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Ideal Versions of the Bolzano-Weierstrass Property

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Abstract. Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω, we say that a space X has $(\mathcal{I}, \mathcal{J})$-*BW* property if every sequence in X contains a $\mathcal{J}$-converging subsequence indexed by an $\mathcal{I}$-positive set. This is a common generalization of BW-like properties types. By modifying some classic notions, we obtain some characterizations of $(\mathcal{I}, \mathcal{J})$-*BW* property.

1. Introduction

We need to recall first some necessary notions in order to formulate problems we will consider in this paper. The letter ω denote the set of all natural numbers, an **ideal** on ω is a family of subsets of ω closed under taking finite unions and subsets of its elements. By *Fin* we denote the ideal of all finite subsets of ω. If not explicitly said we assume that all considered ideals are proper and contain *Fin*.

Let $\mathcal{I}$ be an ideal on ω, and X being a topological space. For sequence $\langle x_n : n \in \omega \rangle$ in X, we say that $\langle x_n : n \in \omega \rangle$ is $\mathcal{I}$**-convergent to** l if for each open neighborhood U of l,

$$\{n : x_n \notin U\} \in \mathcal{I}.$$

The notion of $\mathcal{I}$-convergence is a generalization of the classical one. It was first considered by Steinhaus and Fast [3] in the case of the ideal of sets of statistical density 0:

$$\mathcal{I}_d = \{A \subset \omega : \limsup_{n\to\infty} \tfrac{|A\cap n|}{n} = 0\}.$$

By an $\mathcal{I}$-subsequence of $\langle x_n : n \in \omega \rangle$ we means $\langle x_n : n \in A \rangle$ for some $A \notin \mathcal{I}$. Filipów, Mrożek, Recław and Szuca introduced the following notions ([5], Subsection 2.3):

Definition 1.1. Let $\mathcal{I}$ be an ideal on ω, X being a topological space.

- $(X, \mathcal{I})$ satisfies *BW* if every sequence in X has $\mathcal{I}$-convergent $\mathcal{I}$-subsequence;
- $(X, \mathcal{I})$ satisfies *FinBW* if every sequence in X has convergent $\mathcal{I}$-subsequence;

If $([0, 1], \mathcal{I})$ satisfies *BW* (*FinBW*), we will omit the underlying space $[0, 1]$ and say $\mathcal{I}$ is satisfying *BW* (*FinBW*).

These notions involve two ideals: $\mathcal{I}$ and *Fin*. We are interested in the question how about if we replace *Fin* by another ideal $\mathcal{J}$? Here is the key definition, which is a common generalization of these types.

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Definition 1.2. Let $\mathcal{I}, \mathcal{J}$ be ideals on ω, X being a topological space. We say that X has $(\mathcal{I}, \mathcal{J})$-*BW* property if every sequence in X has $\mathcal{J}$-convergent $\mathcal{I}$-subsequence.

Remark 1.3. It is worthy to point out that if $\mathcal{I} \nsubseteq \mathcal{J}$, then for arbitrary space X, it has $(\mathcal{J}, \mathcal{I})$-*BW* property. Indeed, picking $A \in \mathcal{I} \setminus \mathcal{J}$, A can deal with any sequence in X.

Our considerations are based on the works of Filipów-Mrożek- Recław-Szuca in [4], [5]. In particular, we are motivated by the following results:

$*$: $\mathcal{I}$ satisfies *BW* if, and only if there is no countable $\mathcal{I}$-splitting family.

$**$: If $\mathcal{I}$ is a weak Q-point, then the following conditions are equivalent:

(1) $\mathcal{I}$ is Ramsey;

(2) $\mathcal{I}$ is Mon;

(3) $\mathcal{I}$ is *FinBW*.

In Section 2, some basic notions will be introduced. In Section 3, we generalize the term $*$. In particular, we show that if there is no countable $(\mathcal{I}, \mathcal{J})$-splitting family, then $[0, 1]$ satisfies $(\mathcal{I}, \mathcal{J})$-BW, and this implies that there is no countable $(\mathcal{J}, \mathcal{I})$-splitting family. In Section 4, we introduce *Ramsey**-property, *Mon**-property for pairs $(\mathcal{I}, \mathcal{J})$ and use them to characterize the $(\mathcal{I}, \mathcal{J})$-BW property. In addition, a slightly general ω-diagonalizable property is introduced, and we check its relation among density, *Ramsey** and $(\mathcal{I}, \mathcal{J})$-BW property in this section.

2. Preliminaries

Let $\mathcal{I}$ be an ideal on ω. If $A \notin \mathcal{I}$, we say that A is $\mathcal{I}$-positive. In the next, we will use the following notations:

- $\mathcal{I}^+ = \{A \subseteq \omega : A \notin \mathcal{I}\}$;
- $\mathcal{I}^* = \{A \subseteq \omega : \omega \setminus A \in \mathcal{I}\}$;
- $\mathcal{I}|A = \{I \cap A : I \in \mathcal{I}\}$, for each $A \in \mathcal{I}^+$,

2.1. *Orderings*

Let $\mathcal{I}, \mathcal{J}$ be ideals on ω. For a map $\varphi : \omega \to \omega$, the image of $\mathcal{J}$ is defined by

$$\varphi(\mathcal{J}) = \{A \subseteq \omega : \varphi^{-1}(A) \in \mathcal{J}\}.$$

Clearly, $\varphi(\mathcal{J})$ is closed under subsets and finite unions and $\omega \notin \varphi(\mathcal{J})$. Moreover, if φ is finite-to-one then $\varphi(\mathcal{J})$ is an ideal. Let's recall the following notions:

Definition 2.1. Let $\mathcal{I}, \mathcal{J}$ be ideals on ω,

- $\mathcal{I} \leq_K \mathcal{J}$ if there is a function $\varphi : \omega \to \omega$ such that $\mathcal{I} \subseteq \varphi(\mathcal{J})$, i.e, $\varphi^{-1}(A) \in \mathcal{J}$ for any $A \in \mathcal{I}$ [11];
- $\mathcal{I} \leq_{KB} \mathcal{J}$ if there is a finite-to-one function $\varphi : \omega \to \omega$ such that $\mathcal{I} \leq_K \mathcal{J}$ [11];
- $\mathcal{I} \leq_{RB} \mathcal{J}$ if there is a finite-to-one function $\varphi : \omega \to \omega$ such that $A \in \mathcal{I}$ if, and only if $\varphi^{-1}(A) \in \mathcal{J}$ for every $A \subset \omega$ [9];
- $\mathcal{I} \cong \mathcal{J}$ if there is a bijection $\varphi : \omega \to \omega$ such that $A \in \mathcal{I}$ if, and only if $\varphi^{-1}(A) \in \mathcal{J}$ for every $A \subset \omega$.

The (pre)orderings on ideals, in some sense, are significant in describing some properties of ideals.

2.2. $\mathcal{A}$-dense

Let $\mathcal{I}$ be an ideal on ω. Recall that $\mathcal{I}$ is dense (or tall) if every infinite set $A \subseteq \omega$ contains an infinite subset B that belongs to $\mathcal{I}$.

Definition 2.2. Let $\mathcal{A}, \mathcal{B}$ be sets of subsets of ω. We say that $\mathcal{B}$ is $\mathcal{A}$-dense if for each $A \in \mathcal{A}$, there exists an infinite $B \subseteq A$ such that $B \in \mathcal{B}$.

Evidently, $\mathcal{I}$ being $[\omega]^\omega$-dense coincides with $\mathcal{I}$ being dense. In addition, for any ideal $\mathcal{I}$, $\mathcal{I}^+$ is $[\omega]^\omega$-dense if, and only if $\mathcal{I} = Fin$.

Lots of combinatorial properties of ideals are related to the general density above, we present here some examples.

Example 2.3. Let $\mathcal{I}$ be an ideal on ω with $\mathcal{I} \ncong Fin$. If $\mathcal{I} \ncong Fin \oplus \mathcal{P}(\omega)$, then $\mathcal{I}$ is $\mathcal{I}^*$-dense, where $Fin \oplus \mathcal{P}(\omega)$ is an ideal on $\{0,1\} \times \omega$ defined by

$$Fin \oplus \mathcal{P}(\omega) = \{A \subset \{0,1\} \times \omega : \{n \in \omega : (0,n) \in A\} \in Fin\}.$$

Example 2.4. The following notions are introduced and studied in [12]: For any ideal $\mathcal{I}$, put

$$H(\mathcal{I}) = \{A \subseteq \omega : \mathcal{I}|A \cong \mathcal{I}\}.$$

It is called the *homogeneous family of the ideal* $\mathcal{I}$. An ideal $\mathcal{I}$ is *homogeneous* if $\mathcal{I}^+ = H(\mathcal{I})$; $\mathcal{I}$ is *anti-homogeneous* if $H(\mathcal{I}) = \mathcal{I}^*$. These notions can be reformulated in terms of density as follows:

(1) $\mathcal{I}$ is homogeneous if, and only if $H(\mathcal{I})$ is $\mathcal{I}^+$-dense.

(2) If $\mathcal{I} \ncong Fin \oplus \mathcal{P}(\omega)$, then $\mathcal{I}$ is anti-homogeneous if, and only if $\mathcal{I}^*$ is $H(\mathcal{I})$-dense

The assertion (1) is Corollary 2.2 in [12]. Both proofs rely on the simple fact that if $\mathcal{A}$ is $\mathcal{B}$-dense and $\mathcal{A}$ is closed under supersets (i.e, if $A \subseteq B$ and $A \in \mathcal{A}$, then $B \in \mathcal{A}$), then $\mathcal{B} \subseteq \mathcal{A}$.

Remark 2.5. Let $\mathcal{I}$ be an ideal on ω,

(1) $\mathcal{I}$ is $\mathcal{A}$-dense if and only if $\forall A \in \mathcal{A}$, $\mathcal{I}|A \neq Fin(A)$, where $Fin(A)$ denotes the set of all finite subsets of A.

(2) If $\mathcal{I}$ is dense and $\mathcal{I} \leq_K \mathcal{J}$, then $\mathcal{J}$ is dense.

(3) $H(\mathcal{I})$ is closed under supersets ([12], Theorem 2.1).

2.3. Q-Ideal and Selectivity

Let's recall some combinatorial properties of ideals. Let $\mathcal{I}$ be an ideal on ω,

- $\mathcal{I}$ is local Q if for every partition $\{A_n : n \in \omega\} \subset Fin$ of ω, there exists $A \in \mathcal{I}^+$ such that $|A \cap A_n| \leq 1$ for each $n \in \omega$;
- $\mathcal{I}$ is locally selective if for every partition $\{A_n : n \in \omega\} \subset \mathcal{I}$ of ω, there exists $A \in \mathcal{I}^+$ such that $|A \cap A_n| \leq 1$ for each $n \in \omega$.
- $\mathcal{I}$ is weak Q if for every $A \in \mathcal{I}^+$, $\mathcal{I}|A$ is local Q.
- $\mathcal{I}$ is weakly selective if for every $A \in \mathcal{I}^+$, $\mathcal{I}|A$ is locally selective.

3. $(\mathcal{I},\mathcal{J})$-Splitting Family and $(\mathcal{I},\mathcal{J})$-BW

Let $\mathcal{S} \subseteq [\omega]^\omega$, and $\mathcal{I}$ being an ideal on ω. Recall that a family $\mathcal{S}$ is $\mathcal{I}$-splitting if for every $A \in \mathcal{I}^+$ there exists $S \in \mathcal{S}$ such that $A \cap S \in \mathcal{I}^+$ and $A \setminus S \in \mathcal{I}^+$ [5].

Definition 3.1. Let $\mathcal{I}, \mathcal{J}$ be ideals on ω, and $\mathcal{S} \subset [\omega]^\omega$. We say that $\mathcal{S}$ is an $(\mathcal{I},\mathcal{J})$-splitting family if for every $A \in \mathcal{I}^+$ there exists $X \in \mathcal{S}$ such that both of $A \cap X$ and $A \setminus X$ belong to $\mathcal{J}^+$.

Evidently, when $\mathcal{I}$ is equal to $\mathcal{J}$, the $(\mathcal{I},\mathcal{J})$-splitting family coincides with the $\mathcal{I}$-splitting family mentioned above.

Let $\mathfrak{s}(\mathcal{I},\mathcal{J})$ be the smallest cardinality of an $(\mathcal{I},\mathcal{J})$-splitting family. It is easy to see that the $\mathfrak{s}(Fin, Fin)$ is just the *splitting number* $\mathfrak{s}$ introduced in [1], and $\mathfrak{s}(\mathcal{I},\mathcal{I})$ is just $\mathfrak{s}(\mathcal{I})$ defined in [4].

In terms of cardinality, the assertion $*$ mentioned in Section 1 can be reformulated as the follows: $\mathcal{I}$ satisfies *BW* if, and only if $\mathfrak{s}(\mathcal{I}) > \omega$.

Proposition 3.2. *Let $\mathcal{I}, \mathcal{J}$ be ideals on ω with $\mathcal{I} \subseteq \mathcal{J}$. Then $\mathfrak{s}(\mathcal{I},\mathcal{J}) \geq \mathfrak{s}(\mathcal{J},\mathcal{I})$.*

Let $r \in \omega$, $s \in r^n$ and $i \in \{0, \cdots, r-1\}$, by $s \frown i$ we mean the sequence of length $n+1$ (write $lh(s) = n+1$) which extends s by i. If $x \in r^\omega$ and $n \in \omega$, $x|n$ denotes the initial segment $x|n = \langle x(0), x(1), \cdots, x(n-1)\rangle$.

Now, we are in the position to introduce the main tool, which is a generalization of $\mathcal{I}$-small set used in [5]:

Definition 3.3. Let $\mathcal{I}, \mathcal{J}$ be ideals on ω. $A \subset \omega$ is called an $(\mathcal{I},\mathcal{J})$-small set if there exists $r \in \omega$, and exists a family $\{A_s : s \in r^{<\omega}\}$ such that for all $s \in r^{<\omega}$, we have

S_1 $A_\emptyset = A$,

S_2 $A_s = A_{s\frown 0} \cup \cdots \cup A_{s\frown(r-1)}$,

S_3 $A_{s\frown i} \cap A_{s\frown j} = \emptyset$ for every $i \neq j$,

S_4 for every $b \in r^\omega$, every $X \subset \omega$, if $X \setminus A_{b|n} \in \mathcal{I}$ for each $n \in \omega$, then $X \in \mathcal{J}$.

Let $\mathcal{S}_{(\mathcal{I},\mathcal{J})}$ denote all $(\mathcal{I},\mathcal{J})$-small sets in $\mathcal{P}(\omega)$. Note that $\mathcal{S}_{(\mathcal{I},\mathcal{J})} \neq \emptyset$ if, and only if $\mathcal{I} \subseteq \mathcal{J} \subseteq \mathcal{S}_{(\mathcal{I},\mathcal{J})}$.

The following result can be viewed as a generalization of Proposition 2.9 in [4].

Theorem 3.4. *$\omega \notin \mathcal{S}_{(\mathcal{I},\mathcal{J})}$ if, and only if* $[0,1]$ *satisfies* $(\mathcal{J},\mathcal{I})$*-BW.*

Proof. Thanks to the simple fact that $(\mathcal{J},\mathcal{I})$-BW property is preserved for closed subsets and continuous images, $[0,1]$ has $(\mathcal{J},\mathcal{I})$-BW property if, and only if 2^ω has $(\mathcal{J},\mathcal{I})$-BW property. Thus, we consider the Cantor space 2^ω instead of $[0,1]$.

$\Rightarrow$ Assume that $\omega \notin \mathcal{S}_{(\mathcal{I},\mathcal{J})}$. For every sequence $\langle x_n : n \in \omega\rangle$ in 2^ω, every $s \in 2^{<\omega}$, put

$$A_s = \{n : s \subset x_n\}.$$

Then $\{A_s : s \in 2^{<\omega}\}$ satisfies $S_1 - S_3$. Since $\omega \notin \mathcal{S}_{(\mathcal{I},\mathcal{J})}$, by the condition S_4, there exists $X \notin \mathcal{J}$ and $b \in 2^\omega$ such that $X \setminus A_{b|n} \in \mathcal{I}$ for each $n \in \omega$. Then $\langle x_n : n \in X\rangle$ is $\mathcal{I}$-convergent to b.

$\Leftarrow$ For the sake of contradiction, we may suppose that $\omega \in \mathcal{S}_{(\mathcal{I},\mathcal{J})}$. So there exists $r \in \omega$, $\{A_s : s \in r^{<\omega}\}$ such that the conditions S_1-S_4 are fulfilled. Note that for each $n \in \omega$, there is exactly one $x_n \in 2^\omega$ such that $n \in A_{x_n|l}$ for each $l \in \omega$. Then we obtain a sequence $\langle x_n : n \in \omega\rangle$ in 2^ω. Since 2^ω satisfies $(\mathcal{J},\mathcal{I})$-BW, the sequence has an $\mathcal{I}$-convergent $\mathcal{J}$-subsequence, namely, there is a $x \in 2^\omega$ and $X \subseteq \omega$ with $X \in \mathcal{J}^+$ such that $\langle x_n : n \in X\rangle$ is $\mathcal{I}$-convergent to x. Since for each $l \in \omega$

$$X \setminus A_{x|l} \subseteq \{n \in X : |x - x_n| \geq \tfrac{1}{2^l}\} \in \mathcal{I}.$$

By the condition S_4, $X \in \mathcal{J}$, but this contradicts the fact that $X \in \mathcal{J}^+$. Therefore, we complete the proof. □

Theorem 3.5. *Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω with $\mathcal{J} \subseteq \mathcal{I}$. In the following list of conditions each implies the next:*

(1) $\mathfrak{s}(\mathcal{I}, \mathcal{J}) > \omega$.

(2) $[0, 1]$ *satisfies* $(\mathcal{I}, \mathcal{J})$*-BW.*

(3) $\mathfrak{s}(\mathcal{J}, \mathcal{I}) > \omega$.

Proof. (1) ⇒ (2) Suppose that $[0, 1]$ does not have $(\mathcal{I}, \mathcal{J})$*-BW*. By Theorem 3.4, ω is a $(\mathcal{J}, \mathcal{I})$-small set. We may assume that there exists a $r \in \omega$, and a family $\{A_s : s \in r^{<\omega}\}$ such that the conditions $S_1 - S_3$ are fulfilled. In what follows we will show that $\{A_s : s \in r^{<\omega}\}$ is an $(\mathcal{I}, \mathcal{J})$-splitting family. For the sake of contradiction, suppose that there is $X \in \mathcal{I}^+$ such that for every $s \in r^{<\omega}$ either $X \cap A_s \in \mathcal{J}$ or $X \setminus A_s \in \mathcal{J}$. Put

$$T = \{s \in r^{<\omega} : X \setminus A_s \in \mathcal{J}\}.$$

Then T is a tree on $\{0, \cdots, r-1\}$ with finite branches for every level. In order to see that T is an infinite tree, we need the following Claim:

Claim 3.6. *For any $n \in \omega$, there is $s \in r^n$ such that $X \setminus A_s \in \mathcal{J}$.*

Proof. Suppose that there exists $n \in \omega$ such that for every $s \in r^n$, $X \setminus A_s \in \mathcal{J}^+$, that is, $X \cap A_s \in \mathcal{J}$ for all $s \in r^n$. Note that $\omega = \bigcup_{s \in r^n} A_s$, so

$$X = \bigcup_{s \in r^n} (X \cap A_s) \in \mathcal{J}.$$

This contradicts the assumption that for every $s \in r^n$, $X \setminus A_s \in \mathcal{J}^+$. □

Since T is an infinite tree with finite branches, by König's lemma, there exists $b \in r^\omega$ such that $X \setminus A_{b|n} \in \mathcal{J}$ for every $n \in \omega$. According to the fact that ω is an $(\mathcal{J}, \mathcal{I})$-small set we have that $X \in \mathcal{I}$. Contradiction.

(2) ⇒ (3) Suppose that $\mathfrak{s}(\mathcal{J}, \mathcal{I}) = \omega$, and $\{S_n : n \in \omega\}$ be a $(\mathcal{J}, \mathcal{I})$-splitting family. We will construct a family $\{A_s : s \in 2^{<\omega}\}$ which verifies $\omega \in \mathcal{S}_{(\mathcal{J},\mathcal{I})}$(this implies that $[0, 1]$ does not have $(\mathcal{I}, \mathcal{J})$-BW property).

First, take $A_\emptyset = \omega$, and let $n_\emptyset$ be the smallest n such that S_n splits ω. Put

$$A_0 = A_\emptyset \cap A_{n_\emptyset}; A_1 = A_\emptyset \setminus A_{n_\emptyset}.$$

Then $A_0 \in \mathcal{I}^+$ and $A_1 \in \mathcal{I}^+$.

Suppose that we have already constructed A_s for all $s \in 2^n$. Then for each $s \in 2^n$, $A_s \in \mathcal{I}^+$. Let n_s be the smallest n such that S_n splits A_s. Put

$$A_{s\frown 0} = A_s \cap S_{n_s}, A_{s\frown 1} = A_s \setminus S_{n_s}.$$

According to the definition of $(\mathcal{J}, \mathcal{I})$-splitting family, both of $A_{s\frown 0}$ and $A_{s\frown 1}$ are in $\mathcal{I}^+$. This allows us to keep this proceed going and then we finish our construction. Clearly, the family $\{A_s : s \in 2^{<\omega}\}$ satisfies $S_1 - S_3$, it is enough to show that this family also satisfies the condition S_4. For every $b \in 2^\omega$, every $X \subset \omega$ with $X \setminus A_{b|n} \in \mathcal{J}$ for every $n \in \omega$. Suppose that $X \in \mathcal{I}^+$. Let n_X be the smallest n such that S_n splits X. Since $X \setminus A_{b|n} \in \mathcal{J}$ for every $n \in \omega$, so S_{n_X} splits $A_{b|n}$ for every $n \in \omega$. Hence, there is $k \leq n_X$ such that $S_{n_{b|k}} = S_{n_X}$. Then either $A_{b|k+1} = A_{b|k} \cap S_{n_X}$ or $A_{b|k+1} = A_{b|k} \setminus S_{n_X}$. This implies that S_{n_X} does not split $A_{b|k+1}$, which is a contradiction. Therefore, the family $\{A_s : s \in 2^{<\omega}\}$ also satisfies S_4. □

Remark 3.7. We should point out that the assumption of $\mathcal{J} \subseteq \mathcal{I}$ in the premise is used in the implication (2) ⇒ (3).

4. Ramsey-Like and $(\mathcal{I}, \mathcal{J})$-BW

In this section, we give some characterizations of $(\mathcal{I}, \mathcal{J})$-BW in terms of *Ramsey** property and *Mon** property introduced below.

4.1. Ramsey and Mon* Properties Defined via Pair of Ideals*

Let $\mathcal{I}$ be an ideal on ω, $r \in \omega$, and $c : [\omega]^2 \to \{0, \cdots, r-1\}$ being a coloring. Recall that $A \subset \omega$ is $\mathcal{I}$-homogeneous for c if there is $k \in \{0, \cdots, r-1\}$ such that for every $a \in A$,

$$\{b \in A : c(\{a, b\}) \neq k\} \in \mathcal{I}.$$

Definition 4.1. ([4]) Let $\mathcal{I}$ be an ideal on ω. $\mathcal{I}$ is *Ramsey** if for every finite coloring of $[\omega]^2$ there exists an $\mathcal{I}$-homogeneous $A \in \mathcal{I}^+$.

Definition 4.2. Let $\mathcal{I}, \mathcal{J}$ be ideals on ω. We say that the pair $(\mathcal{I}, \mathcal{J})$ is *Ramsey** if for every finite coloring of $[\omega]^2$ there exists $A \in \mathcal{I}^+$ that is $\mathcal{J}$-homogeneous.

When $\mathcal{I} = \mathcal{J}$ we say that $\mathcal{I}$ has *Ramsey** instead of $(\mathcal{I}, \mathcal{I})$ having *Ramsey**. It is not hard to see that for any ideals $\mathcal{I}, \mathcal{J}$ on ω, if $\mathcal{I} \not\subset \mathcal{J}$, then the pair $(\mathcal{J}, \mathcal{I})$ is *Ramsey**. Indeed, picking $A \in \mathcal{I} \setminus \mathcal{J}$, we have that for every finite coloring c of $[\omega]^2$, A is $\mathcal{I}$-homogeneous for c.

Let $\mathcal{I}$ be an ideal on ω. Recall that a sequence $\langle x_n : n \in A\rangle$ in $[0, 1]$ is $\mathcal{I}$-increasing if for every $N \in A$

$$\{n \in A : x_N \geq x_n\} \in \mathcal{I}.$$

Analogously, we can define $\mathcal{I}$-decreasing, $\mathcal{I}$-nonincreasing and $\mathcal{I}$-nondecreasing sequences. A sequence $\langle x_n : n \in \omega\rangle$ in $[0, 1]$ is $\mathcal{I}$-monotone if it is $\mathcal{I}$-nonincreasing or $\mathcal{I}$-nondecreasing.

Definition 4.3. ([4]) Let $\mathcal{I}$ be an ideal on ω, we say that $\mathcal{I}$ is *Mon** if for every sequence $\langle x_n : n \in \omega\rangle$ in $[0, 1]$ there exists $A \in \mathcal{I}^+$ such that $\langle x_n : n \in A\rangle$ is $\mathcal{I}$-monotone.

Remark 4.4. The *Mon** property of $\mathcal{I}$ is a generalization of the *Mon* property which says that for every infinite sequence of real numbers there exists a monotone subsequence which is indexed by some member of $\mathcal{I}^+$. It has been showed that *Mon* implies *local selectivity* ([4], Lemma 3.9), but we point out that *Mon** does not necessary imply *local selectivity*, and the ideal $\mathcal{ED}$ is a counterexample, where

$$\mathcal{ED} = \{A \subseteq \omega \times \omega : (\exists m, n \in \omega)(\forall k \geq n)(|A_{(k)}| \leq m)\}.$$

To see this, note first that $\mathcal{ED} \leq_K \mathcal{I}$ if and only if $\mathcal{I}$ is not *local selective* (p. 51, [10]). On the other hand, $\mathcal{ED}$ is an F_σ-ideal, and every F_σ-ideal satisfies *FinBW* ([5], Proposition 3.4), then $\mathcal{ED}$ satisfies *FinBW*, which implies *Mon** ([4], Theorem 4.3).

Definition 4.5. Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω. We say that the pair $(\mathcal{I}, \mathcal{J})$ is *Mon** if every sequence in $[0, 1]$ contains a $\mathcal{J}$-monotone $\mathcal{I}$-subsequence. That is, for every sequence $\langle x_n : n \in \omega\rangle$ in $[0, 1]$, there exists $A \in \mathcal{I}^+$ such that $\langle x_n : n \in A\rangle$ is $\mathcal{J}$-monotone.

By modifying the proof of Theorem 4.3 in [4], we get the following characterization of $(\mathcal{I}, \mathcal{J})$-BW.

Theorem 4.6. *Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω, then the following conditions are equivalent:*

(1) $(\mathcal{I}, \mathcal{J})$ *is Ramsey**,

(2) $(\mathcal{I}, \mathcal{J})$ *is Mon**,

(3) $[0, 1]$ *has* $(\mathcal{I}, \mathcal{J})$*-BW.*

Proof. (1) ⇒ (2) Let $\langle x_n : n \in \omega\rangle$ be a sequence in $[0, 1]$, define a coloring c: $[\omega]^2 \to \{0, 1\}$ by

$$c(\{n, m\}) = 0 \text{ if } n < m \text{ and } x_n \leq x_m;\ c(\{n, m\}) = 1, \text{ otherwise.}$$

Since $(\mathcal{I}, \mathcal{J})$ is *Ramsey**, there exists $A \in \mathcal{I}^+$ such that A is $\mathcal{J}$-homogeneous for c. So we may assume that for every $n \in A$,

$$\{m : c(\{n, m\}) = 1\} \in \mathcal{J}.$$

Therefore, $\langle x_n : n \in A\rangle$ is $\mathcal{J}$-increasing.

(2) ⇒ (3) Assume that $(\mathcal{I}, \mathcal{J})$ is *Mon**. For a given sequence $\langle x_n : n \in \omega\rangle$ in $[0, 1]$, there exists $A \in \mathcal{I}^+$ such that $\langle x_n : n \in A\rangle$ is $\mathcal{J}$-monotone. We may assume that $\langle x_n : n \in A\rangle$ is $\mathcal{J}$-nondecreasing. Let

$$x = sup_{n\in A} x_n.$$

For any $\varepsilon > 0$, there is $x_N \in A$ such that $x_N > x - \varepsilon$. Then

$$\{n \in A : |x_n - x| \geq \varepsilon\} \subseteq \{n \in A : x_N > x_n\} \in \mathcal{J}.$$

Thus, $\langle x_n : n \in A\rangle$ is $\mathcal{J}$-convergent to x.

(3) ⇒ (1) Let $r \in \omega$, and c: $[\omega]^2 \to \{0, \cdots, r - 1\}$ being a coloring of $[\omega]^2$. We shall define a family $\{A_s : s \in r^{<\omega}\}$ that satisfies S_1-S_3 as follows

- $A_\emptyset = \omega$,
- $A_{s\frown i} = \{n \in A_s : c(lh(s \frown i), n) = i\}, i \in \{0, \cdots, r - 1\}$.

Note that $[0, 1]$ has $(\mathcal{I}, \mathcal{J})$-BW, so ω is not a $(\mathcal{J}, \mathcal{I})$-small set, this implies that there are $x \in r^\omega$ and $B \in \mathcal{I}^+$ such that $B \setminus A_{x|n} \in \mathcal{J}$ for all $n \in \omega$. Then there exists $i \in \{0, \cdots, r - 1\}$, and $C \subseteq B$ with $C \in \mathcal{I}^+$ such that $x(k - 1) = i$ for every $k \in C$. It is not hard to see that for every $n \in C$,

$$\{k \in C : c(\{n, k\}) \neq i\} \subseteq C \setminus A_{x|n} \in \mathcal{J}.$$

This implies that C is $\mathcal{J}$-homogeneous as desired. □

Recall that an ideal $\mathcal{I}$ is called a P-ideal if for every countable $\mathcal{A} \subseteq \mathcal{I}$, there exists $B \in \mathcal{I}$ such that $A \subseteq^* B$ for each $A \in \mathcal{A}$. The following results are showed in [4].

Corollary 4.7. *Let $\mathcal{I}$ be an ideal on ω. Then the following statements hold:*

(1) $[0, 1]$ has $(\mathcal{I}, Fin)$-BW if, and only if $(\mathcal{I}, Fin)$ has Ramsy*.

(2) If $\mathcal{I}$ is a P-ideal, then $(\mathcal{I}, \mathcal{I})$ has Ramsey* if, and only if $(\mathcal{I}, Fin)$ has Ramsey*.

Proof. Assertion (1) follows by replacing $\mathcal{J}$ by *Fin*. As for assertion (2), it is enough to notice that for every P-ideal $\mathcal{I}$, $(\mathcal{I}, Fin)$-BW is equal to $(\mathcal{I}, \mathcal{I})$-BW. □

4.2. Q-Property and Selectivity Defined via Pair of Ideals

As mentioned previously, our aim is to seek for characterizations of $(\mathcal{I}, \mathcal{J})$-BW, so it becomes natural to extend the notions of Q-ideal and selectivity to some general ones. In order to do so, we need the following notations:

- $Q(\mathcal{I}) = \{A \subseteq \omega : \mathcal{I}|A$ *is a local Q-ideal*$\}$;
- $Se(\mathcal{I}) = \{A \subseteq \omega : \mathcal{I}|A$ *is locally selective*$\}$.

Using these notations, $\mathcal{I}$ is *weak Q* if and only if $Q(\mathcal{I}) = \mathcal{I}^+$; $\mathcal{I}$ is *weakly selective* if and only if $Se(\mathcal{I}) = \mathcal{I}^+$. Now, we introduce the following definitions.

Definition 4.8. Let $\mathcal{I}, \mathcal{J}$ be ideals on ω, then

- $(\mathcal{I}, \mathcal{J})$ is *weak Q* if $Q(\mathcal{J}) = \mathcal{I}^+$;
- $(\mathcal{I}, \mathcal{J})$ is *weakly selective* if $Se(\mathcal{J}) = \mathcal{I}^+$.

Clearly, $(\mathcal{I}, \mathcal{J})$ is *weak selective* ⇒ $(\mathcal{I}, \mathcal{J})$ is weak Q ⇒ $\mathcal{J} \subseteq \mathcal{I}$. Moreover, we observe the following simple facts.

Proposition 4.9. *Let $\mathcal{I}$ be an ideal on ω with $\mathcal{I} \not\cong Fin \oplus \mathcal{P}(\omega)$.*

(1) If it is locally selective, then $\mathcal{I}^* \subseteq Se(\mathcal{I})$.

(2) If it is local Q, then $\mathcal{I}^* \subseteq Q(\mathcal{I})$.

Proof. Note that $\mathcal{I} \ncong Fin \oplus \mathcal{P}(\omega)$ implies $\mathcal{I}^* \subseteq H(\mathcal{I})$, this is proved in Proposition 1.2 in [12]. In addition, $H(\mathcal{I}) \subseteq Se(\mathcal{I})$ if $\mathcal{I}$ is locally selective and $H(\mathcal{I}) \subseteq Q(\mathcal{I})$ if $\mathcal{I}$ is local Q. Therefore, both of (1) and (2) hold. □

Remark 4.10. If $\mathcal{I} \cong Fin \oplus \mathcal{P}(\omega)$, then $\mathcal{I}^*$ does not necessary contained in $H(\mathcal{I})$. But we also have that $\mathcal{I}^* \subseteq Se(\mathcal{I})$ whenever $\mathcal{I}$ is locally selective: Let $A \in \mathcal{I}^*$. For any separation $\{I_n : n \in \omega\}$ of A with sets from $\mathcal{I}$, then $\{I_n : n \in \omega\} \cup \{\omega \setminus A\}$ is a partition of ω into sets from $\mathcal{I}$. So there exists $S \in \mathcal{I}^+$ such that $|S \cap (\omega \setminus A)| \leq 1$ and $|S \cap I_n| \leq 1$ for every $n \in \omega$. Note that $S \cap A \in \mathcal{I}^+$ since $|S \cap (\omega \setminus A)| \leq 1$, so $S \cap A$ is a desired selector for $\{I_n : n \in \omega\}$.

Note that both of $Q(\mathcal{I})$ and $Se(\mathcal{I})$ are closed under supersets, we observe the following:

Proposition 4.11. *The following are hold for any ideal $\mathcal{I}$ on ω:*

(1) $\mathcal{I}$ is weak Q if, and only if $Q(\mathcal{I})$ is $\mathcal{I}^+$-dense,

(2) $\mathcal{I}$ is weak selective if, and only if $Se(\mathcal{I})$ is $\mathcal{I}^+$-dense,

Theorem 4.12. *Let $\mathcal{I}, \mathcal{J}$ be ideals on ω such that $(\mathcal{I}, \mathcal{J})$ is weak selective. For the following conditions:*

(1) $[0, 1]$ has $(\mathcal{I}, \mathcal{J})$-BW;

(2) For every $r \in \omega$, every family $\{A_s : s \in r^{<\omega}\}$ fulfilling conditions S_1-S_3, there are $x \in r^\omega$ and $C \in \mathcal{J}^+$ such that $C \subseteq^* A_{x|n}$ for each $n \in \omega$;

(3) $[0, 1]$ has $(\mathcal{J}, \mathcal{I})$-BW

it holds that (1) ⇒ (2) ⇒ (3).

Proof. (1) ⇒ (2) Note that $[0, 1]$ has $(\mathcal{I}, \mathcal{J})$-BW implies that $\omega \notin \mathcal{S}_{(\mathcal{J},\mathcal{I})}$. So for every $r \in \omega$, every family $\{A_s : s \in r^{<\omega}\}$ fulfilling conditions S_1-S_3, there are $x \in r^\omega$ and $B \in \mathcal{I}^+$ such that $B \setminus A_{x|n} \in \mathcal{J}$ for every $n \in \omega$. It is easy to see that

$$B \setminus A_{x|1}, B \cap (A_{x|2} \setminus A_{x|1}), \cdots, B \cap (A_{x|n+1} \setminus A_{x|n}), \cdots$$

is a partition of B into sets from $\mathcal{J}$. Note that $(\mathcal{I}, \mathcal{J})$ is weak selective, so $\mathcal{J}|B$ is locally selective. Thus, there exists $C \subset B$ with $C \in \mathcal{J}^+$ such that $|C \cap B \setminus A_{x|1}| \leq 1$, $|C \cap B \cap (A_{x|2} \setminus A_{x|n})| \leq 1$ for every $n \in \omega$. It is easy to check that the set C is desired.

(2) ⇒ (3) It is enough to show that ω is not an $(\mathcal{I}, \mathcal{J})$-small set. To this end, for every $r \in \omega$, for any family $\{A_s : s \in 2^{<\omega}\}$ satisfying S_1-S_3. By (2), there are $x \in r^\omega$ and $C \in \mathcal{J}^+$ such that for each $n \in \omega$, $C \setminus A_{x|n} \in Fin \subseteq \mathcal{I}$. □

Remark 4.13. Recall that an ideal $\mathcal{I}$ is *selective*, if for any decreasing sequence

$$F_1 \supset F_2 \supset F_3 \supset \cdots$$

from $\mathcal{I}^+$, there exists a *diagonalization* F (i.e, for all $i, j \in F$ with $i < j$, $j \in F_i$). Evidently, if $\bigcap_{n\in\omega} F_n$ is nonempty, then it is a diagonalization. If we replace 'weak selective' by 'selective' in the previous result, the set C existing in (2) can be chosen as a diagonalization of $\langle A_{x|n} : n \in \omega\rangle$.

Corollary 4.14. *Let $\mathcal{I}$ be an ideal on ω which is weak selective. Then following conditions are equivalent:*

(1) $[0, 1]$ has $(\mathcal{I}, \mathcal{I})$-BW;

(2) For every $r \in \omega$, every family $\{A_s : s \in r^{<\omega}\}$ fulfilling conditions S_1-S_3, there are $x \in r^\omega$ and $C \in \mathcal{I}^+$ such that $C \subseteq^* A_{x|n}$ for each $n \in \omega$.

Definition 4.15. ([10]) Let $\mathcal{I}$ be an ideal on ω. Recall that $\mathcal{I}$ satisfies $\omega \to (\omega, \mathcal{I}^+)^2_2$ if for every coloring c: $[\omega]^2 \to \{0,1\}$ either there is an infinite 0-homogeneous set X or there is an $\mathcal{I}$-positive 1-homogeneous.

Remark 4.16. It is easy to see that both $\omega \to (\omega, \mathcal{I}^+)^2_2$ and *Ramsy** are weaker than *Ramsey* property, so it is a natural question to ask what is the relation between $\omega \to (\omega, \mathcal{I}^+)^2_2$ and *Ramsy**. Unfortunately, there is no directed relation between them. In fact, $\mathcal{I}$ being *Ramsey** does not imply $\omega \to (\omega, \mathcal{I}^+)^2_2$. To see this, let's consider the ideal $\mathcal{ED}_{fin}$, where

$$\mathcal{ED}_{fin} = \{A \subset \{\langle n,m\rangle \in \omega \times \omega, m \leq n\} : (\exists m, n \in \omega)(\forall k \geq n)(|A_{(k)}| \leq m)\}.$$

It is easy to see that $\mathcal{ED}_{fin}$ is defined as the restriction of $\mathcal{ED}$ to $\triangle = \{\langle n,m\rangle \in \omega \times \omega : m \leq n\}$. Note that $[0,1]$ has $(\mathcal{ED}_{fin}, \mathcal{ED}_{fin})$-BW property since $\mathcal{ED}_{fin}$ is an F_σ-ideal, so $\mathcal{ED}_{fin}$ is *Ramsey** by Theorem 5.6. But $\omega \nrightarrow (\omega, \mathcal{ED}^+_{fin})^2_2$ ([10], Lemma 2.3.8).

4.3. $\mathcal{A}$-Dense and ω-Diagonalizable

Let $\mathcal{I}$ be an ideal on ω. For a certain $\mathcal{A} \subseteq [\omega]^\omega$, recall that $\mathcal{I}$ is ω-diagonalizable by elements of $\mathcal{A}$ if there is a sequence $\{A_n : n \in \omega\} \subseteq \mathcal{A}$ such that for every $I \in \mathcal{I}$, there exists $n \in \omega$ such that $I \cap A_n = \emptyset$. This notion was introduced in [8] and was useful in characterizing selectivity and density of ideals (see, [13]).

Definition 4.17. Let $\mathcal{A} \subseteq [\omega]^\omega$, and $\mathcal{I}$ being an ideal on ω,

- $non^*(\mathcal{A}, \mathcal{I}) = min\{|\mathcal{H}| : \mathcal{H} \subseteq \mathcal{A} \wedge (\forall I \in \mathcal{I})(\exists H \in \mathcal{H})(I \cap H \text{ is finite})\}$
- $non(\mathcal{A}, \mathcal{I}) = min\{|\mathcal{H}| : \mathcal{H} \subseteq \mathcal{A} \wedge (\forall I \in \mathcal{I})(\exists H \in \mathcal{H})(I \cap H = \emptyset)\}$.

It is easy to see that $non(\mathcal{A}, \mathcal{I}) = \omega$ is equal to saying that $\mathcal{I}$ is ω-diagonalizable by elements of $\mathcal{A}$, and $non^*([\omega]^\omega, \mathcal{I})$ coincides with $non^*(\mathcal{I})$ introduced in [6] whenever $\mathcal{I}$ is dense. In addition, if $\mathcal{I}$ is dense, then $non^*([\omega]^\omega, \mathcal{I})$ is equal to $non([\omega]^\omega, \mathcal{I})$ ([10], Remark 1.3.1).

The following examples show that $non^*(\mathcal{A}, \mathcal{I})$ and $non(\mathcal{A}, \mathcal{I})$ are not defined for all pairs $(\mathcal{A}, \mathcal{I})$. The first one is a dense ideal, and the second is not dense.

Example 4.18. Let $\mathcal{I}$ be a dense P-ideal, and $\mathcal{A} \subset [\omega]^\omega$ with $|\mathcal{A}| = \omega$. For any $\mathcal{H} \subseteq \mathcal{A}$, since $\mathcal{I}$ is dense, there exists for each $H \in \mathcal{H}$ an infinite $A_H \subseteq H$ such that $A_H \in \mathcal{I}$. Since $\mathcal{I}$ is a P-ideal, there is $I \in \mathcal{I}$ such that $A_H \subseteq^* I$ for all $H \in \mathcal{H}$. Clearly, I intersects with each member of $\mathcal{H}$ infinitely.

Example 4.19. Let $A \subset \omega$ be an infinite set such that $\omega \setminus A$ is infinite. Put $\langle A\rangle^* = \{B \subset \omega : B \subseteq^* A\}$. Then $\langle A\rangle^*$ is a P-ideal that is not dense. It is easy to see that the notion of $non(\mathcal{A}, \mathcal{I})$ fails for the pair $(\{A\}, \langle A\rangle^*)$.

Lemma 4.20. *Let $\mathcal{I}$ be an ideal on ω, and $\mathcal{A} \subseteq [\omega]^\omega$. For the following conditions:*

(1) $\mathcal{I}$ is not $\mathcal{A}$-dense;

(2) $non^*(\mathcal{A}, \mathcal{I}) = 1$;

(3) $non(\mathcal{A}, \mathcal{I}) = \omega$.

(1) $\Leftrightarrow$ (2) $\Rightarrow$ (3).

Proof. (1) $\Leftrightarrow$ (2) Since $\mathcal{I}$ is not $\mathcal{A}$-dense, there exists $A \in \mathcal{A}$ such that $[A]^\omega \cap \mathcal{I} = \emptyset$. Therefore, $non^*(\mathcal{A}, \mathcal{I}) = 1$. The converse is obvious.

(2) $\Rightarrow$ (3) Assume that $non^*(\mathcal{A}, \mathcal{I}) = 1$, there exists $A \in \mathcal{A}$ fulfilling this. For each $n \in \omega$, let

$$A_n = A \setminus \{0, 1, \cdots, n-1\}.$$

Then the family $\{A_n : n \in \omega\}$ verifies $non(\mathcal{A}, \mathcal{I}) = \omega$. □

Corollary 4.21. ([13], Proposition 3.4) *Let $\mathcal{I}$ be an ideal on ω. Then $non(\mathcal{I}^*, \mathcal{I}) = \omega$ if and only if $\mathcal{I}$ is not $\mathcal{I}^*$-dense;*

Let T be a tree, $\mathcal{A} \subseteq \mathcal{P}(\omega)$. For each $s \in T$, let $succ_T(s) = \{n > max(s) : s \cup \{n\} \in T\}$. Recall that a tree T is an $\mathcal{A}$-tree if for every $s \in T$, $succ_T(s) \in \mathcal{A}$, where $\mathcal{A} \subseteq \mathcal{P}(\omega)$.

With the similar discussion of Remark 1.3.1 in [10], we observe that for any family $\mathcal{A} \subseteq [\omega]^\omega$ closed under finite modifications, if $\mathcal{I}$ is $\mathcal{A}$-dense then $non^*(\mathcal{A}, \mathcal{I}) = non(\mathcal{A}, \mathcal{I})$. So, together with Proposition 3.1 in [13], we have the following.

Proposition 4.22. *For any ideals $\mathcal{I}$ and $\mathcal{J}$, if $\mathcal{I}$ is $\mathcal{J}^+$-dense, then $non^*(\mathcal{J}^+, \mathcal{I}) = \omega$ if and only if there exists a $\mathcal{J}^+$-tree with all branches in $\mathcal{I}^+$.*

Proposition 4.23. *Let $\mathcal{A} \subset [\omega]^\omega$ such that $\mathcal{A}$ is dense. Then the following conditions are equivalent:*

(1) $non^*(\mathcal{I}) = \omega$;

(2) $non^*(\mathcal{A}, \mathcal{I}) = \omega$.

Proof. (2) ⇒ (1) Together with $\mathcal{I}$ being $\mathcal{A}$-dense and $\mathcal{A}$ being dense, we have that $\mathcal{I}$ is dense. In addition, $non^*(\mathcal{A}, \mathcal{I}) = \omega$ implies that $\mathcal{I}$ is $\mathcal{A}$-dense. So $\mathcal{I}$ is dense, and so $\omega \leq non^*(\mathcal{I}) \leq non^*(\mathcal{A}, \mathcal{I})$.

(1) ⇒ (2) To check the converse, assume that $A_0, A_1, \cdots, A_n, \cdots$ be a countable family in $[\omega]^\omega$ which meet that $non^*(\mathcal{I}) = \omega$. Since $\mathcal{A}$ is dense, there are, for all $n \in \omega$, $B_n \subseteq A_n$ such that $B_n \in \mathcal{A}$. It is easy to verify that the sequence $B_n, n \in \omega$ meet that $non^*(\mathcal{A}, \mathcal{I}) = \omega$. □

Recall that $\mathcal{I}$ is h-Ramsey (respectively, *h-Ramsey**) if for every $A \in \mathcal{I}^+$, $\mathcal{I}|A$ is Ramsey (respectively, $\mathcal{I}|A$ is *Ramsey**)[4]

Theorem 4.24. *Let $\mathcal{I}, \mathcal{J}$ be ideals on ω and $\mathcal{J}$ being a weak Q-ideals such that $\mathcal{I} \leq_{RB} \mathcal{J}$,*

(1) If $\mathcal{J}$ is h-Ramsey*, then $\mathcal{I}$ is h-Ramsey*;

(2) If $\mathcal{J}$ is h-Ramsey, then $\mathcal{I}$ is h-Ramsey.

Proof. The assertion (1) follows from the facts that *h-Ramsey** is equal to h-BW property ([4], Theorem 4.3) and the h-BW property is preserved under the $\leq_{RB}$-order in the realm of Q-ideals ([5], Theorem 6.2).

The key in the proof of the assertion (2) is that $\mathcal{I}$ is h-Ramsey if, and only if $\mathcal{I}$ is h-Fin-BW and being a weak Q-ideal ([4], Theorem 3.16). So we need the following Claims:

Claim 4.25. *Let $\mathcal{I}, \mathcal{J}$ be ideals on ω, and $\mathcal{J}$ being a Q-ideal. If $\mathcal{I} \leq_{KB} \mathcal{J}$ then $\mathcal{I}$ is also a Q-ideal.*

Proof. Let $f\colon \omega \to \omega$ be a finite to one function meeting $\mathcal{I} \leq_{KB} \mathcal{J}$. Let $\{I_n : n \in \omega\}$ be a partition of ω into finite sets. Put $A_n = \{f^{-1}(m) : m \in I_n\}$. Then $\{A_n : n \in \omega\}$ is also a partition of ω into finite sets. It is easy to check that if S is a selector for $\{A_n : n \in \omega\}$, then $f(S)$ is a selector for $\{I_n : n \in \omega\}$, this end the proof. □

Claim 4.26. *Let $\mathcal{I}, \mathcal{J}$ be ideals on ω, and $\mathcal{J}$ being a weak Q-point. If $\mathcal{I} \leq_{RB} \mathcal{J}$ then $\mathcal{I}$ is also a weak Q-ideal.*

Proof. Assume $f\colon \omega \to \omega$ witness $\mathcal{I} \leq_{RB} \mathcal{J}$, so for $A \in \mathcal{I}^+$, $f^{-1}(A) \in \mathcal{J}^+$. It is easy to see that $\mathcal{I}|A \leq_{KB} \mathcal{J}|f^{-1}(A)$. Note that $\mathcal{J} \leq_{KB} \mathcal{J}|f^{-1}(A)$ and $\mathcal{J}$ is a weak Q-ideal, so is $\mathcal{J}|f^{-1}(A)$. By Claim 2 above we have that $\mathcal{I}|A$ is a Q-ideal as well. □

Therefore, $\mathcal{I}$ is a weak Q-ideal. In addition, $\mathcal{I}$ is h-Fin-BW by Theorem 6.1 and Theorem 6.2 in [5]. □

Remark 4.27. In Claim 2, if we replaced $\mathcal{I} \leq_{KB} \mathcal{J}$ by $\mathcal{I} \leq_K \mathcal{J}$, and $\mathcal{I}, \mathcal{J}$ are Borel ideals, then $\mathcal{J}$ being a Q-ideal also implies that $\mathcal{I}$ is a Q-ideal. To see this, note first that for any Borel ideal $\mathcal{I}$, $non(\mathcal{I}) = \omega$ if, and only if $\mathcal{I}$ is a Q-ideal ([13], Proposition 3.2), so it is enough to show that $non(\mathcal{I}) = \omega$. There are two possible cases: **Case 1**, if $\mathcal{I}$ is not dense, then $non(\mathcal{I}) = \omega$; **Case 2**, if $\mathcal{I}$ is dense, then $non^*(\mathcal{I}) \geq \omega$, and $\mathcal{J}$ is also dense since $\mathcal{I} \leq_K \mathcal{J}$. Since $\mathcal{J}$ being a Q-ideal, we have that $non^*(\mathcal{J}) = \omega$. Moreover, $\mathcal{I} \leq_K \mathcal{J}$ implies that $non^*(\mathcal{I}) \leq non^*(\mathcal{J})$([10], Theorem 1.4.2). Thus, $non^*(\mathcal{I}) = \omega$.

Definition 4.28. ([6]) Let $\mathcal{I}$ be a dense ideal on ω, $cov^*(\mathcal{I}) = min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge (\forall X \in [\omega]^\omega)(\exists A \in \mathcal{A})(|A \cap X| = \omega)\}$.

We end this section with the following result related to $(\mathcal{I}, Fin)$-BW property, which tells us that in the realm of dense ideals, $cov^*(\mathcal{I}) \geq \omega_1$ hold whenever $[0,1]$ satisfying $(\mathcal{I}, Fin)$-BW.

Proposition 4.29. *Let $\mathcal{I}$ be a dense ideal on ω. If $cov^*(\mathcal{I}) = \omega$, then $[0,1]$ does not satisfy $(\mathcal{I}, Fin)$-BW.*

Proof. Assume that $\{A_n : n \in \omega\} \subseteq \mathcal{I}$ is a countable family meeting $cov^*(\mathcal{I}) = \omega$. Without loss of generality, we may assume that they are pairwise disjoint. Define a sequence $\langle x_k : k \in \omega\rangle$ by

$$x_k = \frac{1}{n+1} \text{ for } k \in A_n.$$

Then $\langle x_k : k \in \omega\rangle$ is $\mathcal{I}$-convergent to 0. But for any $A \in \mathcal{I}^+$, there exists $n \in \omega$ such that $|A \cap A_n| = \omega$, so $\langle x_k : k \in A\rangle$ cannot be convergent to 0. □

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References

[1] A. Blass, Combinatorial Cardinal Characteristics of the Continuum, In: M. Foreman, A. Kanamori (eds), Handbook of Set Theory, Springer, Dordrecht, 2010. pp. 395–489.
[2] R. Engelking, General Topology, Heldermann Verlag, Berlin, 1989.
[3] H. Fast, Sur la convergence statistique, Coll. Math. 2 (1951) 241–244.
[4] R. Filipów, N. Mrożek, I. Recław, P. Szuca, Ideal version of Ramsey Theorem, Czech. Math. J. 136 (2011) 289–308.
[5] R. Filipów, N. Mrożek, I. Recław, P. Szuca, Ideal convergence of bounded sequences, J. Symbolic Logic 72 (2007) 501–512.
[6] F. Hernández-Hernández, M. Hrušák, Cardinal invariants of analytic P-ideals, Canadian. J. Math. 59 (2007) 575–595.
[7] M. Hrusak, Combinatorics of filters and ideals, In: Set Theory and its Applications, vol. 533 of Contemp. Math., 2011, pp. 29–69.
[8] C. Laflamme, Filter games and combinatorial properties of winning strategies, Contemp. Math. 192 (1996) 51–67.
[9] C. Laflamme, J. Zhu, The Rudin-Blass ordering of ultrafilters, J. Symbolic Logic 63 (1998) 584–592.
[10] D. Meza-Alcántara, Ideals and filters on countable sets, Ph.D thesis, UNAM México, 2009.
[11] M. Katětov, Products of filters, Comment. Math. Univ. Carolinae 9 (1968) 173–189.
[12] A. Kwela, J. Tryba, Homogeneous ideals on countable sets, Acta. Math. Hungarica 151 (2017) 139–161.
[13] A. Kwela, P. Zakrzewski, Combinatorics of ideals - selectivity versus density, Comment. Math. Univ. Carolinae 58 (2017) 261–266.