



Skew m -Complex Symmetric Operators

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Abstract. In this paper we study skew m -complex symmetric operators. In particular, we show that if $T \in \mathcal{L}(\mathcal{H})$ is a skew m -complex symmetric operator with a conjugation C , then e^{itT} , e^{-itT} , and e^{-itT^*} are (m, C) -isometric for every $t \in \mathbb{R}$. Moreover, we examine some conditions for skew m -complex symmetric operators to be skew $(m - 1)$ -complex symmetric.

1. Introduction

Let $\mathcal{L}(\mathcal{H})$ be the algebra of bounded linear operators on a separable complex Hilbert space \mathcal{H} . Let \mathbb{N} be the set of natural numbers and let \mathbb{C} be the set of complex numbers. A *conjugation* on \mathcal{H} is an antilinear operator $C : \mathcal{H} \rightarrow \mathcal{H}$ with $C^2 = I$ which satisfies $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$. For a conjugation C , there is an orthonormal basis $\{e_n\}_{n=0}^\infty$ for \mathcal{H} such that $Ce_n = e_n$ for all n . Note that if $T \in \mathcal{L}(\mathcal{H})$ and C is a conjugation on \mathcal{H} , then $(CTC)^k = CT^kC$ and $(CTC)^* = CT^*C$ for every $k \in \mathbb{N}$ (see [8] or [9]). An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *skew complex symmetric* if there exists a conjugation C on \mathcal{H} such that $CTC = -T^*$.

Less attention has been paid to skew complex symmetric operators. There are various motivations for such operators. In particular, skew symmetric matrices have many applications in pure mathematics, applied mathematics, and even in engineering disciplines. Real skew symmetric matrices are very important in applications, including function theory, the solution of linear quadratic optimal control problems, robust control problems, model reduction, crack following in anisotropic materials, and others. In light of these applications, it is natural to study skew symmetric operators on a Hilbert space \mathcal{H} (see [13]-[16]).

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be a *skew m -complex symmetric operator* if there exists some conjugation C such that $\Gamma_m(T; C) = 0$, where

$$\Gamma_m(T; C) := \sum_{j=0}^m \binom{m}{j} T^{*m-j} C T^j C.$$

2010 *Mathematics Subject Classification.* Primary 47A11, Secondary 47B25.

Keywords. Skew m -complex symmetric operator; (m, C) -Isometric operator.

Received: 15 October 2018; Accepted: 26 June 2019

Communicated by Dragan S. Djordjević

This work was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education(2016R1D1A1B03931937). The third author was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2019R1A2C1002653) and this research is partially supported by Grant-in-Aid Scientific Research No.15K04910.

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In particular, it is obvious that a skew 1-complex symmetric operator is skew complex symmetric. It holds that

$$T^* \Gamma_m(T; C) + \Gamma_m(T; C) \cdot CTC = \Gamma_{m+1}(T; C).$$

Therefore if T is a skew m -complex symmetric operator, then T is skew n -complex symmetric for all $n \geq m$. In general, skew m -complex symmetric operators are not skew $(m - 1)$ -complex symmetric. For example, let

$$Cx = \begin{pmatrix} \overline{x_2} \\ \overline{x_1} \end{pmatrix} \text{ for } x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ and } T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ on } \mathbb{C}^2.$$

Then $T^* = CTC = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and so $CT^2C + 2T^*CTC + T^{*2} = 0$. But, $CTC + T^* = 2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \neq 0$. Hence T is a skew 2-complex symmetric operator which is not skew complex symmetric (see [3]).

In 1990s, J. Agler and M. Stankus ([1]) intensively studied the following operator; for a fixed $m \in \mathbb{N}$, an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be an m -isometric operator if it satisfies the following equation;

$$\sum_{j=0}^m (-1)^j \binom{m}{j} T^{*m-j} T^{m-j} = 0. \tag{1}$$

Using the identity (1) and a conjugation C , we define (m, C) -isometric operators as follows; an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be an (m, C) -isometric operator if there exists some conjugation C such that

$$\sum_{j=0}^m (-1)^j \binom{m}{j} T^{*m-j} C T^{m-j} C = 0$$

for some $m \in \mathbb{N}$. Put $\Lambda_m(T; C) := \sum_{j=0}^m (-1)^j \binom{m}{j} T^{*m-j} C T^{m-j} C$. Then T is an (m, C) -isometric operator if and only if $\Lambda_m(T; C) = 0$. Note that

$$T^* \Lambda_m(T; C)(CTC) - \Lambda_m(T; C) = \Lambda_{m+1}(T; C).$$

Hence, if $\Lambda_m(T; C) = 0$, then $\Lambda_n(T; C) = 0$ for all $n \geq m$. Moreover, it is obvious that T is an (m, C) -isometry if and only if CTC is an (m, C) -isometry (see [5]). Furthermore, T is an (m, C) -isometry if and only if T^* is an (m, C) -isometry.

In this paper we focus on skew m -complex symmetric operators. In particular, we show that if $T \in \mathcal{L}(\mathcal{H})$ is a skew m -complex symmetric operator with a conjugation C , then e^{itT} , e^{-itT} and e^{-itT^*} are (m, C) -isometric for every $t \in \mathbb{R}$. Moreover, we examine some conditions for skew m -complex symmetric operators to be skew $(m - 1)$ -complex symmetric.

2. Skew m -complex symmetric operators

In this section, we give properties of skew m -complex symmetric operators. It is known from [6] that if T is m -complex symmetric, then T^n is also m -complex symmetric for any n . However, the power of a skew m -complex symmetric operator is not skew m -complex symmetric. For example, if $T = \begin{pmatrix} 1 & a & 0 \\ 0 & 0 & a \\ 0 & 0 & -1 \end{pmatrix}$ for some $a \in \mathbb{C}$, then T is a skew complex symmetric operator with the conjugation $C(z_1, z_2, z_3) = (-\overline{z_3}, \overline{z_2}, -\overline{z_1})$ from [13, Example 3.3]. A simple calculation shows that

$$T^2 = \begin{pmatrix} 1 & a & a^2 \\ 0 & 0 & -a \\ 0 & 0 & 1 \end{pmatrix} \text{ and } -CT^2C = \begin{pmatrix} -1 & 0 & 0 \\ -a & 0 & 0 \\ -a^2 & a & -1 \end{pmatrix}.$$

Hence T^2 is not skew complex symmetric with the conjugation C . In [11], J. W. Helton studied the following class of operators; an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be m -symmetric if it satisfies the following equation;

$$\sum_{j=0}^m (-1)^j \binom{m}{j} T^{*m-j} T^j = 0.$$

It is known from [11] that if T is m -symmetric and m is even, then T is $(m - 1)$ -symmetric. For the study of this property for skew m -complex symmetric operators, we prepare the following lemma which is maybe well-known fact (cf. [10, Lemma 3.3]).

Lemma 2.1. ([10]) *Let m be any nonnegative integer. Then the following identities hold.*

- (i)
$$\sum_{j=0}^m (-1)^j \binom{m}{j} j^k = 0 \text{ for all } k = 0, 1, 2, \dots, (m - 1).$$
- (ii)
$$\sum_{j=0}^m (-1)^j \binom{m}{j} j^m = (-1)^m m! \text{ for } m = 1, 2, \dots.$$

Now we will introduce exponential operators $T := e^{-iA}$ which act on a wave function to move it in time and space (see [2]). Note that T is a function of an operator $f(A)$ which is defined its expansion in a Taylor series;

$$T = \exp(-iA) = \sum_{n=0}^{\infty} \frac{(-iA)^n}{n!} = 1 - iA + \frac{(-iA)^2}{2!} + \dots.$$

The most common one is the time-propagator or time-evolution operator U which is the Hamiltonian function and propagates the wave function forward in time;

$$U = \exp\left(\frac{-iHt}{h}\right) = 1 + \frac{-iHt}{h} + \frac{1}{2!} \left(\frac{-iHt}{h}\right)^2 + \dots.$$

For an operator $T \in \mathcal{L}(\mathcal{H})$ and a conjugation C , if $t \in \mathbb{R}$, then

$$e^{itT} = I + itT + \frac{(it)^2}{2!} T^2 + \frac{(it)^3}{3!} T^3 + \dots. \tag{2}$$

Lemma 2.2. *If $T \in \mathcal{L}(\mathcal{H})$ is a skew m -complex symmetric operator with a conjugation C , then*

$$(e^{itT})^{k*} C (e^{itT})^k C = \sum_{j=0}^{m-1} \frac{(-itk)^j}{(j)!} \Gamma_j(T; C) \tag{3}$$

for each $k \in \mathbb{N}$ where $\Gamma_0(T; C) = I$.

Proof. From (2), we have

$$(e^{itT})^* = I + (-it)T^* + \frac{(-it)^2}{2!} T^{*2} + \frac{(-it)^3}{3!} T^{*3} + \dots$$

and

$$C e^{itT} C = I + (-it)CTC + \frac{(-it)^2}{2!} CT^2C + \frac{(-it)^3}{3!} CT^3C + \dots.$$

Therefore, we have

$$(e^{itT})^* C e^{itT} C = I + (-it)(T^* + CTC) + \frac{(-it)^2}{2!} (T^{*2} + 2T^*CTC + CT^2C) + \dots$$

$$= I + (-it)\Gamma_1(T; C) + \frac{(-it)^2}{2!}\Gamma_2(T; C) + \frac{(-it)^3}{3!}\Gamma_3(T; C) + \dots \tag{4}$$

From equation (4), we have for $k \in \mathbb{N}$

$$\begin{aligned} (e^{itT})^{*k}C(e^{itT})^kC &= (e^{-itkT^*})(Ce^{itkT}C) \\ &= I + (-it)k\Gamma_1(T; C) + \frac{(-it)^2}{2!}(k)^2\Gamma_2(T; C) \\ &\quad + \frac{(-it)^3}{3!}(k)^3\Gamma_3(T; C) + \dots + \frac{(-it)^{m-1}}{(m-1)!}(k)^{m-1}\Gamma_{m-1}(T; C) \\ &\quad + \frac{(-it)^m}{m!}(k)^m\Gamma_m(T; C) + \dots \end{aligned} \tag{5}$$

Moreover, since T is a skew m -complex symmetric operator, i.e., $\Gamma_m(T; C) = \Gamma_{m+1}(T; C) = \dots = 0$, it follows from (5) that

$$\begin{aligned} (e^{itT})^{*k}C(e^{itT})^kC &= I + (-i)tk\Gamma_1(T; C) + \frac{(-i)^2}{2!}t^2k^2\Gamma_2(T; C) \\ &\quad + \frac{(-i)^3}{3!}t^3k^3\Gamma_3(T; C) + \dots \\ &\quad + \frac{(-i)^{m-1}}{(m-1)!}t^{m-1}k^{m-1}\Gamma_{m-1}(T; C). \end{aligned}$$

So we complete the proof. \square

As an application of Lemma 2.2, we get the following theorem.

Theorem 2.3. *Let $T \in \mathcal{L}(\mathcal{H})$. If T is a skew m -complex symmetric operator with a conjugation C , then e^{itT} , e^{-itT} , and e^{-itT^*} are (m, C) -isometric for every $t \in \mathbb{R}$.*

Proof. By Lemma 2.2, we get that

$$\begin{aligned} \Lambda_m(e^{itT}; C) &= \sum_{j=0}^m (-1)^j \binom{m}{j} (e^{itT})^{*m-j} C(e^{itT})^{m-j} C \\ &= \sum_{j=0}^m (-1)^j \binom{m}{j} \left(I + (-it)(m-j)\Gamma_1(T; C) \right. \\ &\quad + \frac{(-it)^2}{2!}(m-j)^2\Gamma_2(T; C) + \frac{(-it)^3}{3!}(m-j)^3\Gamma_3(T; C) \\ &\quad \left. + \dots + \frac{(-it)^{m-1}}{(m-1)!}(m-j)^{m-1}\Gamma_{m-1}(T; C) \right) \\ &= \sum_{j=0}^m (-1)^j \binom{m}{j} I + (-it)\Gamma_1(T; C) \sum_{j=0}^m (-1)^j \binom{m}{j} (m-j) \\ &\quad + \frac{(-it)^2}{2!}\Gamma_2(T; C) \sum_{j=0}^m (-1)^j \binom{m}{j} (m-j)^2 \\ &\quad + \frac{(-it)^3}{3!}\Gamma_3(T; C) \sum_{j=0}^m (-1)^j \binom{m}{j} (m-j)^3 \\ &\quad + \dots + \frac{(-it)^{m-1}}{(m-1)!}\Gamma_{m-1}(T; C) \sum_{j=0}^m (-1)^j \binom{m}{j} (m-j)^{m-1}. \end{aligned} \tag{6}$$

By Lemma 2.1, equation (6) becomes $\Lambda_m(e^{itT}; C) = 0$. Hence e^{itT} is (m, C) -isometric for every $t \in \mathbb{R}$. On the other hand, since $\Lambda_m(e^{itT}; C) = 0$ and e^{itT} is invertible, it follows from [5, Theorem 3.7 (i)] that e^{-itT} is (m, C) -isometric for every $t \in \mathbb{R}$. Moreover, since e^{itT} is (m, C) -isometry, it follows that e^{-itT^*} is (m, C) -isometry. \square

Remark 2.4. In general, the converse of Theorem 2.3 may not be hold. But, if e^{itT} is a $(1, C)$ -isometric operator and T is a skew 2-complex symmetric operator with the conjugation C , then T is a skew complex symmetric operator.

Indeed, in this case, equation (4) gives that

$$0 = \Lambda_1(e^{itT}; C) = (e^{itT})^* C e^{itT} C - I = -it\Gamma_1(T; C).$$

Therefore $\Gamma_1(T; C) = 0$. Hence T is a skew complex symmetric operator.

If $T \in \mathcal{L}(\mathcal{H})$, we write $\sigma_p(T)$ and $\sigma_{ap}(T)$ for the point spectrum and the approximate point spectrum of T , respectively. As some applications of Theorem 2.3, we obtain the following corollaries.

Corollary 2.5. *If $T \in \mathcal{L}(\mathcal{H})$, then the following statements hold:*

- (i) *Let T be skew m -complex symmetric with a conjugation C . If $\lambda \in \sigma_{ap}(e^{itT})$, then $\frac{1}{\lambda} \in \sigma_{ap}(e^{-itT^*})$. In particular, If $\lambda \in \sigma_p(e^{itT})$, then $\frac{1}{\lambda} \in \sigma_p(e^{-itT^*})$.*
- (ii) *If T is skew m -complex symmetric with a conjugation C , then e^{itnT} is an (m, C) -isometric operator for any $n \in \mathbb{N}$.*
- (iii) *Let $\{T_k\}$ be a sequence of skew m -complex symmetric operators with a conjugation C such that $\lim_{k \rightarrow \infty} \|e^{itT_k} - e^{itT}\| = 0$. Then e^{itT} is an (m, C) -isometric operator.*

Proof. If T is a skew m -complex symmetric operator with a conjugation C , then e^{itT} is an (m, C) -isometric operator from Theorem 2.3. The proofs of (i), (ii) follow from [6, Theorems 3.4 and 3.7].

(iii) Suppose that $\{T_k\}$ is a sequence of skew m -complex symmetric operators with a conjugation C such that $\lim_{k \rightarrow \infty} \|e^{itT_k} - e^{itT}\| = 0$. Then e^{itT_k} is an (m, C) -isometric operator from Theorem 2.3. Hence e^{itT} is an (m, C) -isometric operator from [6, Theorem 3.7 (iii)]. \square

Corollary 2.6. *If $T \in \mathcal{L}(\mathcal{H})$ is skew m -complex symmetric and complex symmetric with a conjugation C , i.e., $T^* = CTC$, then the following statements hold:*

- (i) *e^{itT} is an algebraic operator of order at most $2m$.*
- (ii) *$C \ker(\Gamma_{m-1}(e^{itT}; C))$ is invariant for e^{itT} .*

Proof. Let T be a skew m -complex symmetric operator with a conjugation C . Then e^{itT} is an (m, C) -isometric operator from Theorem 2.3.

(i) Since

$$(e^{itT})^* = I + (-it)T^* + \frac{(-it)^2}{2!} T^{*2} + \frac{(-it)^3}{3!} T^{*3} + \dots$$

and

$$C e^{itT} C = I + (-it)CTC + \frac{(-it)^2}{2!} CT^2C + \frac{(-it)^3}{3!} CT^3C + \dots,$$

it follows that e^{itT} is complex symmetric with a conjugation C . Hence e^{itT} is an algebraic operator of order at most $2m$ from [5, Theorem 3.7(ii)].

(ii) Since e^{itT} is an (m, C) -isometric operator, it follows that

$$e^{-itT^*} \Gamma_{m-1}(e^{itT}; C) C e^{itT} C = \Gamma_{m-1}(e^{itT}; C). \tag{7}$$

Let $x \in \ker(\Gamma_{m-1}(e^{itT}; C))$. Then by (7), we have

$$0 = \Gamma_{m-1}(e^{itT}; C)x = e^{-itT^*} \Gamma_{m-1}(e^{itT}; C) C e^{itT} C x.$$

Therefore $\Gamma_{m-1}(e^{itT}; C) C e^{itT} C x = 0$. Thus $C e^{itT} C x \in \ker(\Gamma_{m-1}(e^{itT}; C))$, which means that $C \ker(\Gamma_{m-1}(e^{itT}; C))$ is invariant for e^{itT} . \square

Corollary 2.7. *If $T \in \mathcal{L}(\mathcal{H})$ is skew m -complex symmetric and complex symmetric with a conjugation C , then the following statements hold.*

(i) e^{iT} is unitarily equivalent to a finite operator matrix of the form:

$$\begin{pmatrix} \alpha_1 & A_{12} & \cdots & \cdots & \cdots & A_{1,2m} \\ 0 & \alpha_2 & A_{23} & \cdots & \cdots & A_{2,2m} \\ 0 & 0 & \alpha_3 & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \ddots & A_{2m-1,2m} \\ 0 & 0 & \cdots & \cdots & \cdots & \alpha_{2m} \end{pmatrix}$$

where α_j are the roots of the polynomial $p(z)$ of degree at most $2m$.

(ii) The dimension of $\bigvee_{k=0}^{\infty} \{(e^{iT})^k x\}$ is less than or equals to $2m$.

Proof. (i) The proof follows from Corollary 2.6 and [12].

(ii) If p is a nonzero polynomial $p(z)$ of degree at most $2m$, then $p(e^{iT}) = 0$. Hence we complete the proof. \square

Recall that $\cos(tT) = \frac{e^{iT} + e^{-iT}}{2}$ and $\sin(tT) = \frac{e^{iT} - e^{-iT}}{2i}$ for every $t \in \mathbb{R}$.

Corollary 2.8. *Let $T \in \mathcal{L}(\mathcal{H})$ be skew complex symmetric with a conjugation C and let $t \in \mathbb{R}$. Then the following statements hold.*

- (i) $\cos(tT)$ is a $(1, C)$ -isometric operator if and only if $\cos(2tT^*) = I$.
- (ii) $\sin(tT)$ is a $(1, C)$ -isometric operator if and only if $\cos(2tT^*) = -I$.

Proof. (i) Let $T \in \mathcal{L}(\mathcal{H})$ be skew complex symmetric with a conjugation C . Then e^{iT} and e^{-iT^*} are $(1, C)$ -isometric operators by Theorem 2.3. Since $e^{-iT^*} C e^{iT} C = I$, it follows that $C e^{iT} C = e^{iT^*}$. Therefore we have

$$\begin{aligned} & I - (\cos(tT))^* C (\cos(tT)) C \\ &= I - \left(\frac{e^{iT} + e^{-iT}}{2}\right)^* C \left(\frac{e^{iT} + e^{-iT}}{2}\right) C \\ &= I - \frac{1}{4} (e^{-iT^*} + e^{iT^*}) (C e^{iT} C + C e^{-iT} C) \\ &= I - \frac{1}{4} (e^{-iT^*} C e^{iT} C + e^{iT^*} C e^{iT} C + e^{-iT^*} C e^{-iT} C + e^{iT^*} C e^{-iT} C) \\ &= \frac{1}{2} I - \frac{1}{4} (e^{iT^*} C e^{iT} C + e^{-iT^*} C e^{-iT} C) \\ &= \frac{1}{2} I - \frac{1}{4} (e^{i2tT^*} + e^{-i2tT^*}) \\ &= \frac{1}{2} I - \frac{1}{2} \cos(2tT^*). \end{aligned}$$

Hence $\cos(tT)$ is a $(1, C)$ -isometric operator if and only if $\cos(2tT^*) = I$.

(ii) As the proof of (i), we get that

$$\begin{aligned} & I - (\sin(tT))^* C (\sin(tT)) C \\ &= I - \left(\frac{e^{iT} - e^{-iT}}{2i}\right)^* C \left(\frac{e^{iT} - e^{-iT}}{2i}\right) C \\ &= I - \frac{1}{4} (e^{-iT^*} - e^{iT^*}) (C e^{iT} C - C e^{-iT} C) \\ &= I - \frac{1}{4} (e^{-iT^*} C e^{iT} C - e^{iT^*} C e^{iT} C - e^{-iT^*} C e^{-iT} C + e^{iT^*} C e^{-iT} C) \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2}I + \frac{1}{4}(e^{itT^*}Ce^{itT}C + e^{-itT^*}Ce^{-itT}C) \\
 &= \frac{1}{2}I + \frac{1}{4}(e^{i2tT^*} + e^{-i2tT^*}) \\
 &= \frac{1}{2}I + \frac{1}{2}\cos(2tT^*).
 \end{aligned}$$

Hence $\sin(itT)$ is a $(1, C)$ -isometric operator if and only if $\cos(2tT^*) = -I$. \square

Remark that if e^{itT} and e^{-itT^*} are skew complex symmetric operators, then

$$\cos(itT)^* + C\cos(itT)C = \frac{1}{2}(e^{-itT^*} + Ce^{itT}C + e^{itT^*} + Ce^{-itT}C) = 0$$

and

$$\sin(itT)^* + C\sin(itT)C = \frac{1}{2i}(e^{-itT^*} + Ce^{itT}C - e^{itT^*} - Ce^{-itT}C) = 0.$$

Therefore $\cos(itT)$ and $\sin(itT)$ is skew complex symmetric operators.

Remark that if an invertible operator T is an (m, C) -isometric operator with $CTC = T$ and m is even, then T is an invertible m -isometric operator. Hence it follows from [4] that T is an $(m - 1)$ -isometric operator. Thus T is an $(m - 1, C)$ -isometric operator. However, we still do not know the answer to the following.

Question If $T \in \mathcal{L}(\mathcal{H})$ is skew m -complex symmetric with a conjugation C and m is even, is it skew $(m - 1)$ -complex symmetric?

Finally, we give a partial solution for the above question. To this, we need the following lemma.

Lemma 2.9. Let $T \in \mathcal{L}(\mathcal{H})$ be an invertible (m, C) -isometric operator and let m be even. If $\Lambda_{m-1}(T; C)$ and $(T^*)^{m-1}\Lambda_{m-1}(T^{-1}; C)CT^{m-1}C$ are nonnegative, then T is an $(m - 1, C)$ -isometric operator.

Proof. By the hypothesis, T is an (m, C) -isometric operator and

$$\Lambda_{m-1}(T; C) = \sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} T^{*m-1-j} C T^{m-1-j} C \geq 0.$$

Since $m - 1$ is odd, it holds that

$$\begin{aligned}
 -\Lambda_{m-1}(T; C) &= \sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} T^{*j} C T^j C \\
 &= \sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} T^{*m-1} ((T^{-1})^{*m-1-j} C (T^{-1})^{m-1-j} C) C T^{m-1} C \\
 &= T^{*m-1} \cdot \Lambda_{m-1}(T^{-1}; C) \cdot C T^{m-1} C \geq 0.
 \end{aligned}$$

Therefore $\Lambda_{m-1}(T; C) \leq 0$ and so $\Lambda_{m-1}(T; C) = 0$. Hence T is an $(m - 1, C)$ -isometric operator. \square

Theorem 2.10. Let $T \in \mathcal{L}(\mathcal{H})$ and let C be a conjugation on \mathcal{H} . Suppose that $((e^{itT})^*)^{m-1}\Lambda_{m-1}(e^{-itT}; C)C(e^{itT})^{m-1}C$ and $\Lambda_{m-1}(e^{itT}; C)$ are nonnegative. If T is a skew m -complex symmetric operator with a conjugation C where m is even, then T is skew $(m - 1)$ -complex symmetric and e^{itT} is an $(m - 1, C)$ -isometric operator for all $t \in \mathbb{R}$.

Proof. Suppose that T is a skew m -complex symmetric operator and m is even. Then e^{iT} is (m, C) -isometry from Theorem 2.3. Using equation (5) we have

$$\begin{aligned}
 0 &= \Lambda_{m-1}(e^{iT}; C) = \sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} (e^{iT})^{*m-1-j} C (e^{iT})^{m-1-j} C \\
 &= \sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} \left(I + (-it)(m-1-j)\Gamma_1(T; C) \right. \\
 &\quad + \frac{(-it)^2}{2!} (m-1-j)^2 \Gamma_2(T; C) + \frac{(-it)^3}{3!} (m-1-j)^3 \Gamma_3(T; C) \\
 &\quad + \dots + \frac{(-it)^{m-2}}{(m-2)!} (m-1-j)^{m-2} \Gamma_{m-2}(T; C) \\
 &\quad \left. + \frac{(-it)^{m-1}}{(m-1)!} (m-1-j)^{m-1} \Gamma_{m-1}(T; C) + \frac{(-it)^m}{m!} (m-1-j)^m \Gamma_m(T; C) + \dots \right) \\
 &= \sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} \cdot I + (-it)\Gamma_1(T; C) \left(\sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} (m-1-j) \right) \\
 &\quad + \frac{(-it)^2}{2!} \Gamma_2(T; C) \left(\sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} (m-1-j)^2 \right) \\
 &\quad + \dots + \frac{(-it)^{m-2}}{(m-2)!} \Gamma_{m-2}(T; C) \left(\sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} (m-1-j)^{m-2} \right) \\
 &\quad + \frac{(-it)^{m-1}}{(m-1)!} \Gamma_{m-1}(T; C) \left(\sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} (m-1-j)^{m-1} \right) \\
 &\quad + \frac{(-it)^m}{m!} \Gamma_m(T; C) \left(\sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} (m-1-j)^m \right) + \dots .
 \end{aligned}$$

Moreover, since T is a skew m -complex symmetric operator, it follows from Lemma 2.1 and $\Gamma_k(T; C) = 0$ for some $k \geq m$ that

$$\left(\sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} (m-1-j)^{m-1} \right) \frac{(-it)^{m-1}}{(m-1)!} \Gamma_{m-1}(T; C) = 0. \tag{8}$$

Since

$$\sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} (m-1-j)^{m-1} = (-1)^{m-1} (m-1)! \neq 0$$

by Lemma 2.1, it follows that (8) implies $\Gamma_{m-1}(T; C) = 0$. Hence T is a skew $(m-1)$ -complex symmetric operator. On the other hand, since m is even and e^{iT} is invertible, it follows from Lemma 2.9 and the hypothesis that e^{iT} is an $(m-1, C)$ -isometric operator. \square

Corollary 2.11. *If $T \in \mathcal{L}(\mathcal{H})$ is skew m -complex symmetric with a conjugation C , m is even, and $T = CTC$, then T is skew $(m-1)$ -complex symmetric.*

Proof. Suppose that T is skew m -complex symmetric and m is even. Since $T = CTC$, it follows that

$$\Lambda_{m-1}(e^{iT}; C) \text{ and } ((e^{iT})^*)^{m-1} \Lambda_{m-1}(e^{-iT}; C) C (e^{iT})^{m-1} C$$

are nonnegative. Hence T is skew $(m-1)$ -complex symmetric from Theorem 2.10. \square

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