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Certain q-Difference Operators and Their Applications to the Subclass of Meromorphic q-Starlike Functions

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Abstract. The main aim of this work is to find some coefficient inequalities and sufficient condition for some subclasses of meromorphic starlike functions by using q-difference operator. Here we also define the extended Ruscheweyh differential operator for meromorphic functions by using q-difference operator. Several properties such as coefficient inequalities and Fekete-Szego functional of a family of functions are investigated.

1. Introduction

Let $\mathcal{H}(\mathrm{E})$ denote the class of functions which are analytic in the open unit disk $\mathrm{E} = \{z : z \in \mathbb{C}, |z| < 1\}$. Also let $\mathcal{A}$ denote a subclass of analytic functions f in $\mathcal{H}(\mathrm{E})$, satisfying the normalization conditions $f(0) = f'(0) - 1 = 0$. In other words, a function f in $\mathcal{A}$ has Taylor-Maclaurin series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (z \in \mathrm{E}). \tag{1}$$

We denote $\mathcal{S}$ by a subclass of $\mathcal{A}$, consisting of univalent functions. Furthermore, we denote the class of starlike functions by $\mathcal{S}^*$. A function $f \in \mathcal{A}$ is in the class $\mathcal{S}^*$ of starlike functions if it satisfies the relation

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in \mathrm{E}).$$

A function f is said to be subordinate to a function g written as $f \prec g$, if there exists a schwarz function w with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. In particular if g is univalent in E and $f(0) = g(0)$, then $f(\mathrm{E}) \subset g(\mathrm{E})$.

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For two analytic functions f of the form (1) and g of the form

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (z \in \mathrm{E}),$$

the convolution (Hadamard product) of f and g is defined as:

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \quad (z \in \mathrm{E}).$$

We now recall some essential definitions and concepts of the q -calculus, which are useful in our investigations. We suppose throughout the paper that $0 < q < 1$ and

$$\mathbb{N} := \{1, 2, 3, ...\} = \mathbb{N}_0 \setminus \{0\}, \quad \mathbb{N}_0 := \{0, 1, 2, 3, ...\}.$$

Definition 1.1. *Let $q \in (0,1)$ and define the q-number $[\lambda]_q$ by*

$$[\lambda]_q = \begin{cases} \frac{1-q^{\lambda}}{1-q}, & \lambda \in \mathbb{C}, \\ \sum_{k=0}^{n-1} q^k = 1 + q + q^2 + ... + q^{n-1}, & \lambda = n \in \mathbb{N}. \end{cases}$$

Definition 1.2. *Let $q \in (0,1)$ and define the q-factorial $[n]_q!$ by*

$$[n]_q! = \begin{cases} 1, & n = 0, \\ \prod_{k=1}^{n-1} [k]_q, & n \in \mathbb{N}. \end{cases}$$

Definition 1.3. *Let $q \in (0,1)$ and define q-generalized Pochhammer symbol by*

$$[t]_{q,n} = \begin{cases} 1, & n = 0, \\ \prod_{k=0}^{n-1} [t+k]_q, & n \in \mathbb{N}. \end{cases}$$

Definition 1.4. *For $t > 0$, let the q-gamma function be defined as:*

$$\Gamma_q(t+1) = [t]_q \Gamma_q(t) \quad \text{and} \quad \Gamma_q(1) = 1.$$

Definition 1.5. (*see [5] and [6]*) *The q-derivative (or q-difference) of a function f of the form* (1) *is denoted by D_q and defined in a given subset of $\mathbb{C}$ by*

$$D_q f(z) = \begin{cases} \frac{f(z) - f(qz)}{(1-q)z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases} \tag{2}$$

When $q \longrightarrow 1^-$, the difference operator D_q approaches to the ordinary differential operator. That is

$$\lim_{q \longrightarrow 1^-} \left(D_q f\right)(z) = f'(z).$$

The operator D_q provides an important tool that has been used in order to investigate the various subclasses of analytic functions of the form given in Definition 1.5. A q-extension of the class of starlike functions was first introduced in [4] by means of the q-difference operator, a firm footing of the usage of the q-calculus in the context of Geometric Functions Theory was actually provided and the basic (or q-) hypergeometric functions were first used in Geometric Function Theory by Srivastava (see, for details [14]). After that, wonderful research work has been done by many mathematicians which has played an important role in the development of Geometric Function Theory. In particular, Srivastava and Bansal [17] studied the close-to-convexity of q-Mittag-Leffler functions. The authors in [16] have investigated the Hankel determinant of a subclass of bi-univalent functions defined by using symmetric q-derivative. Mahmood *et al.* [10] studied the class of q-starlike functions in the conic region, while in [9], the authors studied the class of q-starlike functions related with Janowski functions. The upper bound of third Hankel determinant for the class of q-starlike functions has been investigated in [11]. Recently Srivastava *et al.* [15] have investigated the Hankel and Toeplitz determinants of a subclass of q-starlike functions, while the authors in [18] have introduced and studied a generalized class of q-starlike functions. Motivated by the above mentioned work, in this paper our aim is to present some subclasses of meromorphic starlike functions by using q-difference operator. We also introduce Ruscheweyh differential operator for meromorphic functions by using q-difference operator.

Definition 1.6. *(see [4]) A function* $f \in \mathcal{H}(\mathrm{E})$ *is said to belong to the class* $\mathcal{PS}_q$*, if*

$$f(0) = f'(0) - 1 = 0 \tag{3}$$

and

$$\left|\frac{z}{f(z)}\left(D_q f\right)(z) - \frac{1}{1-q}\right| \leq \frac{1}{1-q} \quad (z \in \mathrm{E}). \tag{4}$$

It is readily observed that as $q \to 1^-$, the closed disk

$$\left|w - (1-q)^{-1}\right| \leq (1-q)^{-1}$$

becomes the right−half plane and the class $\mathcal{PS}_q$ reduces to $\mathcal{S}^*$. Equivalently, by using the principle of subordination between analytic functions, we can rewrite the conditions in (3) and (4) as follows (see [19]) :

$$\frac{z}{f(z)}\left(D_q f\right)(z) \prec \widehat{p}(z), \qquad \widehat{p}(z) = \frac{1+z}{1-qz}.$$

Let $\mathcal{M}$ denote the class of functions f of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n, \tag{5}$$

which are analytic in the punctured open unit disk

$$\mathrm{E}^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = \mathrm{E} - \{0\}.$$

A function $f \in \mathcal{M}$ is said to be in the class $\mathcal{MS}^*(\alpha)$ of meromorphically starlike functions of order α, if it satisfies the inequality

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathrm{E}); \quad 0 \leqq \alpha < 1.$$

Let $\mathcal{P}$ denote the class of analytic functions p normalized by

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{6}$$

such that

$$\Re(p(z)) > 0 \quad (z \in \mathrm{E}).$$

Next, we extend the idea of q-difference operator analogous to the Definition 1.5 to a function f given by (5) and introduce the class $\mathcal{MS}_q(\beta, \lambda)$.

Definition 1.7. *Let $f \in \mathcal{M}$. Then the q-derivative operator or q-difference operator for the function f of the form* (5) *is defined by*

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z} = -\frac{1}{qz^2} + \sum_{n=0}^{\infty} [n]_q a_n z^{n-1} \quad (z \in \mathrm{E}^*).$$

Definition 1.8. *Let $f \in \mathcal{M}$. Then $f \in \mathcal{MS}_q(\beta, \lambda)$, if it satisfies the condition*

$$\left| \frac{-z\frac{D_q f(z)}{f(z)} - \beta z^2 \frac{D_q\left(D_q f(z)\right)}{f(z)} - \gamma}{1-\gamma} - \frac{1}{1-q} \right| \leqq \frac{1}{1-q}, \tag{7}$$

which by using subordination can be written as:

$$\frac{-zD_q f(z) - \beta z^2 D_q\left(D_q f(z)\right)}{f(z)\left(\frac{1}{q} - \Upsilon(\beta, q)\right)} \prec \frac{1 + (1 - \gamma(1+q))z}{1-qz}. \tag{8}$$

Remark 1.9. *It can easily be seen that*

$$\lim_{q \longrightarrow 1^-} \mathcal{MS}_q(\beta, \lambda) = \mathcal{H}(\beta, \lambda).$$

The class $\mathcal{H}(\beta, \lambda)$ was introduced and studied by Wang et al. [20, 21]. Secondly, we have

$$\lim_{q \longrightarrow 1^-} \mathcal{MS}_q(0, \lambda) = \mathcal{H}(0, \lambda) = \mathcal{MS}^*(\lambda),$$

introduced and studied by Wang et al. See [21].

Throughout this paper unless otherwise stated the parameters β and λ are considered as follows:

$$\beta \geqq 0 \quad \text{and} \quad \frac{1}{2} \leqq \lambda < 1 \tag{9}$$

and

$$\Lambda_q(n, \beta, \gamma) = [n]_q + \beta [n]_q [n-1]_q + \gamma, \tag{10}$$

$$\gamma = \lambda - \beta\lambda\left(\lambda + \frac{1}{2}\right) - \frac{\beta}{2}, \tag{11}$$

$$\Upsilon(\beta, q) = \beta\frac{(1+q)}{q^2}. \tag{12}$$

2. Preliminary Results

Lemma 2.1. *[8] If a function p of the form* (6) *is in class $\mathcal{P}$, then*

$$\left|p_2 - \upsilon p_1^2\right| \leqq \begin{cases} -4\upsilon + 2, & \upsilon \leqq 0, \\ 2, & 0 \leqq \upsilon \leqq 1, \\ 4\upsilon - 2, & \upsilon \geqq 1. \end{cases} \tag{13}$$

When $\upsilon < 0$ or $\upsilon > 1$, equality holds true in (13) *if and only if $p(z)$ is $\frac{1+z}{1-z}$ or one of its rotations. If $0 < \upsilon < 1$, then equality holds true in* (13) *if and only if $p(z)$ is $\frac{1+z^2}{1-z^2}$ or one of its rotations. If $\upsilon = 0$, equality holds true in* (13) *if and only if*

$$p(z) = \left(\frac{1+\rho}{2}\right)\left(\frac{1+z}{1-z}\right) + \left(\frac{1-\rho}{2}\right)\left(\frac{1-z}{1+z}\right), \quad 0 \leqq \rho \leqq 1, \ z \in \mathrm{E},$$

or one of its rotations. For $\upsilon = 1$, equality holds true in (13) *if and only if $p(z)$ is the reciprocal of one of the functions such that the equality holds true in* (13) *in the case when $\upsilon = 0$.*

Remark 2.2. *Although the above upper bound in* (13) *is sharp, it can be improved as follows*:

$$\left|p_2 - \upsilon p_1^2\right| + \upsilon \left|p_1\right|^2 \leqq 2, \qquad 0 < \upsilon \leqq \frac{1}{2}, \tag{14}$$

and

$$\left|p_2 - \upsilon p_1^2\right| + (1-\upsilon) \left|p_1\right|^2 \leqq 2, \qquad \frac{1}{2} \leqq \upsilon < 1. \tag{15}$$

Lemma 2.3. *[12] Let a function p has the form* (6) *and subordinate to a function H of the form*

$$H(z) = 1 + \sum_{n=1}^{\infty} C_n z^n.$$

If H is univalent in E and H (E) *is convex, then*

$$\left|p_n\right| \leqq |C_1|, \quad n \geqq 1.$$

Lemma 2.4. *[2] If a function p of the form* (6) *is in the class $\mathcal{P}$, then*

$$\left|p_n\right| \leqq 2, \qquad n \in \mathbb{N}.$$

This inequality is sharp.

3. Main Results

In this section, we prove our main results.

Theorem 3.1. *If $f \in \mathcal{MS}_q(\beta, \lambda)$, then for any complex number μ*

$$\left|a_1 - \mu a_0^2\right| \leqq \begin{cases} \frac{\mu(\beta-q)\eta^2+(\eta-q)(1-\gamma)\sigma}{(q-\beta)}, & \mu \leqq \frac{(q-1-\eta)\sigma}{(\beta-q)(1+q)\eta}, \\ \frac{\sigma(1-\gamma)}{(\beta-q)}, & \frac{(q-1-\eta)\sigma}{(\beta-q)(1+q)\eta} \leqq \mu \leqq \frac{(1+q-\eta)\sigma}{(\beta-q)(1+q)\eta}, \\ \frac{\mu(\beta-q)\eta^2+(\eta-q)(1-\gamma)\sigma}{(\beta-q)}, & \mu \geqq \frac{(1+q-\eta)\sigma}{(\beta-q)(1+q)\eta}. \end{cases}$$

Furthermore, for $\frac{(q-1-\eta)\sigma}{(\beta-q)(1+q)\eta} < \mu \leqq \frac{(q-\eta)\sigma}{(\beta-q)(1+q)\eta}$, we have

$$|a_1 - \mu a_0^2| + \left(\frac{\mu(\beta-q)\eta^2 + (\eta+1-q)(1-\gamma)\sigma}{(\beta-q)\eta^2}\right)|a_0|^2 \leqq \frac{\sigma(1-\gamma)}{(\beta-q)},$$

and $\frac{(q-\eta)\sigma}{(\beta-q)(1+q)\eta} \leqq \mu < \frac{(1+q-\eta)\sigma}{(\beta-q)(1+q)\eta}$,

$$|a_1 - \mu a_0^2| + \left(\frac{(1+q-\eta)(1-\gamma)\sigma - \mu(\beta-q)\eta^2}{(\beta-q)\eta^2}\right)|a_0|^2 \leqq \frac{\sigma(1-\gamma)}{(\beta-q)},$$

where

$$\sigma = q - \beta(1+q), \tag{16}$$

$$\eta = (1+q)(1-\gamma). \tag{17}$$

These results are sharp.

Proof. If $f \in \mathcal{MS}_q(\beta, \lambda)$, then it follows from (8) that:

$$\frac{-zD_qf(z) - \beta z^2 D_q\left(D_qf(z)\right)}{f(z)\left(\frac{1}{q} - \Upsilon(\beta,q)\right)} \prec \phi(z), \tag{18}$$

where

$$\phi(z) = \frac{1 + (1 - \gamma(1+q))z}{1 - qz}.$$

Define a function

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots,$$

then it is clear that $p \in \mathcal{P}$. This implies that

$$w(z) = \frac{p(z) - 1}{p(z) + 1}.$$

From (18), we have

$$\frac{-zD_qf(z) - \beta z^2 D_q\left(D_qf(z)\right)}{f(z)\left(\frac{1}{q} - \Upsilon(\beta,q)\right)} = \phi(w(z)),$$

with

$$\phi(w(z)) = \frac{1 + p(z) + (1 - \gamma(1+q))(p(z) - 1)}{p(z) + 1 - q(p(z) - 1)}.$$

Now

$$\begin{aligned}\frac{1 + p(z) + (1 - \gamma(1+q))(p(z) - 1)}{p(z) + 1 - q(p(z) - 1)} &= 1 + \left[\frac{1}{2}(1+q)(1-\gamma)p_1\right]z + \left[\frac{1}{2}(q+1)(1-\gamma)p_2\right. \\ &\quad \left. + \frac{1}{4}(q^2-1)(1-\gamma)p_1^2\right]z^2 + \ldots,\end{aligned}$$

and

$$\begin{aligned}\frac{-zD_qf(z) - \beta z^2 D_q\left(D_qf(z)\right)}{f(z)} &= \left(\frac{1}{q} - \Upsilon(\beta,q)\right)\left\{1 + \left[\frac{1}{2}(1+q)(1-\gamma)p_1\right]z\right. \\ &\quad \left.\left[\frac{1}{2}(q+1)(1-\gamma)p_2 + \frac{1}{4}(q^2-1)(1-\gamma)p_1^2\right]z^2 + \ldots\right\}. \end{aligned} \tag{19}$$

From (5) and (19), we have

$$a_0 = -\frac{\eta}{2}p_1 \tag{20}$$

$$a_1 = \frac{\sigma\eta}{2(\beta - q)(1+q)}\left[p_2 - (\eta + 1 - q)\frac{p_1^2}{2}\right]. \tag{21}$$

Thus, clearly we find that:

$$\left|a_1 - \mu a_0^2\right| = \frac{\sigma(1-\gamma)}{2(\beta - q)}\left|p_2 - \nu p_1^2\right|, \tag{22}$$

where

$$\nu = \frac{\mu(\beta - q)(1+q)\eta + (\eta + 1 - q)\sigma}{2\sigma}.$$

By using Lemma 2.1 in (22), we obtain the required result. □

Theorem 3.2. *Let γ be defined by* (11). *If $f \in \mathcal{MS}_q(\beta, \lambda)$ and of the form* (5) *with $0 < \beta < \frac{2}{5}$, then*

$$|a_0| \leqq \frac{\sigma\eta}{Q_q(0,\beta)}$$

and

$$|a_n| \leqq \frac{\sigma\eta}{Q_q(n,\beta)} \prod_{j=0}^{n-1} \left(1 + \frac{\sigma\eta}{Q_q(j,\beta)}\right), \qquad n \in \mathbb{N}, \tag{23}$$

where σ, η are given by (16) *and* (17) *respectively with*

$$Q_q(n,\beta) = [n]_q \left(1 + [n-1]_q \beta\right) q^2 + q - \beta(1+q). \tag{24}$$

Proof. *Since $f \in \mathcal{MS}_q(\beta, \lambda)$, therefore*

$$\frac{-zD_q f(z) - \beta z^2 D_q\left(D_q f(z)\right)}{f(z)\left(\frac{1}{q} - \Upsilon(\beta, q)\right)} = p(z), \tag{25}$$

where

$$p(z) \prec 1 + \left[\frac{1}{2}(1+q)(1-\gamma)p_1\right]z + \left[\frac{1}{2}(q+1)(1-\gamma)p_2 + \frac{1}{4}(q^2-1)(1-\gamma)p_1^2\right]z^2 + \ldots.$$

Also

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

by using Lemma 2.3 and Lemma 2.4, we obtain

$$\left|p_n\right| \leqq \eta, \qquad n \in \mathbb{N}. \tag{26}$$

Now the relation (25) *can be written as:*

$$-zD_q f(z) - \beta z^2 D_q\left(D_q f(z)\right) = \left(\frac{1}{q} - \Upsilon(\beta, q)\right) p(z) f(z).$$

Which implies

$$\left(\frac{1}{q} - \Upsilon(\beta,q)\right)\frac{1}{z} - \sum_{n=0}^{\infty}\left([n]_q + \beta[n]_q[n-1]_q\right)a_n z^n = \left(\frac{1}{q} - \Upsilon(\beta,q)\right)\left(1 + \sum_{n=1}^{\infty} p_n z^n\right)\left(\frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n\right). \tag{27}$$

Equating the coefficients of z and z^{n+1} on both sides of (27) *, we obtain*

$$-a_0 = p_1$$

and

$$-\left(Q_q(n,\beta)\right)a_n = \sigma\left(p_{n+1} + \sum_{j=1}^{n} a_{n-j}p_j\right),$$

or equivalently

$$a_0 = -p_1$$

and

$$a_n = -\left(\frac{\sigma}{Q_q\,(n,\beta)}\right)\left(p_{n+1} + \sum_{j=1}^{n} a_{n-j}p_j\right).$$

Using (26) *, we have*

$$|a_0| \leqq \frac{\sigma\eta}{Q_q\,(0,\beta)} \tag{28}$$

and also

$$|a_n| \leqq \frac{\sigma\eta}{Q_q(n,\beta)}\left(1 + \sum_{j=1}^{n} \left|a_{n-j}\right|\right), \qquad n \in \mathbb{N}. \tag{29}$$

For $n = 1$, the relation (29) *yields*

$$\begin{aligned} |a_1| &\leqq \frac{\sigma\eta}{Q_q\,(1,\beta)}\,(1 + |a_0|) \\ &\leqq \frac{\sigma\eta}{Q_q(1,\beta)}\left(1 + \frac{\sigma\eta}{\left(Q_q(0,\beta)\right)}\right). \end{aligned}$$

To prove (23) *, we apply mathematical induction. For $n = 2$,* (29) *yields*

$$|a_2| \leqq 1 + |a_0| + |a_1|\,.$$

That is

$$\begin{aligned} |a_2| &\leqq \frac{\sigma\eta}{Q_q(2,\beta)}\left\{1 + \frac{\sigma\eta}{\left(Q_q(0,\beta)\right)} + \frac{\sigma\eta}{Q_q(1,\beta)}\left(1 + \frac{\sigma\eta}{\left(Q_q(0,\beta)\right)}\right)\right\} \\ &= \frac{\sigma\eta}{Q_q(2,\beta)}\left(1 + \frac{\sigma\eta}{\left(Q_q(0,\beta)\right)}\right)\left(1 + \frac{\sigma\eta}{Q_q(1,\beta)}\right) \\ &= \frac{\sigma\eta}{Q_q(2,\beta)}\prod_{j=0}^{1}\left(1 + \frac{\sigma\eta}{\left(Q_q(j,\beta)\right)}\right), \end{aligned}$$

which implies that (23) *holds true for $n = 2$. Let us assume that* (23) *is true for $n \leqq k$. That is*

$$|a_k| \leqq \frac{\sigma\eta}{Q_q(k,\beta)}\prod_{j=0}^{k-1}\left(1 + \frac{\sigma\eta}{\left(Q_q(j,\beta)\right)}\right).$$

Consider

$$\begin{aligned} |a_{k+1}| &\leqq \frac{\sigma\eta}{Q_q(k+1,\beta)}\left(1+|a_0|+|a_1|+...+|a_k|\right) \\ &\leqq \frac{\sigma\eta}{Q_q(k+1,\beta)}\left[1+\frac{\sigma\eta}{Q_q(0,\beta)}+\frac{\sigma\eta}{Q_q(1,\beta)}\left(1+\frac{\sigma}{Q_q(0,\beta)}\right)+...+\frac{\sigma\eta}{Q_q(k,\beta)}\prod_{j=0}^{k-1}\left(1+\frac{\sigma\eta}{\left(Q_q(j,\beta)\right)}\right)\right] \\ &= \frac{\sigma\eta}{Q_q(k+1,\beta)}\prod_{j=0}^{k}\left(1+\frac{\sigma\eta}{Q_q(j,\beta)}\right). \end{aligned}$$

Therefore, the result is true for $n = k+1$. Consequently (23) *holds true for all $n \in \mathbb{N}$.* □

The following equivalent form of Definition 1.8 is potentially useful in further investigation of the class $\mathcal{MS}_q(\beta,\lambda)$,

$$f \in \mathcal{MS}_q(\beta,\lambda) \Longleftrightarrow \left|-z\frac{D_qf(z)}{f(z)} - \beta z^2\frac{D_q\left(D_qf(z)\right)}{f(z)} - \frac{1-\gamma q}{1-q}\right| \leqq \frac{1-\gamma}{1-q}. \tag{30}$$

Theorem 3.3. *Let*

$$\frac{1}{q} - \Upsilon(\beta,q) - \gamma > 0. \tag{31}$$

Also suppose that $f \in \mathcal{M}$ and of the form (5). *If*

$$\sum_{n=0}^{\infty}\left(\Lambda_q(n,\beta,\gamma)\right)|a_n| \leqq \frac{1}{q} - \Upsilon(\beta,q) - \gamma, \tag{32}$$

then $f \in \mathcal{MS}_q(\beta,\lambda)$, where $\Upsilon(\beta,q)$ and γ are stated in (12) *and* (11) *respectively.*

Proof. Assuming that (32) holds true, it suffices to show that

$$\left|-z\frac{D_qf(z)}{f(z)} - \beta z^2\frac{D_q\left(D_qf(z)\right)}{f(z)} - \frac{1-\gamma q}{1-q}\right| \leqq \frac{1-\gamma}{1-q}. \tag{33}$$

Let us consider

$$\begin{aligned} &\left|-z\frac{D_qf(z)}{f(z)} - \beta z^2\frac{D_q\left(D_qf(z)\right)}{f(z)} - \frac{1-\gamma q}{1-q}\right| \\ &= \left|\frac{\left(-\frac{1}{q}+\Upsilon(\beta,q)+\frac{1-\gamma q}{1-q}\right)+\sum_{n=0}^{\infty}\left([n]_q+[n]_q[n-1]_q\beta\right)a_nz^{n+1}}{1+\sum_{n=0}^{\infty}a_nz^{n+1}} + \frac{\sum_{n=0}^{\infty}\left(\frac{1-\gamma q}{1-q}\right)a_nz^{n+1}}{1+\sum_{n=0}^{\infty}a_nz^{n+1}}\right|. \end{aligned}$$

Last expression is bounded above by $\frac{1-\gamma}{1-q}$ if

$$\left(-\frac{1}{q}+\Upsilon(\beta,q)+\frac{1-\gamma q}{1-q}\right)+\sum_{n=0}^{\infty}\left([n]_q+[n]_q[n-1]_q\beta+\frac{1-\gamma q}{1-q}\right)|a_n| \leqq \frac{1-\gamma}{1-q}\left(1+\sum_{n=0}^{\infty}|a_n|\right).$$

After some simplee calculations, we have

$$\sum_{n=0}^{\infty}\left(\Lambda_q(n,\beta,\gamma)\right)|a_n| \leqq \left(\frac{1}{q}-\Upsilon(\beta,q)-\gamma\right).$$

This complete the require proof. □

When $q \longrightarrow 1^-$, Theorem 3.3 reduces to the following known result.

Corollary 3.4. *(see [20]) Let*

$$1 + \beta\lambda\left(\lambda + \frac{1}{2}\right) - \lambda - \frac{3}{2}\beta > 0.$$

Also suppose that $f \in \mathcal{M}$ is given by (5)*. If*

$$\sum_{n=0}^{\infty} (n + \beta n (n-1) + \gamma) |a_n| \leqq 1 - \gamma - 2\beta,$$

then $f \in \mathcal{H}(\beta, \lambda)$.

4. Ruscheweyh q-Difference Operator for Meromorphic Functions.

Ruscheweyh derivatives for analytic function was defined by Ruscheweyh [13] and named as m-th order Ruscheweyh derivative by Al-Amiri (see [1]). Ganigi and Uralegaddi introduced the meromorphic analogy of Ruscheweyh derivative in [3]. Recently Kanas et al. (see [7]) introduced the Ruscheweyh derivative operator for analytic functions by using q-differential operator. We here define the meromorphic analogy of Ruscheweyh derivative by using q-differential operator. In this section, we define and study a new class of functions from class $\mathcal{M}$ by using meromorphic analogy of Ruscheweyh q-difference operator. We also investigate the similar kind of results which have been proved in the above section.

Definition 4.1. *Let $f \in \mathcal{M}$. Then the meromorphic analogue of Ruscheweyh q-differential operator is defined as*

$$\mathcal{MR}_q^{\delta} f(z) = f(z) * \phi(q, \delta+1; z) = \frac{1}{z} + \sum_{n=1}^{\infty} \psi_n a_n z^n, \qquad z \in \mathrm{E}^*, \quad \delta > -1, \tag{34}$$

where

$$\phi(q, \delta+1; z) = \frac{1}{z} + \sum_{n=0}^{\infty} \psi_n z^n$$

and

$$\psi_n = \frac{[\delta+n+1]_q!}{[n+1]_q! \, [\delta]_q!}. \tag{35}$$

From (34) , we have

$$\mathcal{MR}_q^0 f(z) = f(z), \quad \mathcal{MR}_q^1 f(z) - [2]_q \, \mathcal{MR}_q^0 f(qz) = zD_q f(z)$$

and

$$\mathcal{MR}_q^m f(z) = \frac{z^{-1} D_q\left(z^{m+1} f(z)\right)}{[m]_q!}, \qquad m \in \mathbb{N}.$$

Note that

$$\lim_{q \to 1^-} \phi(q, \delta+1; z) = \frac{1}{z(1-z)^{\delta+1}}$$

and

$$\lim_{q\to 1^-} \mathcal{MR}_q^{\delta} f(z) = f(z) * \frac{1}{z(1-z)^{\delta+1}},$$

which is the well-known Ruscheweyh differential operator for meromorphic functions introduced and studied by Ganigi and Uralegaddi [3].

Definition 4.2. *Let $f \in \mathcal{M}$. Then $f \in \mathcal{MS}_q^{\delta}(\beta,\lambda)$, if it satisfies the condition*

$$\left| \frac{-z\frac{D_q\left(\mathcal{MR}_q^{\delta} f(z)\right)}{\mathcal{MR}_q^{\delta} f(z)} - \beta z^2 \frac{D_q\left(D_q \mathcal{MR}_q^{\delta} f(z)\right)}{\mathcal{MR}_q^{\delta} f(z)} - \gamma}{1-\gamma} - \frac{1}{1-q} \right| \leqq \frac{1}{1-q}, \tag{36}$$

which by using subordination can be written as

$$\frac{-zD_q\left(\mathcal{MR}_q^{\delta} f(z)\right) - \beta z^2 D_q\left(D_q \mathcal{MR}_q^{\delta} f(z)\right)}{\left(\frac{1}{q} - \Upsilon(\beta,q)\right)\mathcal{MR}_q^{\delta} f(z)} \prec \frac{1+(1-\gamma(1+q))z}{1-qz}. \tag{37}$$

Remark 4.3. *Firstly, it can easily be seen that*

$$\mathcal{MS}_q^{0}(\beta,\lambda) = \mathcal{MS}_q(\beta,\lambda),$$

where $\mathcal{MS}_q(\beta,\lambda)$ is the class of functions defined in Definition 1.8*. Secondly, we have*

$$\lim_{q\longrightarrow 1^-} \mathcal{MS}_q^{0}(\beta,\lambda) = \mathcal{H}(\beta,\lambda),$$

where the class $\mathcal{H}(\beta,\lambda)$ was introduced and studied by Wang et al. For detail see [20, 21].

The following results can be proved by using the similar arguments as in Section 3, so we choose to omit the details of proofs.

Theorem 4.4. *If $f \in \mathcal{MS}_q^{\delta}(\beta,\lambda)$, then for any complex number μ*

$$\left|a_1 - \mu a_0^2\right| \leqq \begin{cases} \frac{\mu(\beta-q)\eta^2\psi_1 + (\eta-q)(1-\gamma)\sigma\psi_0^2}{(q-\beta)\psi_0^2\psi_1}, & \mu \leqq \frac{(q-1-\eta)\sigma\psi_0^2}{(\beta-q)(1+q)\eta\psi_1}, \\ \frac{\sigma(1-\gamma)}{(\beta-q)\psi_1}, & \frac{(q-1-\eta)\sigma\psi_0^2}{(\beta-q)(1+q)\eta\psi_1} \leqq \mu \leqq \frac{(1+q-\eta)\sigma\psi_0^2}{(\beta-q)(1+q)\eta\psi_1}, \\ \frac{\mu(\beta-q)\eta^2\psi_1 + (\eta-q)(1-\gamma)\sigma\psi_0^2}{(\beta-q)\psi_1}, & \mu \geqq \frac{(1+q-\eta)\sigma\psi_0^2}{(\beta-q)(1+q)\eta\psi_1}. \end{cases}$$

Furthermore for $\frac{(q-1-\eta)\sigma\psi_0^2}{(\beta-q)(1+q)\eta\psi_1} < \mu \leqq \frac{(q-\eta)\sigma\psi_0^2}{(\beta-q)(1+q)\eta\psi_1}$,

$$|a_1 - \mu a_0^2| + \left(\frac{\mu(\beta-q)\eta^2\psi_1 + (\eta+1-q)(1-\gamma)\sigma\psi_0^2}{(\beta-q)\eta^2\psi_1}\right)|a_0|^2 \leqq \frac{\sigma(1-\gamma)}{(\beta-q)\psi_1}$$

and $\frac{(q-\eta)\sigma\psi_0^2}{(\beta-q)(1+q)\eta\psi_1} \leqq \mu < \frac{(1+q-\eta)\sigma\psi_0^2}{(\beta-q)(1+q)\eta\psi_1}$,

$$|a_1 - \mu a_0^2| + \left(\frac{(1+q-\eta)(1-\gamma)\sigma\psi_0^2 - \mu(\beta-q)\eta^2\psi_1}{(\beta-q)\eta^2\psi_1}\right)|a_0|^2 \leqq \frac{\sigma(1-\gamma)}{(\beta-q)\psi_1},$$

where σ, η and ψ_n are given by (16), (17) *and* (35) *respectively. These results are sharp.*

By putting $\psi_n = 1$, the above result is proved in Theorem 3.1.

Theorem 4.5. *Let γ be defined by* (11). *If $f \in \mathcal{MS}_q^{\delta}(\beta,\lambda)$ of the* (5) *with $0 < \beta < \frac{2}{5}$, then*

$$|a_0| \leqq \frac{\sigma\eta}{Q_q(0,\beta)\,\psi_0}$$

and

$$|a_n| \leqq \frac{\sigma\eta}{Q_q(n,\beta)\,\psi_n} \prod_{j=0}^{n-1}\left(1+\frac{\sigma\eta}{Q_q(j,\beta)}\right), \qquad n \in \mathbb{N}, \tag{38}$$

where σ, η and $Q_q(n,\beta)$ are given by (16), (17) *and* (24) *respectively.*

By choosing $\psi_n = 1$, the above result is proved in Theorem 3.2.

Theorem 4.6. *Let*

$$\frac{1}{q} - \Upsilon(\beta,q) - \gamma > 0. \tag{39}$$

Also suppose that $f \in \mathcal{M}$ is given by (5). *If*

$$\sum_{n=0}^{\infty} \psi_n\left(\Lambda_q(n,\beta,\gamma)\right)|a_n| \leqq \frac{1}{q} - \Upsilon(\beta,q) - \gamma, \tag{40}$$

then $f \in \mathcal{MS}_q^{\delta}(\beta,\lambda)$ where $\Upsilon(\beta,q)$, ψ_n and γ are given in (12)*,* (35) *and* (11) *respectively.*

When $\delta = 0$ and $q \longrightarrow 1^-$, Theorem 4.6 reduces to the following known result.

Corollary 4.7. *(See [20]) Let*

$$1 + \beta\lambda\left(\lambda + \frac{1}{2}\right) - \lambda - \frac{3}{2}\beta > 0.$$

Also suppose that $f \in \mathcal{M}$ is given by (5). *If*

$$\sum_{n=0}^{\infty}(n + \beta n(n-1) + \gamma)|a_n| \leqq 1 - \gamma - 2\beta,$$

then $f \in \mathcal{H}(\beta,\lambda)$.

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