Filomat 33:10 (2019), 3013–3022
https://doi.org/10.2298/FIL1910013L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Completeness of a Normed Space via Strong p-Cesàro Summability

F. León-Saavedra[a], S. Moreno-Pulido[b], A. Sala-Pérez[b]

[a] *Departamento de Matemáticas, Universidad de Cádiz, Facultad de Ciencias Sociales y de la Comunicación, Avenida de la Universidad s/n, 11403-Jerez de la Frontera (Cádiz)*
[b] *Departamento de Matemáticas, Universidad de Cádiz, Escuela Superior de Ingeniería, Avenida de la Universidad s/n, Puerto Real (Cádiz)*

Abstract. In this paper we will characterize the completeness and barrelledness of a normed space through the strong p-Cesáro summability of series. A new characterization of weakly unconditionally Cauchy series and unconditionally convergent series through the strong p-Cesàro summability is obtained.

1. Introduction

Let X be a normed space and $0 < p < \infty$, a sequence (x_k) is said to be strongly p-Cesàro summable to $L \in X$ if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \|x_k - L\|^p = 0.$$

The strong 1-Cesàro summability for real numbers was introduced by Hardy-Littlewood [9] and Fekete [6] in connection with the convergence of Fourier series (see [17], for historical notes, and the most recent monograph [2]).
Some years later, in 1935, Professor A. Zygmund (see [18] for one of the reprints) introduced the idea of statistical convergence in a independently way. A sequence (x_n) is statistically convergent to L if for any $\varepsilon > 0$ the subset $\{k : \|x_k - L\| < \varepsilon\}$ has density 1 on the natural numbers.
Both concepts were developed independently and surprisingly enough, both are related thanks to a result by J. Connor ([5]). Since then, in this circle of ideas, a significant number of deep and beautiful results have been obtained by Connor, Fridy, Mursaleen...and many others (see [1, 4, 8, 11, 12, 14–16])
There are also results that obtain characterizations of properties of Banach spaces through convergence types. For instance, Kolk [10] was one of the pionnering contributors. Connor, Ganichev and Kadets [3] obtained important results that relate the statistical convergence to classical properties of Banach spaces.
The aim of this paper is to obtain properties of a Banach space studying properties of strong p-Cesàro summability of a series. Let X be a normed space, and set $\sum x_i$ a series in X. In [13] the spaces of convergence $S(\sum x_i)$ associated to the series $\sum x_i$ are introduced. $S(\sum x_i)$ is defined as the sequences $(a_j) \in \ell_\infty$ such that $\sum a_i x_i$ converges. The space X is complete if and only if for every weakly unconditionally Cauchy

2010 *Mathematics Subject Classification*. Primary 40A05; Secondary 46B15
Keywords. statistical convergence, strong p-Cesàro summability, weak undonditionally Cauchy series
Received: 18 October 2018; Revised 03 July 2019; Accepted: 09 July 2019
Communicated by Eberhard Malkowsky
All authors contributed equally to the work.
Email addresses: fernando.leon@uca.es (F. León-Saavedra), soledad.moreno@uca.es (S. Moreno-Pulido), antonio.sala@uca.es (A. Sala-Pérez)

series $\sum x_i$, the space $S(\sum x_i)$ is complete. Moreover, the space X is barrelled if and only if each series $\sum_i x_i^*$ in X^* if the corresponding space of weak-$*$ convergence associated to $\sum x_i^*$ is the entire space ℓ_∞, that is, $S_{w^*}(\sum x_i^*) = \ell_\infty$.

In this paper we explore this structure for the strong-p Cesàro summability. At first glance, it seems that in order to show that a sequence is strongly $p-$Cesàro summable it is necessary to know the value of its limit previously. However, thanks to the results by Connor [5] and Fridy [7], we can avoid this difficulty. Section 2 is an expository section where we will show examples and preliminary aspects related to the strong p-Cesàro summability. Section 3 deals with space of strong p-Cesàro summability. It is shown that a series in a Banach space is weakly unconditionally Cauchy if and only if its space of strong p-Cesàro summability is complete. Moreover, if $p \geq 1$ and this equivalence is true for each series in a normed space X, then the space X must be complete. In Section 4 and 5 we will begin by defining reasonably, the strong p-Cesàro summability for the weak and the weak-$*$ topology in a Banach space X and its dual X^* respectively. After this, we will show analogous results for the strong p-Cesàro summability in these topologies. We also prove a characterization of barrelledness which is similar to the aforementioned one, but replacing weak-$*$ convergence by our concept of strong p-Cesàro summability for the weak-$*$ topology.

2. Some preliminary results

We begin this section by recalling some preliminaries we will need throughout this work. If $A \subset \mathbb{N}$, the density of A is denoted by $d(A) = \lim_n \frac{1}{n}\text{card}(\{k \leq n : k \in A\})$, whenever this limit exists.

Let X be a normed space and $x = (x_k)_k$ a sequence in X. The sequence x is said to be *statistically convergent* if there is $L \in X$ such that for every $\varepsilon > 0$, $d\Big(\{k : \|x_k - L\| \geq \varepsilon\}\Big) = 0$ or equivalently $d\Big(\{k : \|x_k - L\| < \varepsilon\}\Big) = 1$ and we will write $(x_k) \xrightarrow{st} L$ and $L = st - \lim_n x_n$. The sequence x is said to be *statistically Cauchy* if for each $\varepsilon > 0$ and $n \in \mathbb{N}$, there exists $p \geq n$ such that $d\Big(\{k : \|x_k - x_p\| \geq \varepsilon\}\Big) = 0$ or equivalently $d\Big(\{k : \|x_k - x_p\| < \varepsilon\}\Big) = 1$.

Fridy [7, Theorem 1] proved that in a Banach space, a sequence is statistically convergent if and only if it is statistically Cauchy.

Let us consider now $0 < p < +\infty$. The sequence x is said to be *strongly $p-$Cesàro or w_p summable* if there is $L \in X$ such that $\displaystyle\lim_n \frac{1}{n}\sum_{k=1}^{n} \|x_k - L\|^p = 0$, in which case we say that x is strongly $p-$Cesàro summable to L, and we will write $(x_k) \xrightarrow{\mathrm{w}_p} L$ and $L = \mathrm{w}_p - \lim_n x_n$.

Although the convergent sequences are w_p summable, it is easy to see that this kind of convergence is weaker than the usual, as we will show in the next example:

Example 2.1. *There exist unbounded sequences that are strong $p-$Cesàro summable.*
Define the real-valued sequence (x_k) by

$$x_k = \begin{cases} 0, & k \neq j^3 \text{ for all } j \in \mathbb{N}. \\ j, & k = j^3 \text{ for some } j \in \mathbb{N}. \end{cases}$$

Observe that the sequence (x_k) is unbounded. Let $n \in \mathbb{N}$ be given and suppose that $r^3 \leq n < (r+1)^3$ for some $r \in \mathbb{N}$. Now note that

$$\frac{1}{n}\sum_{k=1}^{n} |x_k| = \frac{1}{n}\sum_{k=1}^{n} x_k \leq \frac{1}{r^3}\sum_{k=1}^{r^3} x_k = \frac{1 + \cdots + r}{r^3}$$

tends to 0 as n tends to infinity.

However, a w_p summable $X-$valued sequence (x_k) necessarily satisfies that $(\frac{1}{n}\sum_{k=1}^{n} \|x_k\|^p)$ is a bounded sequence, as the following proposition shows:

Proposition 2.2. *Let $0 < p < \infty$ and $(x_k)_k$ be a w_p summable sequence to L in a normed space X. Then, $(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|^p)_n$ is a bounded sequence.*

Proof. Since (x_k) is w_p summable sequence to $L \in X$,

$$\lim_n \frac{1}{n}\sum_{k=1}^{n}\|x_k - L\|^p = 0.$$

Then, for all $n \in \mathbb{N}$:

$$\frac{1}{n}\sum_{k=1}^{n}\|x_k\|^p \leq \frac{1}{n}\sum_{k=1}^{n}(\|x_k - L\| + \|L\|)^p \leq \frac{1}{n}\sum_{k=1}^{n}\|x_k - L\|^p + \|L\|^p,$$

which implies that $(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|^p)_n$ is a bounded sequence. □

Connor [5, Theorem 2.1] discovered that the real bounded sequences w_p convergent are exactly the statistically convergent sequences. This fact also holds for normed spaces and we include the proof for the sake of completeness.

Proposition 2.3 (Connor [5]). *Set $0 < p < \infty$ and let X be a normed space. If a sequence is strongly $p-$Cesàro summable to L, then it is statistically convergent to L. Additionally, if the sequence is bounded, the converse is also true.*

Proof. Let us consider $(x_k)_k$ a sequence which is strongly $p-$Cesàro summable to $L \in X$ and $\varepsilon > 0$. For any $n \in \mathbb{N}$,

$$\sum_{k=1}^{n}\|x_k - L\|^p \geq \sum_{\substack{k=1\\ \|x_k - L\| \geq \varepsilon}}^{n}\|x_k - L\|^p \geq \sum_{\substack{k=1\\ \|x_k - L\| \geq \varepsilon}}^{n}\varepsilon^p = \mathrm{card}\Big(\{k \leq n : \|x_k - L\|^p \geq \varepsilon\}\Big)\,\varepsilon^p.$$

Since $(x_k)_k$ is strongly $p-$Cesàro summable to L, we have that $\lim_n \frac{1}{n}\sum_{k=1}^{n}\|x_k - L\|^p = 0$, so for every $\varepsilon > 0$, $\lim_n \frac{1}{n}\mathrm{card}\{k \leq n : \|x_k - L\| \geq \varepsilon\} = 0$ which shows that $(x_k)_k$ is statistically convergent to L.

Suppose now that $x = (x_k)_k$ is a bounded sequence which is statistically convergent to $L \in X$ and set $K = \|x\|_\infty + \|L\|$, where $\|x\|_\infty = \sup_k \|x_k\|$. Given $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that

$$\frac{1}{n}\mathrm{card}\left(\left\{k \leq n : \|x_k - L\| \geq \left(\frac{\varepsilon}{2}\right)^{1/p}\right\}\right) < \frac{\varepsilon}{2K^p},$$

for every $n \geq N_\varepsilon$. Set $L_n = \{k \leq n : \|x_k - L\| \geq \left(\frac{\varepsilon}{2}\right)^{1/p}\}$. For every $n \geq N_\varepsilon$, we have:

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n}\|x_k - L\|^p &= \frac{1}{n}\left[\sum_{k\in L_n}\|x_k - L\|^p + \sum_{\substack{k\leq n\\ k\notin L_n}}\|x_k - L\|^p\right]\\
&< \frac{1}{n}\Big(K^p\mathrm{card}(L_n) + \frac{n\varepsilon}{2}\Big)\\
&< \frac{1}{n}\left((K^p)\left(\frac{n\varepsilon}{2K^p}\right) + \frac{n\varepsilon}{2}\right) = \varepsilon.
\end{aligned}$$

Thus, $(x_k)_k$ is strongly $p-$Cesàro summable to L. □

Next, we show that for the converse, boundedness is necessary.

Example 2.4. *There exist unbounded statistically convergent sequences which are not strongly $p-$Cesàro summable. Indeed, set $n_j = j^2$ and let us define*

$$x_k = \begin{cases} 0, & k \neq j^2 \text{ for all } j. \\ j^{2/p}, & k = j^2 \text{ for some } j. \end{cases}$$

The sequence $(x_k)_k$ is unbounded. Take $\varepsilon > 0$, it is easy to see that $d(\{k \leq n : |x_k| \geq \varepsilon\}) = 0$, so $(x_k)_k$ is statistically convergent to zero. Let us Observe that:

$$\frac{1}{n_j}\sum_{k=1}^{n_j}|x_k|^p = \frac{1}{n_j}\sum_{k=1}^{n_j}x_k^p = \frac{1+2^2+3^2+\cdots+j^2}{j^2},$$

which diverges as $j \to \infty$. Hence, by applying Proposition 2.2, we deduce that $(x_k)_k$ is not strongly $p-$Cesàro summable.

Let us recall that a sequence $x = (x_k)_k$ in a normed space X is said to be *Cesàro convergent* if there is $L \in X$ such that $\lim_n \left\| \frac{1}{n}\sum_{k=1}^{n} x_k - L \right\| = 0$. The w_p summability is related to the Cesàro convergence in a natural way:

Proposition 2.5. *Let X be a normed space and $(x_k)_k$ a sequence in X. If $p \geq 1$ and $(x_k)_k$ is strongly $p-$summable to L, then $(x_k)_k$ is Cesàro convergent to L.*

Proof. Let us observe that

$$\begin{aligned} 0 &\leq \left\| \frac{1}{n}\sum_{k=1}^{n} x_k - L \right\| = \frac{1}{n}\left\| \sum_{k=1}^{n} x_k - nL \right\| = \frac{1}{n}\left\| \sum_{k=1}^{n}(x_k - L) \right\| \\ &\leq \frac{1}{n}\sum_{k=1}^{n}\|x_k - L\| \leq \frac{1}{n}\sum_{k=1}^{n}\|x_k - L\|^p \underset{n\to\infty}{\to} 0 \end{aligned}$$

□

Remark 2.6. *Let us observe that the condition $p \geq 1$ is sharp. Indeed, the sequence*

$$x_k = \begin{cases} 0, & k \neq r^3 \text{ for all } r. \\ r^2, & k = r^3 \text{ for some } r. \end{cases}$$

is $\frac{1}{2}-$ Cesàro summable to zero, and the Cesàro means do not converge to zero.

The converse of Proposition 2.5 is clearly not true, as we show in the next example.

Example 2.7. *There exist Cesàro convergent sequences which are not $p-$Cesàro summable. Let us define*

$$x_k = \begin{cases} 1 & \text{if } k \text{ is odd,} \\ 0 & \text{if } k \text{ is even.} \end{cases}$$

$(x_k)_k$ is not strong $p-$Cesàro summable to any $L \in \mathbb{R}$ because it is not statistically convergent to any L. However, observe that:

$$\left| \frac{1}{n}\sum_{k=1}^{n} x_k \right| = \frac{n/2}{n} \to \frac{1}{2},$$

so $(x_k)_k$ is Cesàro convergent to $\frac{1}{2}$.

Finally, for future references, we have:

Proposition 2.8. *Let $(x_k)_k$ be a sequence in $\mathbb{R}$ such that $\sum_{k=1}^{\infty} x_k = L \in \mathbb{R}\cup\{\pm\infty\}$. If $S_k = \sum_{j=1}^{k} x_j$, then $\lim_n \frac{1}{n}\sum_{k=1}^{n} S_k = L$.*

3. The strong $p-$Cesàro summability space

Let $\sum_i x_i$ be a series in a real Banach space X, set $0 < p < +\infty$ and let us define

$$S_{w_p}\left(\sum_i x_i\right) = \left\{(a_i)_i \in \ell_\infty : \sum_i a_i x_i \text{ is } w_p \text{ summable}\right\}$$

endowed with the supremum norm. This space will be called the space of w_p summability associated to the series $\sum_i x_i$. The following theorem characterizes the completeness of the space $S_{w_p}\left(\sum_i x_i\right)$.

Theorem 3.1. *Let X be a real Banach space and $0 < p < +\infty$. The following conditions are equivalent:*

(1) $\sum_i x_i$ is a weakly unconditionally Cauchy series (wuc).

(2) $S_{w_p}(\sum_i x_i)$ is a complete space.

(3) $c_0 \subset S_{w_p}(\sum_i x_i)$.

Proof. Let us show that (1)⇒(2). Since $\sum x_i$ is wuc, the following supremum is finite:

$$H = \sup\left\{\left\|\sum_{i=1}^{n} a_i x_i\right\| : |a_i| \le 1, 1 \le i \le n, n \in \mathbb{N}\right\} < +\infty.$$

Let $(a^m)_m \subset S_{w_p}(\sum_i x_i)$ such that $\lim_m \|a^m - a^0\|_\infty = 0$, with $a^0 \in \ell_\infty$. We will prove that $a^0 \in S_{w_p}(\sum_i x_i)$. Let us suppose without any loss of generality that $\|a^0\|_\infty \le 1$. Then, the partial sums $S_k^0 = \sum_{i=1}^k a_i^0 x_i$ satisfy $\|S_k^0\| \le H$ for every $k \in \mathbb{N}$, that is, the sequence (S_k^0) is bounded. Then, $a^0 \in S_{w_p}(\sum_i x_i)$ if and only if (S_k^0) is w_p summable to some $L \in X$. Since (S_k^0) is bounded, according to Connor's Theorem [5, Theorem 2.1] (Proposition 2.3), (S_k^0) is w_p summable if and only if (S_k^0) is statistically convergent to some $L \in X$. According to [7, Theorem 1], (S_k^0) is statistically convergent to $L \in X$ if and only if (S_k^0) is a statistically Cauchy sequence.
Set $\varepsilon > 0$ and $n \in \mathbb{N}$. Then, we obtain statement (2) if we show that there exists $p_0 \ge n$ such that

$$d\Big(\{k : \|S_k^0 - S_{p_0}^0\| < \varepsilon\}\Big) = 1.$$

Given $\varepsilon > 0$, since $a^m \to a^0$ in ℓ_∞, there exists $m_0 > n$ such that $\|a^m - a^0\|_\infty < \dfrac{\varepsilon}{4H}$ for all $m > m_0$, and since $S_k^{m_0}$ is statistically Cauchy, there exists $p_0 \ge n$ such that the density $d\left(\left\{k : \|S_k^{m_0} - S_{p_0}^{m_0}\| < \frac{\varepsilon}{2}\right\}\right) = 1$. Fix k such that

$$\|S_k^{m_0} - S_{p_0}^{m_0}\| < \frac{\varepsilon}{2}. \tag{3.1}$$

We will show that $\|S_k^0 - S_{p_0}^0\| < \varepsilon$, and this will prove that

$$\left\{k : \|S_k^{m_0} - S_{p_0}^{m_0}\| < \frac{\varepsilon}{2}\right\} \subset \{k : \|S_k^0 - S_{p_0}^0\| < \varepsilon\}.$$

Since the first set has density 1, the second will also have density 1 and we will be done.

Let us observe first that for every $j \in \mathbb{N}$,

$$\left\| \sum_{i=1}^{j} \frac{4H}{\varepsilon}(a_i^p - a_i^{m_0})x_i \right\| \leq H,$$

therefore

$$\left\| S_j^0 - S_j^{m_0} \right\| = \left\| \sum_{i=1}^{j} (a_i^0 - a_i^{m_0})x_i \right\| \leq \frac{\varepsilon}{4}. \tag{3.2}$$

Then, by applying the triangular inequality,

$$\begin{aligned} \left\| S_k^0 - S_{p_0}^0 \right\| &\leq \left\| S_k^0 - S_k^{m_0} \right\| + \left\| S_k^{m_0} - S_{p_0}^{m_0} \right\| + \left\| S_{p_0}^0 - S_{p_0}^{m_0} \right\| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon. \end{aligned}$$

where the last inequality follows by applying (3.1) and (3.2), which yields to the desired result.
Now, let us observe that if $S_{w_p}(\sum_i x_i)$ is a complete space, it contains the space of eventually zero sequences c_{00} and therefore we get (2) ⇒ (3).

Finally, let us show (3) ⇒ (1). If the series $\sum x_i$ is not wuc, there exists $f \in X^*$ such that $\sum_{i=1}^{\infty} |f(x_i)| = +\infty$. Inductively, we will construct a sequence $(a_i)_i \in c_0$ such that $\sum_i a_i f(x_i) = +\infty$ and $a_i f(x_i) \geq 0$. If we denote by $S_k = \sum_{i=1}^{k} a_i f(x_i)$, then by applying Proposition 2.8, $\lim_n \frac{1}{n} \sum S_k = +\infty$. This implies that, by applying Proposition 2.2, $(S_k)_k$ is not w_p summable to any $L \in \mathbb{R}$, which is a contradiction with statement (3).
Since $\sum_{i=1}^{\infty} |f(x_i)| = +\infty$, there exists m_1 such that $\sum_{i=1}^{m_1} |f(x_i)| > 2 \cdot 2$. We define $a_i = \frac{1}{2}$ if $f(x_i) \geq 0$ and $a_i = -\frac{1}{2}$ if $f(x_i) < 0$ for $i \in \{1, 2, \ldots, m_1\}$. This implies that $\sum_{i=1}^{m_1} a_i f(x_i) > 2$ and $a_i f(x_i) \geq 0$ if $i \in \{1, 2, \ldots, m_1\}$.
Let $m_2 > m_1$ be such that $\sum_{i=m_1+1}^{m_2} |f(x_i)| > 2^2 \cdot 2^2$. We define $a_i = \frac{1}{2^2}$ if $f(x_i) \geq 0$ and $a_i = -\frac{1}{2^2}$ if $f(x_i) < 0$ for $i \in \{m_1 + 1, \ldots, m_2\}$. Then, $\sum_{i=m_1+1}^{m_2} a_i f(x_i) > 2^2$ and $a_i f(x_i) \geq 0$ if $i \in \{m_1 + 1, \ldots, m_2\}$.
Inductively we obtain a sequence $(a_i)_i \in c_0$ with the above properties which lead us to a contradiction. □

Remark 3.2. *Let us observe that in the above proof, the completeness hypothesis is used in the implication* (1) ⇒ (2). *Specifically, when we use Fridy's result ([7, Theorem 1]). On the other hand, the implication* (2) ⇒ (3) *that we will use in Theorem 3.5 does not use the completeness of the space X.*

Remark 3.3. *Let $\sum_i x_i$ be a series in a normed space X and let*

$$S\left(\sum_i x_i\right) = \left\{ (a_i)_i \in \ell_\infty : \sum_i a_i x_i \text{ converges} \right\},$$

endowed with the supremum norm. Clearly, $S(\sum_i x_i)$ is a subspace of ℓ_∞ and $S(\sum_i x_i) \subseteq S_{w_p}(\sum_i x_i)$. If X is a Banach space, then $\sum_i x_i$ is wuc if and only if $S(\sum_i x_i)$ is complete [13]. Theorem 3.1 gives us a similar characterization by considering w_p summability.

Corollary 3.4. *Let X be a Banach space, $\sum_i x_i$ a series in X and $p \geq 1$. The following properties are equivalent:*

(1) $\sum_i x_i$ is wuc.

(2) $S(\sum_i x_i)$ is a complete space.

(3) $c_0 \subseteq S(\sum_i x_i)$.

(4) $S_{\mathrm{w}_p}(\sum_i x_i)$ *is a complete space.*

(5) $c_0 \subseteq S_{\mathrm{w}_p}(\sum_i x_i)$.

(6) $\sum |f(x_i)|$ is w_p *summable for every* $f \in X^*$.

Proof. The first three equivalence properties (1), (2) and (3) can be found in [13] and the rest of equivalences are consequences of Theorem 3.1. □

Now let us show another main theorem.

Theorem 3.5. *Let X be a normed space and $p \geq 1$. Then X is complete if and only if $S_{\mathrm{w}_p}(\sum_i x_i)$ is a complete space for every $\sum_i x_i$.*

Proof. By Theorem 3.1, the condition is necessary. Now if X is not complete, there exists $\sum x_i$ a series in X such that $\|x_i\| \leq \frac{1}{i2^i}$ and $\sum x_i = x^{**} \in X^{**} \setminus X$. We will construct a wuc series $\sum_n y_n$ such that $S_{\mathrm{w}_p}(\sum_n y_n)$ is not complete, a contradiction.

Indeed, since X^{**} is a Banach space with the dual topology, if $S_k = \sum_{j=1}^{k} x_j$, $\sup_{\|y^*\|\leq 1} |y^*(S_n) - x^{**}(y^*)| \to 0$, that is, $\sum_{i=1}^{\infty} y^*(x_i) = x^{**}(y^*)$, for all $\|y^*\| \leq 1$. By applying Proposition 2.8, we have:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} y^*(S_k) = x^{**}(y^*) \tag{3.3}$$

Set $y_n = nx_n$ and let us observe that since $\|y_n\| < \frac{1}{2^n}$, $\sum y_n$ is absolutely convergent, and hence weakly unconditionally Cauchy. We claim that the series $\sum_n \frac{1}{n} y_n$ is not w_p summable in X.

On the contrary, let us suppose that $S_N = \sum_{n=1}^{N} \frac{1}{n} y_n$ is w_p summable in X. That is, there exists $L \in X$ such that $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \|S_n - L\|^p = 0$. In particular, for every $y^* \in X^*$ with $\|y^*\| \leq 1$ we have that $\sup_{\|y^*\|\leq 1} \frac{1}{N} \sum_{k=1}^{N} |y^*(S_k - L)|^p \to 0$. By applying Proposition 2.5, since $p \geq 1$, we have that

$$\frac{1}{N} \sum_{k=1}^{N} y^*(S_k) = y^*(L), \text{ for every } \|y^*\| \leq 1. \tag{3.4}$$

From equations 3.3 and 3.4 and the uniqueness of the limit, we have that $x^{**}(y^*) = y^*(L)$ for every $\|y^*\| \leq 1$, so we obtain $x^{**} = L \in X$, which is a contradiction. This means that $S_N = \sum_{n=1}^{N} \frac{1}{n} y_n$ is not w_p summable to any $L \in X$.

Finally, let us observe that since $\sum_n y_n$ is a weakly unconditionally Cauchy series and $S_N = \sum_{n=1}^{N} \frac{1}{n} y_n$ is not w_p summable, we have that $(\frac{1}{n}) \notin S_{w_p}(\sum_n y_n)$ and this means that $c_0 \not\subseteq S_{\mathrm{w}_p}(\sum_n y_n)$ which is a contradiction according to (3) in Theorem 3.1 (see Remark 3.2) and the proof is completed. □

Theorem 3.6. *Let $\sum_i x_i$ be a series in a Banach space. The series $\sum_i x_i$ is wuc if and only if the operator $T : S_{\mathrm{w}_p}(\sum_i x_i) \to X$ defined by $T((a_i)_i) = \mathrm{w}_p - \lim_n S_n$ is continuous (where $S_n = \sum_{i=1}^{n} a_i x_i$).*

Proof. Suppose that T is continuous and let us show that $\sum_i x_i$ is wuc. Since $c_{00} \subset S_{\mathrm{w}_p}(\sum_i x_i)$, for every $(a_i)_i \in c_{00}$, $\|T\big((a_i)_i\big)\| \leq \|T\| \|(a_i)_i\|_\infty$. Hence,

$$\sup_{n\in\mathbb{N}} \left\{ \left\| \sum_{i=1}^{n} a_i x_i \right\| : |a_i| \leq 1 \right\} \leq \|T\|,$$

and this implies that the series $\sum x_i$ is wuc.
Let us suppose that $\sum_i x_i$ is wuc. Then,

$$H = \sup_{n\in\mathbb{N}} \left\{ \left\| \sum_{i=1}^{n} a_i x_i \right\| : |a_i| \leq 1 \right\} < +\infty.$$

Set $a = (a_i)_i \in S_{\mathrm{w}_p}(\sum_i x_i)$ such that $\|a\|_\infty = 1$. Then, $S_n = \sum_{i=1}^{n} a_i x_i$ is w_p summable and since it is a bounded sequence, by applying Connor's Theorem 2.3, it is statistically convergent to some L, and $L = st - \lim_n S_n = \mathrm{w}_p - \lim_n S_n = T\Big((a_i)_i\Big)$. By applying Fridy's result [7, Theorem 1], there exists $A \subset \mathbb{N}$ of density 1 such that $\lim_{\substack{n\\ n\in A}} \|S_n - L\| = 0$. For every $k \in A$, $\|S_k\| \leq H$, so

$$\|T\Big((a_i)_i\Big)\| = \|L\| = \lim_{\substack{k\\ k\in A}} \|S_k\| \leq \|H\|.$$

which proves that T is continuous and this completes the desired result. □

4. The space of weak w_p− summability

In this section we study a similar structure with respect to the weak w_p− summability.
Let X be a normed space. Set $0 < p < +\infty$, a sequence $(x_k)_k$ is said to be *weak* w_p− *summable* to $L \in X$ if for every $f \in X^*$, $f(x_k) \stackrel{\mathrm{w}_p}{\to} f(L)$, that is,

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} |f(x_k) - f(L)|^p = 0,$$

and we will write $(x_k) \stackrel{w-\mathrm{w}_p}{\to} L$ and $L = w - \mathrm{w}_p - \lim_n x_n$.
Let $\sum_i x_i$ be a series in a Banach space X, $0 < p < +\infty$. We now consider the space of $w - \mathrm{w}_p$ summability given by:

$$S_{w-\mathrm{w}_p}\left(\sum_i x_i\right) = \left\{ (a_i)_i \in \ell_\infty : \sum_i a_i x_i \text{ is } w - \mathrm{w}_p \text{ summable} \right\}$$

endowed with the supremum norm.

Theorem 4.1. *Let $0 < p < +\infty$. The following conditions are equivalent:*

(1) $\sum_i x_i$ is a weakly unconditionally Cauchy series (wuc).

(2) $S_{w-\mathrm{w}_p}(\sum_i x_i)$ is a complete space.

(3) $c_0 \subset S_{w-\mathrm{w}_p}(\sum_i x_i)$.

Proof. Since $\sum_i x_i$ is wuc,

$$H = \sup \left\{ \left\| \sum_{i=1}^{n} a_i x_i \right\| : |a_i| \leq 1, n \in \mathbb{N} \right\} < +\infty.$$

Let $(a^m)_m \subset S_{w-\mathrm{w}_p}(\sum_i x_i)$ be such that $\lim_m \|a^m - a^0\|_\infty = 0$, with $a^0 \in \ell_\infty$; we will prove that $a^0 \in S_{w-\mathrm{w}_p}(\sum_i x_i)$. Suppose without any loss of generality that $\|a^0\|_\infty \leq 1$. The sequence $S_k^0 = \sum_{i=1}^{k} a_i^0 x_i$ is bounded, and for every $f \in X^*$, we have that $f(S_k^0) = \sum_{i=1}^{k} a_i f(x_i)$ is a bounded sequence. We will show that $f(S_k^0)$ is w_p summable. By applying again Connor's Theorem 2.3 and Fridy's result, it is sufficient to prove that $\Big(f(S_k^0)\Big)$ is statistically convergent or equivalently, $\Big(f(S_k^0)\Big)$ is statistically Cauchy.

Given $\varepsilon > 0$, we will show that for every $n \in \mathbb{N}$ there exists $p_0 \geq n$ such that

$$d(\{k : |f(S_k^0) - f(S_{p_0}^0)| < \varepsilon\}) = 1.$$

Since $a^m \to a^0$ in ℓ_∞, there exists m_0 such that $\|a^m - a^0\|_\infty \leq \dfrac{\varepsilon}{4H\|f\|}$ for every $m \geq m_0$. And since $\left(f(S_k^{m_0})\right)$ is statistically Cauchy, for every $n \in \mathbb{N}$, there exists $p_0 \geq n$ such that the set $\left\{k : |f(S_k^{m_0}) - f(S_{p_0}^{m_0})| \leq \dfrac{\varepsilon}{2}\right\}$ has density 1. Let us consider $k \leq n$ such that

$$|f(S_k^{m_0}) - f(S_{p_0}^{m_0})| < \frac{\varepsilon}{2}. \tag{4.1}$$

Let us observe that, for every j, $\left\| \displaystyle\sum_{i=1}^{j} \frac{\varepsilon}{4H\|f\|}(a_i^0 - a_i^{m_0})x_i \right\| \leq H$, so we deduce that

$$\|S_j^0 - S_j^{m_0}\| = \left\| \sum_{i=1}^{j} (a_i^0 - a_i^{m_0})x_i \right\| \leq \frac{\varepsilon}{4\|f\|}. \tag{4.2}$$

Then, using (4.1) and (4.2) and the triangular inequality,

$$\begin{aligned} |f(S_k^0) - f(S_{p_0}^0)| &\leq |f(S_k^0 - S_k^{m_0})| + |f(S_{p_0}^0 - S_{p_0}^{m_0})| + |f(S_k^{m_0} - S_{p_0}^{m_0})| \\ &\leq \|f\|\frac{\varepsilon}{4\|f\|} + \|f\|\frac{\varepsilon}{4\|f\|} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

which implies that

$$\left\{k : |f(S_k^{m_0}) - f(S_{p_0}^{m_0})| \leq \frac{\varepsilon}{2}\right\} \subseteq \left\{k : |f(S_k^0) - f(S_{p_0}^0)| \leq \frac{\varepsilon}{2}\right\}$$

and since the first set has density 1, the second has also density 1 and we are done.
Finally, let us observe that implication (2) $\Rightarrow$ (3) is obvious and (3) $\Rightarrow$ (1) follows by a similar argument like in Theorem 3.1, and this finishes the proof. □

5. The weak* w_p- summability space

We begin this section by defining a reasonable concept for weak* w_p- summability. This convergence provides a different result due to the singular structure of this new topology.
Let X be a normed space, $0 < p < +\infty$ and $(f_k)_k$ a sequence in X^*. The sequence $(f_k)_k$ is said to be weak* w_p- summable to $f \in X^*$ if for every $x \in X$, $f_k(x) \stackrel{\mathrm{w}_p}{\to} f(x)$, that is,

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} |f_k(x) - f(x)|^p = 0,$$

and we will write $(f_k) \stackrel{w^*-\mathrm{w}_p}{\to} f$ and $f = w^* - \mathrm{w}_p - \lim_n f_n$.
Let $\sum_i f_i$ be a series in the dual space X^* of a Banach space X, $0 < p < +\infty$. We now consider the space of $w^* - \mathrm{w}_p$ summability defined by:

$$S_{w^*-\mathrm{w}_p}\left(\sum_i f_i\right) = \left\{(a_i)_i \in \ell_\infty : \sum_i a_i f_i \text{ is } w^* - \mathrm{w}_p \text{ summable}\right\}$$

endowed with the supremum norm.

Theorem 5.1. *Let X be a normed space and $\sum f_i$ a series in X^*. Let us consider the following statements:*

(1) $\sum_i f_i$ is a weakly unconditionally Cauchy series (wuc).

(2) $S_{w^*-w_p}(\sum_i f_i) = \ell_\infty$.

(3) If $x \in X$ and $M \subset \mathbb{N}$, then $\sum_{i\in M} f_i(x)$ is w_p convergent.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3)*, and if X is barrelled, then* (3) $\Rightarrow$ (1)*.*

Proof. If $(a_i)_i \in \ell_\infty$, since the unit ball of X^* is weak-star compact, the series $\sum_i a_i f_i$ is weak-star convergent in X^*. Hence, there exists $f \in X^*$ such that $\sum_{i=1}^n a_i f_i \overset{w^*}{\to} f$. This implies that for every $x \in X$, $\sum_i f_i(x) = f(x)$, and it is easily shown that $\sum_{i=1}^n a_i f_i(x) \overset{\mathrm{w}_p}{\to} f(x)$, which implies that $(a_i)_i \in S_{w^*-\mathrm{w}_p}$.
The implication (2) $\Rightarrow$ (3) follows directly.
Now, if X is barrelled, let us define

$$E = \left\{ \sum_{i=1}^n a_i f_i : n \in \mathbb{N}, |a_i| \le 1 \right\}.$$

In order to prove (3) $\Rightarrow$ (1), it is sufficient to show that E is pointwise bounded. Suppose on the contrary that there exists $x_0 \in X$ such that $\sum_i |f_i(x_0)|$ diverges. If $M^+ = \{i \in \mathbb{N} : f_i(x_0) \ge 0\}$ and $M^- = \{i \in \mathbb{N} : f_i(x_0) < 0\}$, then either $\sum_{i\in M^+} f_i(x_0)$ diverges or $\sum_{i\in M^-}(-f_i)(x_0)$ diverges. Then, by applying Proposition 2.8 and Proposition 2.2, we obtain that the series is not w_p convergent, which is a contradiction with (3). □

Finally, from the preceding results, the following questions arise:

Remark 5.2. *1. What happens if $p \in (0, 1)$? Is Theorem 3.5 still valid?*

2. For any non negative regular summability method A, the connection between strong A-summability and A-statistical convergence holds. Is it possible to generalize the results in this manuscript in that framework?

3. If we consider $\theta = (k_r)_r \subset \mathbb{N}$ a lacunar sequence, i.e. $k_r - k_{r-1} \to \infty$ as $r \to \infty$. What happens with the strong lacunary convergence N_θ and the statistical lacunary convergence S_θ?

Acknowledgements:
The authors would like to express their deepest gratitude towards the referee for valuable comments and suggestions. This work is supported by the FQM-257 research group of the University of Cádiz and the Research Grant PGC-101514-B-100 awarded by the Spanish Ministry of Science, Innovation and Universities and partially funded by the European Regional Development Fund.

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