \cup B$ where $B = A\backslash\mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}_\gamma A))$. Suppose that $x \in A\backslash\mathrm{int}_\gamma A$. First we notice that $B \in \mathcal{T}_j$, $\mathrm{int}\,(\mathrm{cl}_\gamma B) = \emptyset$ and so $B \in RO(\mathcal{T}_\gamma)$ by (1). On the other hand, $\mathrm{int}\,B = \emptyset$ implies that $B \in PC(\mathcal{T}) \cap SC(\mathcal{T}_\gamma)$ and thus $A\backslash B = A\backslash(A\backslash\mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}_\gamma A))) = A \cap \mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}_\gamma A)) \in PO(\mathcal{T}) \cap SO(\mathcal{T}_\gamma)$. Clearly, $x \notin B$ that is $x \in A \cap \mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}_\gamma A))$, and thus $x \in \mathrm{int}\,(\mathrm{cl}_\gamma A)$ by 4.15. Therefore $x \in \mathrm{int}_\gamma(\mathrm{cl}_\gamma A)$ and so $A \in PO(\mathcal{T}_\gamma)$. □

Corollary 4.17. *Let* $(X,\mathcal{T})$ *be a space. Then* $\mathcal{T}_j = \mathcal{T}_{\gamma\alpha}$.

(G) $A \subset$ **int (cl (int**$_\gamma$**cl**$_\gamma A$**))**

It follows from 1.12(2) that the class of sets satisfying this condition coincides with $PO(\mathcal{T}) \cap SPO(\mathcal{T}_\gamma)$. Denote the topology generated by this class by $\mathcal{T}_k$, that is $\mathcal{T}_k = \{G \in PO(\mathcal{T}) \cap SPO(\mathcal{T}_\gamma) \mid G \cap A \in PO(\mathcal{T}) \cap SPO(\mathcal{T}_\gamma)$ whenever $A \in PO(\mathcal{T}) \cap SPO(\mathcal{T}_\gamma)\}$. It is clear that $\mathcal{T}_\gamma \subset \mathcal{T}_k \subset PO(\mathcal{T}) \cap SPO(\mathcal{T}_\gamma)$.

Proposition 4.18. *Let* $(X,\mathcal{T})$ *be a space,* $A \in PO(\mathcal{T}) \cap SO(\mathcal{T}_\gamma)$ *and* $B \in \Gamma PO(\mathcal{T})$. *Then* $A \cap B \in PO(\mathcal{T}) \cap SPO(\mathcal{T}_\gamma)$.

Proof. Let $A \subset \mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}_\gamma A))$ and $B \subset \mathrm{int}\,(\mathrm{cl}_\gamma B)$. Then $\mathrm{cl}_\gamma(A \cap B) \supset \mathrm{cl}_\gamma(\mathrm{int}_\gamma A \cap B) \supset \mathrm{int}_\gamma A \cap \mathrm{cl}_\gamma B$, and hence $\mathrm{int}_\gamma\mathrm{cl}_\gamma(A \cap B) \supset \mathrm{int}_\gamma A \cap \mathrm{int}_\gamma\mathrm{cl}_\gamma B \supset \mathrm{int}_\gamma A \cap \mathrm{int}\,(\mathrm{cl}_\gamma B)$. This implies $\mathrm{cl}\,(\mathrm{int}_\gamma\mathrm{cl}_\gamma(A \cap B)) \supset \mathrm{cl}\,(\mathrm{int}_\gamma A \cap \mathrm{int}\,(\mathrm{cl}_\gamma B)) \supset \mathrm{cl}\,(\mathrm{int}_\gamma A) \cap \mathrm{int}\,(\mathrm{cl}_\gamma B)$ and finally $\mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}_\gamma\mathrm{cl}_\gamma(A \cap B))) \supset \mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}_\gamma A)) \cap \mathrm{int}\,(\mathrm{cl}_\gamma B) \supset A \cap B$, that is $A \cap B \in PO(\mathcal{T}) \cap SPO(\mathcal{T}_\gamma)$. □

Proposition 4.19. *Let* $(X,\mathcal{T})$ *be a space. Then* $A \in PO(\mathcal{T}) \cap SPO(\mathcal{T}_\gamma)$ *if and only if* $A = B \cap C$ *with* $B \in PO(\mathcal{T}) \cap SO(\mathcal{T}_\gamma)$ *and* $C \in \Gamma PO(\mathcal{T})$.

Proof. Let $A \in PO(\mathcal{T}) \cap SPO(\mathcal{T}_\gamma)$ and put $B = \mathrm{cl}_\gamma A \cap \mathrm{int}\,(\mathrm{cl}\,A)$, $C = A \cup (X\backslash\mathrm{cl}_\gamma A)$. It is not difficult to see that $B \in PO(\mathcal{T}) \cap SO(\mathcal{T}_\gamma)$, $C \in \Gamma PO(\mathcal{T})$ and $A = B \cap C$. □

Proposition 4.20. *Let* $(X,\mathcal{T})$ *be a space. Then* $\mathcal{T}_{\gamma\alpha} \subset \mathcal{T}_k$.

Proof. Suppose that $G \in \mathcal{T}_{\gamma\alpha}$ and let $A \in \mathrm{PO}(\mathcal{T}) \cap \mathrm{SPO}(\mathcal{T}_\gamma)$. Then by 4.19, $A = B \cap C$ with $B \in \mathrm{PO}(\mathcal{T}) \cap \mathrm{SO}(\mathcal{T}_\gamma)$ and $C \in \Gamma\mathrm{PO}(\mathcal{T})$. It follows from 4.17 that $G \cap B \in \mathrm{PO}(\mathcal{T}) \cap \mathrm{SO}(\mathcal{T}_\gamma)$ and hence $G \cap A = (G \cap B) \cap C \in \mathrm{PO}(\mathcal{T}) \cap \mathrm{SPO}(\mathcal{T}_\gamma)$ by 4.18. Thus $G \in \mathcal{T}_k$. □

Proposition 4.21. *Let* $(X, \mathcal{T})$ *be a space. Then* $\mathcal{T}_k \subset \mathrm{SO}(\mathcal{T}_\gamma)$.

Proof. Let $A \in \mathcal{T}_k$ and suppose that $x \in B = A \backslash \mathrm{cl}_\gamma \mathrm{int}_\gamma A$. Then $B \in \mathcal{T}_k$, $\mathrm{int}_\gamma(B\backslash\{x\})= \mathrm{int}\,(B\backslash\{x\}) = \emptyset$ and thus $B\backslash\{x\} \in \mathrm{PC}(\mathcal{T}) \cap \mathrm{SPC}(\mathcal{T}_\gamma)$. Hence $\{x\} = B\backslash(B\backslash\{x\}) \in \mathrm{PO}(\mathcal{T}) \cap \mathrm{SPO}(\mathcal{T}_\gamma)$ and thus $\{x\} \in \mathcal{T}_\gamma$, a contradiction. Therefore $B = \emptyset$, that is $A \in \mathrm{SO}(\mathcal{T}_\gamma)$. □

Lemma 4.22. ([9]) *Let* $(X, \mathcal{T})$ *be a space and* A, $B \in \mathrm{SO}(\mathcal{T})$. *Then* $A \cap B \in \mathrm{SO}(\mathcal{T})$ *if and only if* $A \cap B \in \mathrm{SPO}(\mathcal{T})$.

Proposition 4.23. *Let* $(X, \mathcal{T})$ *be a space. Then* $\mathcal{T}_k = \mathcal{T}_{\gamma\alpha}$.

Proof. Suppose that $A \in \mathcal{T}_k$ and let $B \in \mathrm{PO}(\mathcal{T}) \cap \mathrm{SO}(\mathcal{T}_\gamma)$. Then $A \cap B \in \mathrm{PO}(\mathcal{T}) \cap \mathrm{SPO}(\mathcal{T}_\gamma)$ and so $A \cap B \in \mathrm{PO}(\mathcal{T}) \cap \mathrm{SO}(\mathcal{T}_\gamma)$ by 4.21 and 4.22. Thus $A \in \mathcal{T}_{\gamma\alpha}$ by 4.17 □

At the end of our quest for new topologies let us make a brief recapitulation.

1) Using only the closure and the interior operators in $(X, \mathcal{T})$ we obtain in the natural way four classes of sets which are larger than $\mathcal{T}$ and closed under forming arbitrary unions (Definition 1). One among them, $\mathcal{T}_\alpha$, turns out to be a topology on X.

2) The remaining three classes $\mathrm{SO}(\mathcal{T})$, $\mathrm{PO}(\mathcal{T})$ and $\mathrm{SPO}(\mathcal{T})$ generate a topology by means of the operation $\mathcal{T}(\mathcal{A}) = \{G \in \mathcal{A} \mid G \cap A \in \mathcal{A}$ whenever $A \in \mathcal{A}\}$. In that way we obtain one new topology, $\mathcal{T}_\gamma$. In the next step we apply the operation $\mathcal{T}(\mathcal{A})$ to $\mathrm{SO}(\mathcal{T}_\gamma)$, $\mathrm{PO}(\mathcal{T}_\gamma)$ and $\mathrm{SPO}(\mathcal{T}_\gamma)$ and obtain one new topology which turns out to be $\mathcal{T}_{\gamma\alpha}$.

3) Finally, we introduce new classes of generalized open sets by means of the closure and the interior operators of the topologies $\mathcal{T}$ and $\mathcal{T}_\gamma$. In this way we obtain eight new classes. Four of these classes generate the topology $\mathcal{T}_{\gamma\alpha}$, but the rest give us a new topology $\mathcal{T}_g$. Applying the operation $\mathcal{T}(\mathcal{A})$ to $\mathrm{SO}(\mathcal{T}_g)$, $\mathrm{PO}(\mathcal{T}_g)$ and $\mathrm{SPO}(\mathcal{T}_g)$ we do not obtain any new topology.

4) Now the question arises as to whether we can obtain a new topology by using the other combinations of operators in $\mathcal{T}$, $\mathcal{T}_\alpha$, $\mathcal{T}_\gamma$, $\mathcal{T}_{\gamma\alpha}$ and $\mathcal{T}_g$.

(a) By 1.3 $\mathcal{T}$ and $\mathcal{T}_\alpha$ give us the same as $\mathcal{T}$.

(b) By Lemma 3.16, $\mathrm{int}\,(\mathrm{cl}_{\gamma\alpha} A) = \mathrm{int}\,(\mathrm{cl}_\gamma A)$ while $\mathrm{cl}_{\gamma\alpha}\mathrm{int}\,A = \mathrm{cl}\,(\mathrm{int}\,A)$ follows from 1.2(2) and 1.9(2). Hence $\mathcal{T}$ and $\mathcal{T}_{\gamma\alpha}$ give us the same as $\mathcal{T}$ and $\mathcal{T}_\gamma$.

(c) It follows easily from 2.8 that $\mathrm{int}_g \mathrm{cl}\,A = \mathrm{int}\,\mathrm{cl}\,A$ while by 3.5(2) we have that $\mathrm{cl}\,(\mathrm{int}_g A) = \mathrm{cl}_g \mathrm{int}_g A$. Thus $\mathcal{T}$ and $\mathcal{T}_g$ give us no new topology.

(d) By 1.9 we have that $\mathcal{T}_\alpha$ and $\mathcal{T}_\gamma$ give us the same as $\mathcal{T}$ and $\mathcal{T}_\gamma$.

(e) Similarly, $\mathcal{T}_\alpha$ and $\mathcal{T}_{\gamma\alpha}$ give us the same as $\mathcal{T}$ and $\mathcal{T}_\gamma$.

(f) Since $\mathrm{int}_\alpha \mathrm{cl}_g A = \mathrm{int}\,(\mathrm{cl}_g A) = \mathrm{int}_g \mathrm{cl}_g A$ and $\mathrm{cl}_g \mathrm{int}_\alpha A = \mathrm{cl}\,(\mathrm{int}\,A)$, $\mathcal{T}_\alpha$ and $\mathcal{T}_g$ give us the same as $\mathcal{T}$ and $\mathcal{T}_g$.

(g) From (a) it is clear that $\mathcal{T}_\gamma$ and $\mathcal{T}_{\gamma\alpha}$ give us no new topology.

(h) It follows from 2.6 that $\mathrm{int}_\gamma \mathrm{cl}_g A = \mathrm{int}\,(\mathrm{cl}_g A)$ while 2.8 implies that $\mathrm{cl}_g \mathrm{int}_\gamma A = \mathrm{cl}\,(\mathrm{int}_g A)$. Hence $\mathcal{T}_\gamma$ and $\mathcal{T}_g$ give us no new topology.

(i) It is not difficult to see that $\mathrm{cl}_{\gamma\alpha}\mathrm{int}_g A = \mathrm{cl}\,(\mathrm{int}_g A)$ and $\mathrm{int}_g \mathrm{cl}_{\gamma\alpha} A = \mathrm{int}\,(\mathrm{cl}_\gamma A)$. Thus $\mathcal{T}_{\gamma\alpha}$ and $\mathcal{T}_g$ give us no new topology.

Therefore it seems to me that we may answer our question in the negative.

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