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Two Weak Solutions for a Singular (p, q)-Laplacian Problem

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Abstract. Here, a singular boundary value problem involving the (p, q)-Laplacian operator in a smooth bounded domain in $\mathbb{R}^N$ is considered. Using the variational method and critical point theory, the existence of two weak solutions is proved.

1. Introduction

The study of partial differential equations started in the 18th century and it's area has been growing steadily in the past centuries. It can be used for modeling a wide range of physical phenomena, encountered in statistical mechanics, mathematical physics, theoretical neuroscience, fluid dynamics and mathematical finance. In fact, the theory of partial differential equations is a powerful theory to study a wide variety of physically significant problems arising in very different areas such as physics, engineering and other applied disciplines (see [1, 7, 14, 15, 22, 29–32, 35–40]).

Partial differential equations involving the p-Laplacian are mathematical models occurring in studies of industrial problems, for instance, problems involving electrorheological fluids, image restorations, generalized reaction-diffusion theory, non-Newtonian fluid theory, non-Newtonian filtration and the turbulent flow of a gas in porous medium.
The study of p-Laplacian comes from at least four decades, whereas a deeper study of problems involving the (p, q)-Laplacian operator has only occurred in the last decade. The (p, q)-Laplacian operator generalizes several types of problems and it is important form two points of views:

(I) Physical motivations.
The quasilinear operator (p, q)-Laplacian has been used to model steady-state solutions of reaction-diffusion problems arising in biophysics, in plasma physics and in the study of chemical reactions. More precisely, the prototype for these models can be written in the form

$$u_t = -div[D(u)\nabla u] + f(x, u),$$

where $D(u) = a_p|\nabla u|^{p-2} + b_q|\nabla u|^{q-2}$ and $a_p, b_q \in R^+$ are positive constants. In this framework, the function u generally stands for a concentration, the term $div[D(u)\nabla u]$ corresponds to the diffusion with coefficient $D(u)$, and $f(x, u)$ is the reaction term related to source and loss processes. Typically, in chemical and biological applications, the reaction term $g(x, u)$ is a polynomial of u with variable

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coefficients. The differential operator $\Delta_p + \Delta_q$ is known as the (p, q)-Laplacian operator, if $p \neq q$. The literature on problems involving the single operator p-Laplacian ($p = q$) is very extensive and have been paid much attention by many mathematicians. In particular, they are interested the existence and the uniqueness of solution of this kind of problem (such as [23–26]).

(II) Mathematical techniques.
There is a broad set of purely mathematical techniques which mainly studying the existence of nonnegative nontrivial solutions as well as multiplicity results. Moreover, since the (p, q)-Laplacian operator is not homogeneous, some technical difficulties arise when applying the usual methods of the theory of elliptic equations. In fact, the study of these problems are often very complicated and require relevant topics of nonlinear functional analysis, especially the theory of variable exponent Lebesgue and Sobolev spaces. Moreover, the study of the weak solution in other spaces such as Orlicz–Morrey space and $\dot{B}^{-1}_{\infty,\infty}$ space is also a research problem (see [8, 10–12])

Because of these two notes (p, q)-Laplacian elliptic problems have been studied by many authors, see [3, 5, 6, 16, 17, 21, 28].
Throughout the paper we consider $\Omega \subset \mathbb{R}^N$ is an open bounded domain containing the origin with smooth boundary $\partial\Omega$, $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ and $\Delta_q u = div(|\nabla u|^{q-2}\nabla u)$ are the p-Laplacian and q-Laplacian operators, respectively.
Li and Zhang [20] studied the existence of multiple solutions for the following nonlinear elliptic problem of (p, q)-Laplacian type involving the critical Sobolev exponent

$$\begin{cases} -\Delta_p u - \Delta_q u = |u|^{p^*-2}u + \mu |u|^{r-2}u & x \in \Omega \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{1}$$

where $p^* = \dfrac{Np}{N-p}$ is the critical Sobolev exponent, $\mu > 0$ and $1 < r < q < p < N$.
Liang and Song [21] using Morse theory, studied the existence of solutions for the following $(2, q)$-Laplacian problem:

$$\begin{cases} -\Delta u - \Delta_q u = f(x, u) & x \in \Omega \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{2}$$

where $p > 2$ and $f \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$ is a suitable function.

Here we are interested to study the singular (p, q)-Laplacian elliptic problem as

$$\begin{cases} -\Delta_p u - \Delta_q u + \dfrac{|u|^{p-2}u}{|x|^p} + \dfrac{|u|^{q-2}u}{|x|^q} = \lambda f(x, u) & x \in \Omega \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{3}$$

where $2 \leq q < p < N$, $\lambda > 0$ is a real parameter and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carath*é*odory function such that

(f_1) $|f(x, t)| \leq a_1 t + a_2 |t|^{r-1}$, for all $(x, t) \in \Omega \times \mathbb{R}$,

where a_1 and a_2 are positive constants, $r \in]p, p^*[$.
In order to prove the existence of at least two weak solutions for the problem (3), we recall some definitions and theorems as follow. Let $W_0^{1,p}(\Omega)$ endowed with the norm

$$||u|| := ||u||_p = (\int_\Omega |\nabla u(x)|^p dx)^{\frac{1}{p}}, \tag{4}$$

and the norm in $L^p(\Omega)$ is

$$|u|_p = (\int_\Omega |u(x)|^p dx)^{\frac{1}{p}}. \tag{5}$$

Assume $r \in [1, p^*[$, the compact embedding $W_0^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$ shows that there exists a $c_r > 0$ such that

$$\|u\|_{L^r(\Omega)} \leq c_r \|u\|_r, \qquad \text{for all } u \in W_0^{1,p}(\Omega), \tag{6}$$

where c_r is the best constant of the embedding and it can be estimated by Talenti inequality [42]. Set

$$c := \frac{1}{N\sqrt{\pi}}\Big(\frac{N(p-1)}{N-p}\Big)^{\frac{p-1}{p}}\Big(\frac{N!\Gamma(\frac{N}{2})}{2\Gamma(\frac{N}{p})\Gamma(N+1-\frac{N}{p})}\Big)^{\frac{1}{N}}, \tag{7}$$

where Γ is the Gamma function defined by

$$\Gamma(t) := \int_0^{+\infty} z^{t-1}e^{-z}dz, \qquad \text{for all } t > 0.$$

This is the best constant (see [2]) of the Sobolev embedding theorem (see [33, Proposition B.7]). By Holder's inequality

$$c_r \leq meas(\Omega)^{\frac{p^*-r}{p^*r}} c$$

where $meas(\Omega)$ is the Lebesgue measure of the set Ω.

We recall the classical Hardy's inequality:

$$\int_\Omega \frac{|u(x)|^s}{|x|^s}dx \leq \frac{1}{H}\int_\Omega |\nabla u(x)|^s dx, \qquad \text{for all } u \in W_0^{1,s}(\Omega), \tag{8}$$

where $1 < s < N$ and $H := (\frac{N-s}{s})^s$, see [13].

Definition 1.1. *[41] Let X be a reflexive real Banach space. The operator $T : X \to X^*$ is said to satisfy the $(S+)$ condition if the assumptions* $\limsup_{n\to\infty} < T(u_n) - T(u_0), u_n - u_0 > \leq 0$ *and $u_n \rightharpoonup u_0$ in X imply $u_n \to u_0$ in X(notice that $< .,. >$ denotes the usual inner product in $\mathbb{R}^N$).*

If we set $F(x,\xi) := \int_0^\xi f(x,t)dt$, for every $(x,\xi) \in \Omega \times \mathbb{R}$, then the energy functional $I_\lambda : X \to \mathbb{R}$ associated with (3) can be written

$$I_\lambda := \Phi(u) - \lambda\Psi(u), \qquad \text{for all } u \in X,$$

where

$$\Phi(u) := \Phi_p(u) + \Phi_q(u),$$

such that

$$\begin{aligned}
\Phi_p(u) &:= \tfrac{1}{p}\Big(\int_\Omega |\nabla u|^p dx + \int_\Omega \frac{|u|^p}{|x|^p}dx\Big)\\
\Phi_q(u) &:= \tfrac{1}{q}\Big(\int_\Omega |\nabla u|^q dx + \int_\Omega \frac{|u|^q}{|x|^q}dx\Big)\\
\Psi(u) &:= \int_\Omega F(x,u(x))dx.
\end{aligned}$$

By (8),

$$\frac{\|u\|^p}{p} \leq \Phi_p(u) \leq \Big(\frac{H+1}{pH}\Big)\|u\|^p, \quad \frac{\|u\|^q}{q} \leq \Phi_q(u) \leq \Big(\frac{H+1}{qH}\Big)\|u\|^q, \tag{9}$$

for every $u \in X$.

Definition 1.2. *The function $u : \Omega \to \mathbb{R}$ is a weak solution of (3), if $u \in X$ and*

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx + \int_\Omega \frac{|u|^{p-2}}{|x|^p} uv dx + \int_\Omega |\nabla u|^{q-2}\nabla u \nabla v dx + \int_\Omega \frac{|u|^{q-2}}{|x|^q} uv dx - \lambda \int_\Omega f(x,u) v dx = 0,$$

for every $v \in X$.

Since Ω is bounded and $q < p$, we have $W_0^{1,p}(\Omega) \subset W_0^{1,q}(\Omega)$ and the continuous embedding $W_0^{1,p}(\Omega) \hookrightarrow W_0^{1,q}(\Omega)$. Then for all $u \in W_0^{1,p}(\Omega)$, we have $\frac{1}{p}\int_\Omega |\nabla u|^p dx < \infty$ and $\frac{1}{q}\int_\Omega |\nabla u|^q dx < \infty$.

Definition 1.3. *A Gâteaux differentiable function I satisfies the Palais-Smale condition (in short (PS)-condition) if any sequence $\{u_n\}$ such that*

(I) $\{I(u_n)\}$ *is bounded,*

(II) $\limsup_{n\to\infty} \|I'(u_n)\|_{X^*} = 0$,

has a convergent subsequence.

Here, we need the following proposition and theorem to prove the main result.

Proposition 1.4. *[19] The operator $T : X \to X^*$ defined by*

$$T(u)(v) := \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx + \int_\Omega \frac{|u|^{p-2}}{|x|^p} uv dx + \int_\Omega |\nabla u|^{q-2}\nabla u \nabla v dx + \int_\Omega \frac{|u|^{q-2}}{|x|^q} uv dx,$$

for every $u, v \in X$, is strictly monotone.

Theorem 1.5. *[4, Theorem 3.2] Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$. Fix $\delta > 0$ such that $sup_{\{\Phi(u)<\delta\}}\Psi(u) < +\infty$ and assume that, for each $\lambda \in \Lambda := \left]0, \frac{\delta}{sup_{\{\Phi(u)<\delta\}}\Psi(u)}\right[$, the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies (PS)-condition and it is unbounded from below. Then, for each $\lambda \in \Lambda$ the functional I_λ admits two distinct critical points.*

2. Two weak solutions

In this section the existence of two weak solutions for the problem (3) is studied. The statement of main result is as follows:

Theorem 2.1. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that condition (f_1) holds. Moreover, assume that*

(f_2) *There exist $\theta > p$ and $K > 0$ such that*

$$0 < \theta F(x,t) \leq t f(x,t),$$

for each $x \in \Omega$ and $|t| > K$.

Then for each $\lambda \in]0, \lambda^*[$, problem (3) admits at least two distinct weak solutions, where

$$\lambda^* := \frac{r}{r a_1 c_1 p^{\frac{1}{p}} + a_2 c_r^r p^{\frac{r}{p}}}$$

and c_r is the constant of the embedding $X \hookrightarrow L^r(\Omega)$ for each $r \in [1, p^*[$ in (6).

Proof. Let $X = W_0^{1,p}(\Omega)$, Φ and Ψ be defined as before. Condition (f_1) and the compact embedding $X \hookrightarrow L^r(\Omega)$ implies $\Psi \in C^1(X, \mathbb{R})$ with the following compact derivative

$$\Psi'(u)(v) := \int_\Omega f(x, u(x))v(x)dx,$$

for every $v \in X$. Also $\Phi \in C^1(X, \mathbb{R})$ and $\Phi' : X \to \mathbb{R}$ is strictly monotone (see Proposition 1.4). Notice that the critical points of I_λ are exactly the weak solutions of problem (3).
Now we show that Φ' is a mapping of $(S+)$-type. Let $u_n \rightharpoonup u$ in X and

$$\limsup_{n\to+\infty} < \Phi'(u_n) - \Phi'(u), u_n - u >\leq 0.$$

We denote $L : X \to \mathbb{R}$ by

$$L(u) := \frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{1}{q}\int_\Omega |\nabla u|^q dx,$$

for every $u \in X$. Hence $L' : X \to X^*$ and

$$L'(u)(v) := \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx + \int_\Omega |\nabla u|^{q-2}\nabla u \nabla v dx,$$

for every $v \in X$. Since Φ' is strictly monotone, then

$$\limsup_{n\to+\infty} < L'(u_n) - L'(u), u_n - u >\leq 0.$$

There exist $C_p > 0$ and $C_q > 0$ such that (see [41])

$$\begin{aligned}
0 &\geq \limsup_{n\to+\infty} < L'(u_n) - L'(u), u_n - u > \\
&= \int_\Omega (|\nabla u_n|^{p-2}\nabla u - |\nabla u|^{p-2}\nabla u)\nabla(u_n - u)dx \\
&+ \int_\Omega (|\nabla u_n|^{q-2}\nabla u - |\nabla u|^{q-2}\nabla u)\nabla(u_n - u)dx \\
&\geq C_p \int_\Omega |\nabla u_n - \nabla u|^p dx + C_q \int_\Omega |\nabla u_n - \nabla u|^q dx \\
&\geq C_p \|u_n - u\|^p.
\end{aligned}$$

Then $u_n \to u$ in X, (see [9, Theorem 3.1]). Hence, Φ' is a mapping of $(S+)$-type. Moreover, [9, Theorem 3.1] shows Φ' is a homeomorphism. Now we prove that $I_\lambda = \Phi - \lambda\Psi$ satisfies (PS)-condition for every $\lambda > 0$. we show that any sequence $\{u_n\} \in X$ satisfying

$$m := \sup_n I_\lambda(u_n) < +\infty, \qquad \|I'_\lambda(u_n)\|_{X^*} \to 0, \tag{10}$$

contains a convergent subsequence. For n large enough, from (10) we have

$$m \geq I_\lambda(u_n) = \Phi_p(u_n) + \Phi_q(u_n) - \lambda \int_\Omega F(x, u_n)dx,$$

then since $\theta > p > q$, we have

$$\begin{aligned}
I_\lambda(u_n) \geq & \frac{1}{p}(\int_\Omega |\nabla u_n|^p dx + \int_\Omega \frac{|u_n|^p}{|x|^p} dx) + \frac{1}{q}(\int_\Omega |\nabla u_n|^q dx + \int_\Omega \frac{|u_n|^q}{|x|^q} dx) \\
& - \frac{\lambda}{\theta} \int_\Omega f(x, u_n) u_n dx \\
> & (\frac{1}{p} - \frac{1}{\theta}) \int_\Omega |\nabla u_n|^p dx + (\frac{1}{q} - \frac{1}{\theta}) \int_\Omega |\nabla u_n|^q dx \\
& + \frac{1}{\theta}(\int_\Omega |\nabla u_n|^p dx + \int_\Omega \frac{|u_n|^p}{|x|^p} dx + \int_\Omega |\nabla u_n|^q dx + \int_\Omega \frac{|u_n|^q}{|x|^q} dx \\
& - \lambda \int_\Omega f(x, u_n) u_n dx) \\
\geq & (\frac{1}{p} - \frac{1}{\theta}) \|u_n\|^p + (\frac{1}{q} - \frac{1}{\theta}) \|u_n\|_q^q + \frac{1}{\theta} < I'_\lambda(u_n), u_n > \\
\geq & (\frac{1}{p} - \frac{1}{\theta}) \|u_n\|^p + \frac{1}{\theta} < I'_\lambda(u_n), u_n > .
\end{aligned}$$

By (10), we can assume that $|\frac{1}{\theta} < I'_\lambda(u_n), u_n > | \leq \|u_n\|$. Hence

$$m + \|u_n\| \geq I_\lambda(u_n) - \frac{1}{\theta} < I'_\lambda(u_n), u_n > \geq (\frac{1}{p} - \frac{1}{\theta}) \|u_n\|^p.$$

From this inequality it is easy to see that $\{\|u_n\|\}$ is bounded. Using the Eberlein-Smulyan Theorem, passing to a subsequence if necessary, we can suppose that $u_n \rightharpoonup u$. Compactness of Ψ' implies that $\Psi'(u_n) \to \Psi'(u)$ and since $I'_\lambda(u_n) = \Phi'(u_n) - \lambda \Psi'(u_n) \to 0$, then $\Phi'(u_n) \to \lambda \Psi'(u)$. Since Φ' is a homeomorphism, then $u_n \to u$. This ensures that the functional I_λ verifies (PS)-condition. From (f_2), there exists a positive constant C such that

$$F(x, t) \geq C|t|^\theta, \tag{11}$$

for all $x \in \Omega$ and $|t| > K$. Indeed, setting $h(x) := \min_{\{|\xi| = K\}} F(x, \xi)$ and

$$\varphi_t(s) := F(x, st), \qquad \text{for all } s > 0. \tag{12}$$

By (f_2), for every $x \in \Omega$ and $|t| > K$, we have

$$0 < \theta \varphi_t(s) = \theta F(x, st) \leq st f(x, st) = s\varphi'_t(s), \qquad \text{for all } s > \frac{K}{|t|}.$$

Thus

$$\int_{\frac{K}{|t|}}^1 \frac{\varphi'_t(s)}{\varphi_t(s)} ds \geq \int_{\frac{K}{|t|}}^1 \frac{\theta}{s} ds.$$

And

$$\varphi_t(1) \geq \varphi_t\Big(\frac{K}{|t|}\Big) \frac{|t|^\theta}{K^\theta}.$$

By (12), we get

$$F(x, t) \geq F\Big(x, \frac{K}{|t|} t\Big) \frac{|t|^\theta}{K^\theta} \geq h(x) \frac{|t|^\theta}{K^\theta} \geq C|t|^\theta,$$

where $C > 0$ is a constant. Hence, (11) is proved. Now, for fix $u_0 \in X \setminus \{0\}$ and every $t > 1$ we have

$$I_\lambda(tu_0) \leq \frac{1}{p}t^p\|u_0\|^p + \frac{1}{q}t^q\|u_0\|_q^q - \lambda C t^\theta \int_\Omega |u_0|^\theta dx.$$

Condition $\theta > p$ shows that I_λ is unbounded from below. Moreover, fixed $\lambda \in]0, \lambda^*[$ from (9) we observe that

$$\|u\| < p^{\frac{1}{p}}, \tag{13}$$

for each $u \in X$ such that $u \in \Phi^{-1}(] - \infty, 1[)$. Combining the compact embedding $X \hookrightarrow L^1(\Omega)$, (f_1), (13) and the compact embedding $X \hookrightarrow L^r(\Omega)$, for each $u \in \Phi^{-1}(] - \infty, 1[)$, we obtain

$$\begin{aligned}\Psi(u) \leq & a_1\|u\|_{L^1(\Omega)} + \frac{a_2}{r}\|u\|^r_{L^r(\Omega)} \leq a_1c_1\|u\| + \frac{a_2}{r}(c_r\|u\|)^r \\ \leq & a_1c_1p^{\frac{1}{p}} + \frac{a_2}{r}c_r^r p^{\frac{r}{p}},\end{aligned}$$

Thus,

$$\sup_{\{\Phi(u)<1\}} \Psi(u) \leq a_1c_1p^{\frac{1}{p}} + \frac{a_2}{r}c_r^r p^{\frac{r}{p}} = \frac{1}{\lambda^*} < \frac{1}{\lambda}. \tag{14}$$

From (14), we have

$$\lambda \in]0, \lambda^*[\subseteq \left]0, \frac{1}{\sup_{\{\Phi(u)<1\}} \Psi(u)}\right[.$$

Therefore all hypotheses of Theorem 1.5 (in the case $\delta = 1$) are verified. Hence, for each $\lambda \in]0, \lambda^*[$, the functional I_λ admits two distinct critical points that are weak solutions of problem (3). □

Remark 2.2. *If f is as [4, Example 5.1] and [18, Example 3.4], then all the assumptions in Theorem (2.1) are fulfilled.*

Corollary 2.3. *Assume all hypothesis of Theorem 1.5 hold. Then the problem*

$$\begin{cases} -\Delta_p u - \mu\Delta_q u + \dfrac{|u|^{p-2}u}{|x|^p} + \mu\dfrac{|u|^{q-2}u}{|x|^q} = \lambda f(x,u) & x \in \Omega \\ u = 0 & x \in \partial\Omega. \end{cases} \tag{15}$$

where $\mu > 0$ is the perturbation parameter, has at least two distinct weak solutions for special range of λ.

Proof. Multiply the problem (3) by $\mu^{-\frac{p-1}{p-q}}$, and let $v = \mu^{\frac{-1}{p-q}}u$. Then the problem (3) reduces to

$$\begin{cases} -\Delta_p v - \Delta_q v + \dfrac{|v|^{p-2}v}{|x|^p} + \dfrac{|v|^{q-2}v}{|x|^q} = \lambda_1 g(x,v) & x \in \Omega \\ u = 0 & x \in \partial\Omega. \end{cases} \tag{16}$$

where $\lambda_1 := \lambda\mu^{-\frac{p-1}{p-q}}$ and $g(x,v) := f(x, \mu^{\frac{1}{p-q}}v)$. Thus by Theorem 1.5, for a spacial range of λ_1, the problem (16) has two solutions. Thus the problem (15) has at least two weak solutions for a spacial rage of λ. □

Corollary 2.4. *Assume all hypothesis of Theorem 1.5 hold. Then the problem*

$$\begin{cases} -\Delta_p u - \mu\Delta_q u + \mu^{\frac{1}{q-p}}\dfrac{|u|^{p-2}u}{|x|^p} + \mu^{\frac{q-p+1}{q-p}}\dfrac{|u|^{q-2}u}{|x|^q} = \lambda f(x,u) & x \in \Omega \\ u = 0 & x \in \partial\Omega. \end{cases} \tag{17}$$

where $\mu > 0$ is the perturbation parameter, has at least two distinct weak solutions for a special range of λ.

Proof. Using scaling argument (see [27]), and define $u_s(x) := u(sx)$ where $s := \mu^{\frac{1}{p-q}}$, the problem (17) reduces to

$$\begin{cases} -\Delta_p v - \Delta_q v + \dfrac{|v|^{p-2}v}{|x|^p} + \dfrac{|v|^{q-2}v}{|x|^q} = \lambda \mu^{-\frac{p-1}{p-q}} f(\mu^{\frac{1}{p-q}} x, v) & x \in (\mu^{\frac{1}{p-q}})^{-1}\Omega \\ u = 0 & x \in \partial(\mu^{\frac{1}{p-q}})^{-1}\Omega. \end{cases} \tag{18}$$

where $(\mu^{\frac{1}{p-q}})^{-1}\Omega = \{(\mu^{\frac{1}{p-q}})^{-1}x : x \in \Omega\}$ and $v(x) := u(sx)$. By the same argument in [27, pages 3,4], a solution of the problem (18) actually solves an equation of the type (17). Theorem 2.1 implies the problem (18) admits two weak solutions for a special range of λ. □

Remark 2.5. *The problem (3) can be generalized in the following two formats:*

$$\begin{cases} -a(x)\Delta_p u - b(x)\Delta_q u + \mu_1 \dfrac{|u|^{p-2}u}{|x|^p} + \mu_2 \dfrac{|u|^{q-2}u}{|x|^q} = \lambda f(x, u) & x \in \Omega \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{19}$$

where $a(x)$ and $b(x)$ are (I) continuous and (II) noncontinuous functions. And also

$$\begin{cases} a(x)(-\Delta_p)^s u + b(x)(-\Delta_q)^r u + \mu_1 \dfrac{|u|^{p-2}u}{|x|^p} + \mu_2 \dfrac{|u|^{q-2}u}{|x|^q} = \lambda f(x, u) & x \in \Omega \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{20}$$

where $(-\Delta_p)^s$ is the fractional p-Laplacian.
One may study the existence of multiple solutions of the problems (19) *and* (20).

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