Filomat 33:11 (2019), 3487–3496
https://doi.org/10.2298/FIL1911487C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Reverse Order Laws for Hirano Inverses in Rings

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Abstract. In this paper, various kinds of reverse order laws for the Hirano inverse are characterized in a ring with the unit 1. Under some of conditions $aba = a^2b, ab^2 = bab, a^hab^2 = ba^hab, a^2bb^h = abb^ha$ instead of the condition $ab = ba$, respectively, we present some equivalent conditions of the reverse order laws for Hirano inverses.

1. Introduction

Let R be an associative ring with the unit 1. The set of all nilpotent elements of R will be denoted by $N(R)$, i.e. $N(R) = \{a \in R \mid \exists k \in \mathbb{Z}^+, a^k = 0\}$. The commutator of $a \in R$ is denoted by $\mathrm{comm}(a) = \{x \in R \mid ax = xa\}$. The double commutator of $a \in R$ will be denoted by $\mathrm{comm}^2(a) = \{x \in R \mid yx = xy, \forall y \in \mathrm{comm}(a)\}$.

It is well known that $(ab)^{-1} = b^{-1}a^{-1}$ for invertible elements $a, b \in R$, we call it reverse order law for the ordinary inverse. In general, the previous equality doesn't hold when the ordinary inverse is replaced by generalized inverse. Since the reverse order law for the generalized inverse is a useful computational tool in applications and it is significant from the theoretical point of view, many papers characterized the reverse order laws, such as [2–8, 10–24].

For the readers' convenience, we first recall the definitions of Hirano inverse in [1] and Drazin inverse in [9], respectively.

Definition 1.1. *Let $a \in R$. If there exists $b \in R$ such that*

$$(2)\ bab = b, \qquad (5)\ ab = ba, \qquad (7)\ a^2 - ab \in N(R),$$

then b is called the Hirano inverse of a, and a is Hirano invertible.

Remark 1. *For $a \in R$, the set of elements satisfying the (i)th equation in Definition 1.1 is denoted by $a\{i\}$, where $i \in \{2, 5, 7\}$. And the set of Hirano invertible elements in R is denoted by R^h.*

Definition 1.2. *Let $a \in R$. If there exists $b \in R$ such that*

$$(2)\ bab = b, \qquad (5)\ ab = ba, \qquad (7')\ a - a^2b \in N(R),$$

then b is called the Drazin inverse of a, and a is Drazin invertible.

2010 *Mathematics Subject Classification*. Primary 15A09; Secondary 16W10
Keywords. Hirano inverse; Drazin inverse; reverse order law
Received: 04 January 2019; Accepted: 19 May 2019
Communicated by Dragana Cvetković-Ilić
Research supported by the Education Department of Hubei province key project (No.D20122202), the Education Department of Hubei Province Youth Project (No.B20122203). The second author is supported by China Postdoctoral Science Foundation (No. 2018M632385) and Research Project of Hubei Provincial Department of Education (No. B2019128).
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If the Drazin inverse of $a \in R$ exists, then it is unique and denoted by a^D. The set of Drazin invertible elements in R is denoted by R^D. In [1], the authors gave the relations of the two types of inverses in a ring R, that is $R^h \subsetneq R^D$.

Lemma 1.1. ([1, Corollary 2.2]) *If $a \in R^h$, then $a \in R^D$ and the Hirano inverse of a is exactly the Drazin inverse of a.*

Remark 2. *If the Hirano inverse of a exists, then it is unique and denoted by a^h.*

By Lemma 1.1 and [9, Theorem 1], we have $a^h \in \text{comm}^2(a)$. Applying Definition 1.1, it is easy to get the following remark, which is a useful result.

Remark 3. *If $a, b \in R^h$ and $b \in comm(a)$, then $b^h \in comm(a)$ and $b, b^h \in comm(a^h)$.*

Lemma 1.2. ([1, Corollary 3.2]) *If $a \in R^h$, then $a^h \in R^h$, and $(a^h)^h = a^2a^h$.*

The following lemma gives an equivalent condition of Hirano invertibility.

Lemma 1.3. ([1, Theorem 3.1]) *If $a \in R$, then $a \in R^h$ if and only if $a - a^3 \in N(R)$.*

Lemma 1.4. ([6, Lemma 1.2]) *If $a, b \in N(R)$ and $ab = ba$, then $a + b \in N(R)$.*

Lemma 1.5. ([1, Theorem 4.4]) *If $a, b \in R^h$ and $ab = ba$, then $ab \in R^h$ and $(ab)^h = b^ha^h$.*

The condition $ab = ba$ in Lemma 1.5 is too strong, can we get the reverse order under some weaker conditions? which is our main purpose of this paper.

If $ab = ba$, then $aba = a^2b$ and $ab^2 = bab$, but in general the converse is not true. For example: let $R = \mathbb{C}^{2\times 2}$, $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ *and* $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in R$, it is not difficult to check that $ABA = A^2B$ and $AB^2 = BAB$, but $AB \neq BA$. In this paper, we discuss some equivalent conditions related to reverse order laws for Hirano inverse under several conditions, such as $ab \in \text{comm}(a)$ and (or) $ab \in \text{comm}(b)$.

2. Reverse order laws for the Hirano Inverse

Firstly, we discuss the characterizations of reverse order laws for Hirano inverse under the condition $a, b, ab \in R^h$.

Theorem 2.1. *If $a, b, ab \in R^h$, then the following statements are equivalent:*
(1) $(ab)^h = b^ha^h$;
(2) $(ab)^ha = b^ha^ha$ and $(ab)^h = (ab)^haa^h$;
(3) $b(ab)^h = bb^ha^h$ and $(ab)^h = b^hb(ab)^h$.

Proof. (1) $\Rightarrow$ (2) : It is obvious.
(2) $\Rightarrow$ (1): Using the assumptions $(ab)^ha = b^ha^ha$ and $(ab)^h = (ab)^haa^h$, we conclude $(ab)^h = ((ab)^ha)a^h = b^ha^haa^h = b^ha^h$.
(1) $\Leftrightarrow$ (3): It is similar to the proof of (1) $\Leftrightarrow$ (2). □

Next, in the case that only $a \in R^h$ and $ab \in \text{comm}(a)$, we characterize the mixed reverse order law $(ab)^h = (a^hab)^ha^h$ as follows.

Theorem 2.2. *If $b \in R, a \in R^h$, $ab \in comm(a)$, then the following statements are equivalent:*
(1) $ab, a^hab \in R^h$ and $(ab)^h = (a^hab)^ha^h$;
(2) $a^hab \in R^h$ and $(a^hab)^ha^h \in (ab)\{7\}$;
(3) $ab, a^hab \in R^h$ and $(a^hab)^h = (ab)^ha$;
(4) $ab \in R^h$ and $(ab)^ha \in (a^hab)\{2\}$;
(5) $ab \in R^h$ and $(ab)^ha(1 - a^ha) = 0$;
(6) $aba \in R^h$ and $(aba)^h(a^h)^h \in (ab)\{7\}$;
(7) $ab, aba \in R^h$ and $(ab)^h = (aba)^h(a^h)^h$.

Proof. (1) ⇒ (3): The assumptions $ab, a^hab \in R^h$ and $(ab)^h = (a^hab)^ha^h$ imply $(ab)^haa^h = (a^hab)^ha^haa^h = (a^hab)^ha^h = (ab)^h$. Then $((ab)^ha)(a^hab)((ab)^ha) = (ab)^hab(ab)^ha = (ab)^ha$, i.e. $(ab)^ha \in (a^hab)\{2\}$. Since $ab \in \mathrm{comm}(a)$, we know that $ab \in \mathrm{comm}(a^h)$ by Remark 3, so $(a^hab)^h \in \mathrm{comm}(a^ha)$. Thus

$$\begin{aligned}(a^hab)^2 - (a^hab)((ab)^ha) &= (a^hab)^2 - a^hab(a^hab)^ha^ha = (a^hab)^2 - a^haba^ha(a^hab)^h \\ &= (a^hab)^2 - a^hab(a^hab)^h \in N(R),\end{aligned}$$

i.e. $(ab)^ha \in (a^hab)\{7\}$. In addition, $(ab)^ha \in (a^hab)\{5\}$ is obvious. Hence, $(a^hab)^h = (ab)^ha$.

(3) ⇒ (2): By the hypothesis $(a^hab)^h = (ab)^ha$, and note that $ab \in \mathrm{comm}(a^h)$, hence $(a^hab)^h \in \mathrm{comm}(ab)$ by Remark 3. Then using Lemma 1.2 and Definition 1.1, we obtain

$$\begin{aligned}(ab)^2 - ab(a^hab)^ha^h &= (ab)^2 - (a^hab)^haba^h = (ab)^2 - (a^hab)^ha^hab \\ &= (ab)^2 - a^ha(a^hab)^hb = (ab)^2 - a^haa^hab((a^hab)^h)^2b \\ &= (ab)^2 - a^hab((a^hab)^h)^2b = (ab)^2 - (a^hab)^hb \\ &= (ab)^2 - (ab)^hab = (ab)^2 - ab(ab)^h \in N(R).\end{aligned}$$

(2) ⇒ (1): We note that $(a^hab)^ha^h(ab) = a^hab(a^hab)^h = (ab)a^h(a^hab)^h = (ab)(a^hab)^ha^h$, that is $(a^hab)^ha^h \in (ab)\{5\}$. Obviously, $(a^hab)^ha^h \in (ab)\{2\}$. And by the hypothesis $(a^hab)^ha^h \in (ab)\{7\}$, we obtain $ab \in R^h$ and $(ab)^h = (a^hab)^ha^h$.

(3) ⇒ (4): It is obvious.

(4) ⇒ (5): By the hypotheses $a, ab \in R^h$ and $(ab)^ha \in (a^hab)\{2\}$, we get $(ab)^ha = (ab)^haa^hab(ab)^ha = (ab)^hab(ab)^haa^ha = (ab)^haa^ha$, then $(ab)^ha(1 - a^ha) = (ab)^ha - (ab^h)aa^ha = 0$.

(5) ⇒ (3): By the hypothesis $(ab)^ha = (ab)^haa^ha$, we get $(ab)^haa^hab(ab)^ha = (ab)^hab(ab)^ha = (ab)^ha$, i.e. $(ab)^ha \in (a^hab)\{2\}$.

Since $ab \in \mathrm{comm}(a)$ and $a, ab \in R^h$, we know $(ab)^ha \in (a^hab)\{5\}$ by Remark 3.

Because $ab \in R^h$, we have $(ab)^2 - ab(ab)^h \in N(R)$. Since $ab \in \mathrm{comm}(a)$, we have $(ab)^2 - (ab)(ab)^h \in \mathrm{comm}(aa^h)$ by Remark 3, thus $aa^h((ab)^2 - (ab)(ab)^h) \in N(R)$.

Recall that $a^h \in R^h$ and $ab \in \mathrm{comm}(a^h)$, we have $((a^h)^2 - a^h(a^h)^h)(ab)^2 \in N(R)$, and $aa^h((ab)^2 - (ab)(ab)^h)$ commutes with $((a^h)^2 - a^h(a^h)^h)(ab)^2$. Using Lemma 1.2 and Lemma 1.4, we obtain that

$$\begin{aligned}(a^hab)^2 - a^hab(ab)^ha &= (a^hab)^2 - aa^h(ab)^2 + aa^h(ab)^2 - a^hab(ab)^ha \\ &= (a^h)^2(ab)^2 - a^ha(ab)^2 + aa^h((ab)^2 - ab(ab)^h) \\ &= ((a^h)^2 - a^haa^ha)(ab)^2 + aa^h((ab)^2 - ab(ab)^h) \\ &= ((a^h)^2 - a^h(a^h)^h)(ab)^2 + aa^h((ab)^2 - ab(ab)^h) \in N(R),\end{aligned}$$

i.e. $(ab)^ha \in (a^hab)\{7\}$. Then $(ab)^ha = (a^hab)^h$, the statement (3) is satisfied.

(6) ⇒ (2): By the assumption $a, aba \in R^h$, together with $ab \in \mathrm{comm}(a)$ and $ab \in \mathrm{comm}(a^h)$, we deduce that $a^ha^haba \in R^h$ and $(a^ha^haba)^h = (aba)^h(a^h)^h(a^h)^h$ by Lemma 1.5. Recall that $a^ha^haba = a^hab$, then we obtain $a^hab \in R^h$. The assumption $(aba)^h(a^h)^h \in (ab)\{7\}$ gives

$$\begin{aligned}(ab)^2 - ab(a^hab)^ha^h &= (ab)^2 - ab(a^ha^haba)^ha^h \\ &= (ab)^2 - ab(aba)^h(a^h)^h(a^h)^ha^h \\ &= (ab)^2 - ab(aba)^h(a^h)^h \in N(R),\end{aligned}$$

that is, the statement (2) is satisfied.

(2) ⇒ (6): From the proof of (2) ⇒ (1), we know that $ab \in R^h$ and $(ab)^h = (a^hab)^ha^h$. Also, observe that $a \in R^h$, we get $aba \in R^h$ and $(aba)^h = a^h(ab)^h = a^h(a^hab)^ha^h$. Now, applying the assumption $(a^hab)^ha^h \in (ab)\{7\}$, we obtain

$$\begin{aligned}(ab)^2 - ab(aba)^h(a^h)^h &= (ab)^2 - aba^h(a^hab)^ha^h(a^h)^h \\ &= (ab)^2 - ab(a^hab)^ha^ha^ha^2a^h \\ &= (ab)^2 - ab(a^hab)^ha^h \in N(R),\end{aligned}$$

that is, the statement (6) holds.

Now, the statements (1)–(6) are equivalent.

(7) ⇒ (6): It is obvious.

(1) ⇒ (7): By the assumption $a, ab \in R^h$ and together with $ab \in \mathrm{comm}(a)$, we deduce that $aba \in R^h$ and $(aba)^h = a^h(ab)^h$ by Lemma 1.5. Recall that $(a^hab)^h = (a^ha^haba)^h = (aba)^h(a^h)^h(a^h)^h$. The assumption $(ab)^h = (a^hab)^ha^h$ gives $(aba)^h(a^h)^h = (aba)^h(a^h)^h(a^h)^ha^h = (a^hab)^ha^h = (ab)^h$. Then statement (7) holds. □

If we replace the conditions $a \in R^h$, $ab \in \mathrm{comm}(a)$ in Theorem 2.2 by $b \in R^h$, $ab \in \mathrm{comm}(b)$, we get the dual statements concerning $(ab)^h = b^h(abb^h)^h$ and $(abb^h)^h = b(ab)^h$.

Corollary 2.1. *If $a \in R, b \in R^h$, $ab \in comm(b)$, then the following statements are equivalent:*

(1) $ab, abb^h \in R^h$ and $(ab)^h = b^h(abb^h)^h$;

(2) $abb^h \in R^h$ and $b^h(abb^h)^h \in (ab)\{7\}$;

(3) $ab, abb^h \in R^h$ and $(abb^h)^h = b(ab)^h$;

(4) $ab \in R^h$ and $b(ab)^h \in (abb^h)\{2\}$;

(5) $ab \in R^h$ and $b(1 - bb^h)(ab)^h = 0$;

(6) $bab \in R^h$ and $(b^h)^h(bab)^h \in (ab)\{7\}$;

(7) $ab, bab \in R^h$ and $(ab)^h = (b^h)^h(bab)^h$.

In the following, we give some equivalent characterizations for $(ab)^h = b^ha^h$ under certain conditions.

Theorem 2.3. *If $a, b, ab \in R^h$, $ab \in comm(a)$, then the following statements are equivalent:*

(1) $(ab)^h = b^ha^h$;

(2) $a^hab \in R^h$ and $(a^hab)^h = (ab)^ha = b^ha^ha$;

(3) $a^hab \in R^h$ with $(a^hab)^h = b^ha^ha$ and $(ab)^h = (a^hab)^ha^h$;

(4) $aba \in R^h$ with $(aba)^h = b^ha^ha^h$ and $(ab)^h = (aba)^h(a^h)^h$.

Proof. (2) ⇒ (3): Using the assumption $(a^hab)^h = (ab)^ha = b^ha^ha$, we obtain that $(ab)b^ha^h = abb^ha^haa^h = ab(ab)^haa^h = (ab)^habaa^h = b^ha^habaa^h = b^ha^haa^hab = b^ha^hab$, i.e. $b^ha^h \in (ab)\{5\}$. Also, we have $b^ha^h(ab)b^ha^h = b^ha^habb^ha^haa^h = (ab)^hab(ab)^haa^h = (ab)^haa^h = b^ha^haa^h = b^ha^h$, i.e. $b^ha^h \in (ab)\{2\}$, and $(ab)^2 - (ab)b^ha^h = (ab)^2 - b^ha^hab = (ab)^2 - (ab)^hab \in N(R)$, i.e. $b^ha^h \in (ab)\{7\}$. Thus $(ab)^h = b^ha^h$. So $(a^hab)^ha^h = b^ha^haa^h = b^ha^h = (ab)^h$.

(3) ⇒ (1): It is obvious.

(1) ⇒ (2): By $(ab)^h = b^ha^h$, we obtain $(b^ha^ha)(a^hab)(b^ha^ha) = b^ha^habb^ha^ha = (ab)^hab(ab)^ha = (ab)^ha = b^ha^ha$, i.e. $b^ha^ha \in (a^hab)\{2\}$. Also,

$$\begin{aligned}(b^ha^ha)(a^hab) &= b^ha^hab = (ab)^hab = ab(ab)^h = abb^ha^h = abb^ha^haa^h = abb^ha^ha^ha \\ &= ab(ab)^ha^ha = a^hab(ab)^ha = (a^hab)(b^ha^ha),\end{aligned}$$

i.e. $b^ha^ha \in (a^hab)\{5\}$.

Since $ab \in \mathrm{comm}(a)$ and $ab \in \mathrm{comm}(a^h)$, then $(a^h)^2((ab)^2 - ab(ab)^h)$ commutes with $((a^h)^2 - a^h(a^h)^h)ab(ab)^h$. Note that

$$\begin{aligned}(a^hab)^2 - (a^hab)(b^ha^ha) &= (a^h)^2(ab)^2 - a^hab(ab)^ha \\ &= (a^h)^2(ab)^2 - (a^h)^2ab(ab)^h + (a^h)^2ab(ab)^h - a^haab(ab)^h \\ &= (a^h)^2((ab)^2 - ab(ab)^h) + ((a^h)^2 - a^ha)ab(ab)^h \\ &= (a^h)^2((ab)^2 - ab(ab)^h) + ((a^h)^2 - a^haa^ha)ab(ab)^h \\ &= (a^h)^2((ab)^2 - ab(ab)^h) + ((a^h)^2 - a^ha^2a^h)ab(ab)^h \\ &= (a^h)^2((ab)^2 - ab(ab)^h) + ((a^h)^2 - a^h(a^h)^h)ab(ab)^h.\end{aligned}$$

Applying Lemma 1.4, we have $(a^hab)^2 - (a^hab)(b^ha^ha) \in N(R)$, i.e. $b^ha^ha \in (a^hab)\{7\}$.

Thus $(a^hab)^h = b^ha^ha = (ab)^ha$.

(2) ⇒ (4): Obviously, $ab, a^h ab \in R^h$ and $(a^h ab)^h = (ab)^h a$ yield $aba \in R^h$ and $(ab)^h = (aba)^h (a^h)^h$ by Theorem 2.2. Also, using the assumptions $a, ab \in R^h$, $ab \in \mathrm{comm}(a)$ and $(ab)^h a = b^h a^h a$, we get that $(aba)^h = (ab)^h a^h = (ab)^h a (a^h)^2 = b^h a^h a (a^h)^2 = b^h a^h a^h$.

(4) ⇒ (2): The assumptions $ab, aba \in R^h$ and $(ab)^h = (aba)^h (a^h)^h$ imply $a^h ab \in R^h$ and $(a^h ab)^h = (ab)^h a$ by Theorem 2.2. Also, from the assumption $(aba)^h = b^h a^h a^h$, we get that $(ab)^h a = (aba)^h (a^h)^h a = b^h a^h a^h (a^h)^h a = b^h a^h a^h a^2 a^h a = b^h a^h a$ by Lemma 1.2, that is, $(a^h ab)^h = (ab)^h a = b^h a^h a$. □

If we replace the condition $ab \in \mathrm{comm}(a)$ in Theorem 2.3 by $ab \in \mathrm{comm}(b)$, we get the following dual statements.

Corollary 2.2. *If $a, b, ab \in R^h$, $ab \in comm(b)$, then the following statements are equivalent:*
(1) $(ab)^h = b^h a^h$;
(2) $abb^h \in R^h$ and $(abb^h)^h = b(ab)^h = bb^h a^h$;
(3) $abb^h \in R^h$ with $(abb^h)^h = bb^h a^h$ and $(ab)^h = b^h (abb^h)^h$;
(4) $bab \in R^h$ with $(bab)^h = b^h b^h a^h$ and $(ab)^h = (b^h)^h (bab)^h$.

Now, we discuss the equivalent characterizations for mixed reverse order law $(a^h abb^h)^h = b(a^h ab)^h$ under conditions $a, b, a^h ab \in R^h$, and $a^h ab \in \mathrm{comm}(b)$.

Theorem 2.4. *If $a, b, a^h ab \in R^h$, $a^h ab \in comm(b)$, then the following statements are equivalent:*
(1) $a^h abb^h \in R^h$ and $(a^h abb^h)^h = b(a^h ab)^h$;
(2) $a^h abb^h \in R^h$ and $b^h (a^h abb^h)^h \in (a^h ab)\{7\}$;
(3) $a^h abb^h \in R^h$ and $(a^h ab)^h = b^h (a^h abb^h)^h$;
(4) $b(a^h ab)^h \in (a^h abb^h)\{2\}$;
(5) $b(1 - b^h b)(a^h ab)^h = 0$.

Proof. (1) ⇒ (4): It is obvious by Definition 1.1.

(4) ⇒ (1): Because $a^h ab \in \mathrm{comm}(b)$, we know $(a^h ab)^h \in \mathrm{comm}(b) \cap \mathrm{comm}(b^h)$. Therefore, $a^h abb^h$ commutes with $b(a^h ab)^h$, i.e. $b(a^h ab)^h \in (a^h abb^h)\{5\}$. By Definition 1.1 and Lemma 1.2, we can see that both $(b^h)^2 - b^h (b^h)^h \in N(R)$ and $(a^h ab)^2 - a^h ab(a^h ab)^h \in N(R)$, and $(b^h)^h = b^2 b^h$. Using Lemma 1.4, we have

$$\begin{aligned} (a^h abb^h)^2 - (a^h abb^h) b (a^h ab)^h &= (a^h ab)^2 (b^h)^2 - (a^h ab)^2 b^h (b^h)^h + (a^h ab)^2 b^h (b^h)^h - (a^h abb^h) b (a^h ab)^h \\ &= (a^h ab)^2 ((b^h)^2 - b^h (b^h)^h) + ((a^h ab)^2 - a^h ab (a^h ab)^h) b^h b \in N(R), \end{aligned}$$

i.e. $b(a^h ab)^h \in (a^h abb^h)\{7\}$. And from the the assumption $b(a^h ab)^h \in (a^h abb^h)\{2\}$, we obtain that $a^h abb^h \in R^h$ and $(a^h abb^h)^h = b(a^h ab)^h$.

(4) ⇔ (5): We have

$$\begin{aligned} b(a^h ab)^h \in (a^h abb^h)\{2\} &\Leftrightarrow (b(a^h ab)^h)(a^h abb^h)(b(a^h ab)^h) = b(a^h ab)^h \\ &\Leftrightarrow bb^h b (a^h ab)^h a^h ab (a^h ab)^h = b(a^h ab)^h \\ &\Leftrightarrow bb^h b (a^h ab)^h = b(a^h ab)^h \\ &\Leftrightarrow b(b^h b - 1)(a^h ab)^h = 0. \end{aligned}$$

(2) ⇒ (3): Using the assumption $a^h ab \in \mathrm{comm}(b)$, we have $(a^h ab)(b^h (a^h abb^h)^h) = b^h a^h ab (a^h abb^h)^h = b^h (a^h abb^h)^h (a^h ab)$, i.e. $b^h (a^h abb^h)^h \in (a^h ab)\{5\}$. In addition, it is clear that $(b^h (a^h abb^h)^h)(a^h ab)(b^h (a^h abb^h)^h) = b^h (a^h abb^h)^h$, so $b^h (a^h abb^h)^h \in (a^h ab)\{2\}$. Note that the assumption $b^h (a^h abb^h)^h \in (a^h ab)\{7\}$. Therefore, $a^h ab \in R^h$ and $(a^h ab)^h = b^h (a^h abb^h)^h$.

(3) ⇒ (2): It is obvious by Definition 1.1.

(3) ⇒ (5): Note that $(a^h ab)^h = b^h (a^h abb^h)^h$, we conclude $bb^h b (a^h ab)^h = bb^h bb^h (a^h abb^h)^h = bb^h (a^h abb^h)^h = b(a^h ab)^h$, i.e. $b(1 - b^h b)(a^h ab)^h = 0$.

(1) ⇒ (3): Combining the proofs of (1) ⇔ (5) with Definition 1.1 and the assumption $(a^h abb^h)^h = b(a^h ab)^h$, we obtain

$$\begin{aligned}(a^h ab)^2 - (a^h ab)b^h(a^h abb^h)^h &= (a^h ab)^2 - a^h abb^h b(a^h ab)^h \\ &= (a^h ab)^2 - a^h ab(a^h ab)^h + a^h ab(a^h ab)^h - a^h abb^h b(a^h ab)^h \\ &= (a^h ab)^2 - a^h ab(a^h ab)^h + a^h ab(1 - b^h b)(a^h ab)^h \\ &= (a^h ab)^2 - a^h ab(a^h ab)^h \in N(R).\end{aligned}$$

In addition, it is clear that $b^h(a^h abb^h)^h \in (a^h ab)\{2, 5\}$ by $a^h ab \in \mathrm{comm}(b)$. Then we claim that the statement (3) holds.

□

If we change the condition $a^h ab \in \mathrm{comm}(b)$ of Theorem 2.4 into $abb^h \in \mathrm{comm}(a)$, we get the following dual statements.

Corollary 2.3. *If $a, b, abb^h \in R^h$, $abb^h \in comm(a)$, then the following statements are equivalent:*
(1) $a^h abb^h \in R^h$ and $(a^h abb^h)^h = (abb^h)^h a$;
(2) $a^h abb^h \in R^h$ and $(a^h abb^h)^h a^h \in (abb^h)\{7\}$;
(3) $a^h abb^h \in R^h$ and $(abb^h)^h = (a^h abb^h)^h a^h$;
(4) $(abb^h)^h a \in (a^h abb^h)\{2\}$;
(5) $(abb^h)^h(aa^h - 1)a = 0$.

In [16], D. Mosić and N.Č. Dinčić discussed the mixed reverse order law $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ in rings with involution. In the following result, we discuss the equivalent characterizations for the mixed reverse order law $(ab)^h = b^h(a^h abb^h)^h a^h$ under conditions $a, b \in R^h$, $ab \in \mathrm{comm}(a) \cap \mathrm{comm}(b)$.

Theorem 2.5. *If $a, b \in R^h$, $ab \in comm(a) \cap comm(b)$, then the following statements are equivalent:*
(1) $ab, a^h abb^h \in R^h$ and $(ab)^h = b^h(a^h abb^h)^h a^h$;
(2) $ab \in R^h$ and $b^h b(ab)^h = (ab)^h aa^h = (ab)^h$;
(3) $ab, a^h abb^h \in R^h$, $(a^h abb^h)^h = b(ab)^h a$ and $b^h b(ab)^h aa^h = (ab)^h$;
(4) $a^h abb^h \in R^h$ and $b^h(a^h abb^h)^h a^h \in (ab)\{7\}$;
(5) $ab, a^h ab, abb^h \in R^h$ and $(ab)^h = (a^h ab)^h a^h = b^h(abb^h)^h$;
(6) $ab, a^h ab, abb^h \in R^h$, $(a^h ab)^h = (ab)^h a$ and $(abb^h)^h = b(ab)^h$;
(7) $ab, aba, bab \in R^h$ and $(ab)^h = (aba)^h(a^h)^h = (b^h)^h(bab)^h$;
(8) $aba, bab \in R^h$ and $(aba)^h(a^h)^h, (b^h)^h(bab)^h \in (ab)\{7\}$.
In addition, if any of the previous statements is valid, then $a^h abb^h \in R^h$ and $(a^h abb^h)^h = b(a^h ab)^h = (abb^h)^h a$.

Proof. We know (7) ⇔ (8) by Theorem 2.2 and Corollary 2.1, so we only need to prove the equivalence of (1)–(7).

(1) ⇒ (2): The assumption $(ab)^h = b^h(a^h abb^h)^h a^h$ implies $b^h b(ab)^h = b^h bb^h(a^h abb^h)^h a^h = b^h(a^h abb^h)^h a^h = (ab)^h$, $(ab)^h aa^h = b^h(a^h abb^h)^h a^h aa^h = b^h(a^h abb^h)^h a^h = (ab)^h$.

(2) ⇒ (3): Using the assumption $b^h b(ab)^h = (ab)^h aa^h = (ab)^h$, we get
$(b(ab)^h a)(a^h abb^h)(b(ab)^h a) = b((ab)^h aa^h)ab(b^h b(ab)^h)a = b(ab)^h ab(ab)^h a = b(ab)^h a$,
i.e. $b(ab)^h a \in (a^h abb^h)\{2\}$, and

$$\begin{aligned}(b(ab)^h a)(a^h abb^h) &= b(ab)^h abb^h = b^h b(ab)^h ab \\ &= (ab)^h ab = ab(ab)^h aa^h = a^h ab(ab)^h a = a^h abb^h b(ab)^h a,\end{aligned}$$

i.e. $b(ab)^h a \in (a^h abb^h)\{5\}$.

Since $ab \in \mathrm{comm}(a) \cap \mathrm{comm}(b)$ and $a, b \in R^h$, $a^h b^h \in \mathrm{comm}(ab) \cap \mathrm{comm}((ab)^h)$, so $a^h b^h ab(ab)^h = a^h abb^h(ab)^h = a^h a(ab)^h = (ab)^h aa^h = (ab)^h$. Then applying Lemma 1.2, we have

$$\begin{aligned}a^h b^h(ab)^h - (ab)^h ba &= a^h b^h(ab)^h ab(ab)^h - (ab)^h aa^h ba = (a^h b^h ab(ab)^h)(ab)^h - (ab)^h a^h(ab)a \\ &= (ab)^h(ab)^h - (ab)^h aa^h ab = ((ab)^h)^2 - (ab)^h ab \\ &= ((ab)^h)^2 - (ab)^h ab(ab)^h ab = ((ab)^h)^2 - (ab)^h(ab)^2(ab)^h \\ &= ((ab)^h)^2 - (ab)^h((ab)^h)^h \in N(R),\end{aligned}$$

then by Definition 1.1 and Lemma 1.4, we have

$$\begin{aligned}(a^h abb^h)^2 - (a^h abb^h)(b(ab)^h a) &= (a^h b^h ab)^2 - a^h abb^h b(ab)^h a \\ &= (a^h b^h)^2((ab)^2 - ab(ab)^h) + (a^h b^h)^2 ab(ab)^h - a^h abb^h b(ab)^h a \\ &= (a^h b^h)^2((ab)^2 - ab(ab)^h) + a^h b^h(a^h b^h ab(ab)^h) - (a^h b^h ab(ab)^h)ba \\ &= (a^h b^h)^2((ab)^2 - ab(ab)^h) + (a^h b^h (ab)^h - (ab)^h ba) \\ &= (a^h b^h)^2((ab)^2 - ab(ab)^h) + ((ab)^h)^2 - (ab)^h((ab)^h)^h \in N(R),\end{aligned}$$

which yields $b(ab)^h a \in (a^h abb^h)\{7\}$. Hence $(a^h abb^h)^h = b(ab)^h a$. We note that $b^h b(ab)^h = (ab)^h aa^h = (ab)^h$, thus $b^h b(ab)^h aa^h = (ab)^h aa^h = (ab)^h$.

(3) ⇒ (1): Using the assumptions $(a^h abb^h)^h = b(ab)^h a$ and $b^h b(ab)^h aa^h = (ab)^h$, we have $b^h(a^h abb^h)^h a^h = b^h b(ab)^h aa^h = (ab)^h$.

(1) ⇒ (4): It is obvious using Definition 1.1.

(4) ⇒ (1): Only to prove that $b^h(a^h abb^h)^h a^h \in (ab)\{2,5\}$. Obviously,

$(b^h(a^h abb^h)^h a^h)(ab)(b^h(a^h abb^h)^h a^h) = (b^h(a^h abb^h)^h(a^h abb^h)(a^h abb^h)^h a^h = b^h(a^h abb^h)^h a^h$, i.e. $b^h(a^h abb^h)^h a^h \in (ab)\{2\}$.

Since $ab \in \mathrm{comm}(a) \cap \mathrm{comm}(b)$, $ab \in \mathrm{comm}(a^h) \cap \mathrm{comm}(b^h)$, so $(a^h abb^h) \in \mathrm{comm}(ab)$, and then $(a^h abb^h)^h \in \mathrm{comm}(ab)$, therefore, $b^h(a^h abb^h)^h a^h \in \mathrm{comm}(ab)$, i.e. $b^h(a^h abb^h)^h a^h \in (ab)\{5\}$.

(5) ⇔ (6): It follows directly from Theorem 2.2 (1)(3) and Corollary 2.1(1)(3).

(1) ⇒ (6): By the hypothesis $(ab)^h = b^h(a^h abb^h)^h a^h$, we have $((ab)^h a)(a^h ab)((ab)^h a) = (ab)^h ab(ab)^h aa^h a = (ab)^h aa^h a = b^h(a^h abb^h)^h a^h aa^h a = b^h(a^h abb^h)^h a^h a = (ab)^h a$, i.e. $(ab)^h a \in (a^h ab)\{2\}$.

Since $ab \in \mathrm{comm}(a)$, then $(ab)^h aa^h ab = a^h(ab)^h aab = a^h ab(ab)^h a$, i.e. $(ab)^h a \in (a^h ab)\{5\}$.

Note that $ab \in \mathrm{comm}(a)$, by Definition 1.1 and Lemma 1.4, we have

$(a^h ab)^2 - (a^h ab)((ab)^h a) = (a^h)^2((ab)^2 - ab(ab)^h) + ((a^h)^2 - a^h(a^h)^h)ab(ab)^h \in N(R)$,

i.e. $(ab)^h a \in (a^h ab)\{7\}$.

Hence, $a^h ab \in R^h$ and $(a^h ab)^h = (ab)^h a$ hold. Then $abb^h \in R^h$ and $(abb^h)^h = b(ab)^h$ follow in the similar way.

(6) ⇒ (4): Using $ab, a^h ab, abb^h \in R^h$, $(a^h ab)^h = (ab)^h a$ and $(abb^h)^h = b(ab)^h$, we have $b(a^h ab)^h = b(ab)^h a = (abb^h)^h a$.

Now we want to prove that $a^h abb^h \in R^h$ and $(a^h abb^h)^h = b(a^h ab)^h = (abb^h)^h a$.

Since $ab \in \mathrm{comm}(bb^h)$, and abb^h, $(abb^h)^h \in \mathrm{comm}(bb^h)$, so

$$\begin{aligned}bb^h b(a^h ab)^h &= bb^h b(ab)^h a = bb^h(abb^h)^h a = ((abb^h)^h)^2 abb^h bb^h a = ((abb^h)^h)^2 abb^h a \\ &= (abb^h)^h a = b(a^h ab)^h,\end{aligned}$$

therefore,

$$\begin{aligned}a^h abb^h b(a^h ab)^h &= a^h ab(a^h ab)^h = a^h ab(ab)^h a = a^h a(ab)^h ab \\ &= a^h a(a^h ab)^h b = a^h aa^h ab((a^h ab)^h)^2 b = a^h ab((a^h ab)^h)^2 b \\ &= (a^h ab)^h b = (ab)^h ab.\end{aligned}$$

The equalities $(abb^h)^h aa^h a = (abb^h)^h a$ and $(abb^h)^h aa^h abb^h = (ab)^h ab$ follow in the similar way. Then $b(a^h ab)^h a^h abb^h = b(ab)^h aa^h abb^h = (abb^h)^h aa^h abb^h = (ab)^h ab$, thus $b(a^h ab)^h \in (a^h abb^h)\{5\}$.

Using $bb^h b(a^h ab)^h = b(a^h ab)^h$, we have

$$(b(a^h ab)^h)(a^h abb^h)(b(a^h ab)^h) = b(a^h ab)^h a^h a(bb^h b(a^h ab)^h) = b(a^h ab)^h a^h ab(a^h ab)^h = b(a^h ab)^h,$$

i.e. $b(a^h ab)^h \in (a^h abb^h)\{2\}$.

In order to prove $b(a^h ab)^h \in (a^h abb^h)\{7\}$, first we have

$$\begin{aligned}a^h b^h ab(ab)^h &= a^h ab(ab)^h b^h = a^h a(ab)^h bb^h = a^h a(ab)^h ab(ab)^h bb^h \\ &= (a^h a(a^h ab)^h)((abb^h)^h bb^h) = (a^h ab)^h(abb^h)^h = (ab)^h ab(ab)^h = (ab)^h.\end{aligned}$$

Then,

$$\begin{aligned}(ab)^h a^h b^h - (ab)^h ab &= a^h abb^h((ab)^h)^2 - (ab)^h(ab)^h(ab)^2\\ &= a^h abb^h((ab)^h)^2 - a^h abb^h(ab)^h(ab)^h(ab)^2\\ &= a^h abb^h(((ab)^h)^2 - (ab)^h(ab)^2(ab)^h)\\ &= a^h abb^h(((ab)^h)^2 - (ab)^h((ab)^h)^h) \in N(R).\end{aligned}$$

By Definition 1.1 and Lemma 1.4, we obtain

$$\begin{aligned}(a^h abb^h)^2 - (a^h abb^h)(b(a^h ab)^h) &= a^h abb^h a^h abb^h - (ab)^h ab\\ &= a^h b^h (ab)^2 a^h b^h - a^h b^h ab(ab)^h a^h b^h + (ab)^h a^h b^h - (ab)^h ab\\ &= (a^h b^h)^2((ab)^2 - ab(ab)^h) + ((ab)^h a^h b^h - (ab)^h ab) \in N(R).\end{aligned}$$

So, $(a^h abb^h)^h = b(a^h ab)^h = b(ab)^h a = (abb^h)^h a$.

Now, we prove that $b^h(a^h abb^h)^h a^h \in (ab)\{7\}$. Since $ab, a^h ab \in R^h$, $(a^h ab)^h = (ab)^h a$, we obtain that $(a^h ab)^h a^h = (ab)^h$ by (1) ⇔ (3) in Theorem 2.2. We also note that $(abb^h)^h bb^h = ((abb^h)^h)^2 abb^h bb^h = ((abb^h)^h)^2 abb^h = (abb^h)^h$. Then

$$abb^h(a^h abb^h)^h a^h = abb^h(abb^h)^h aa^h = a(abb^h)^h bb^h aa^h = a(abb^h)^h aa^h = ab(a^h ab)^h a^h = ab(ab)^h,$$

thus $(ab)^2 - abb^h(a^h abb^h)^h a^h = (ab)^2 - ab(ab)^h \in N(R)$.

Then we have proved the equivalence of the statements (1)–(6). By (3) and (6), we have $a^h abb^h \in R^h$ and $(a^h abb^h)^h = b(a^h ab)^h = (abb^h)^h a$.

Now, we will prove the equivalence of (2) and (7).

(2) ⇒ (7): It is easy to obtain $aba, bab \in R^h$ by the hypotheses $a, b, ab \in R^h$, $ab \in \mathrm{comm}(a) \cap \mathrm{comm}(b)$. From the assumption $b^h b(ab)^h = (ab)^h aa^h = (ab)^h$, we deduce that $(aba)^h(a^h)^h = (ab)^h a^h a^2 a^h = (ab)^h aa^h = (ab)^h$ and $(b^h)^h(bab)^h = b^2 b^h b^h (ab)^h = b^h b(ab)^h = (ab)^h$ by Lemma 1.2 and Lemma 1.5.

(7) ⇒ (2): Using the assumptions $(ab)^h = (aba)^h(a^h)^h = (b^h)^h(bab)^h$, we get $(ab)^h = (aba)^h(a^h)^h = a^h(ab)^h a^2 a^h = (ab)^h a^h a^2 a^h = (ab)^h aa^h$, $(ab)^h = (b^h)^h(bab)^h = b^2 b^h(ab)^h b^h = b^h b^2 b^h(ab)^h = b^h b(ab)^h$, we claim that the statement (2) holds. □

Combining the statements of Theorem 2.2 and Corollary 2.1, and applying Theorem 2.5, we obtain more equivalent conditions for $(ab)^h = b^h a^h$. When $ab \in \mathrm{comm}(a) \cap \mathrm{comm}(b)$, we also give characterizations of the reverse order law $(ab)^h = b^h a^h$, which involve the Hirano inverse $(a^h abb^h)^h$.

Theorem 2.6. *If $a, b \in R^h$, $ab \in comm(a) \cap comm(b)$, then $(ab)^h = b^h a^h$ if and only if $a^h abb^h \in R^h$, $(a^h abb^h)^h = bb^h a^h a$ and any one of the following statements equivalent statements holds:*

(1) $ab \in R^h$ and $(ab)^h = b^h(a^h abb^h)^h a^h$;

(2) $ab \in R^h$ and $b^h b(ab)^h = (ab)^h aa^h = (ab)^h$;

(3) $ab \in R^h$ and $(a^h abb^h)^h = b(ab)^h a$ and $b^h b(ab)^h aa^h = (ab)^h$;

(4) $b^h(a^h abb^h)^h a^h \in (ab)\{7\}$;

(5) $ab, a^h ab, abb^h \in R^h$ and $(ab)^h = (a^h ab)^h a^h = b^h(abb^h)^h$;

(6) $ab, a^h ab, abb^h \in R^h$, $(a^h ab)^h = (ab)^h a$ and $(abb^h)^h = b(ab)^h$;

(7) $ab \in R^h$ and $(ab)^h = (aba)^h(a^h)^h = (b^h)^h(bab)^h$;

(8) $ab \in R^h$ and $(aba)^h(a^h)^h, (b^h)^h(bab)^h \in (ab)\{7\}$.

Proof. ⇒: The assumption $(ab)^h = b^h a^h$ implies that

$$(bb^h a^h a)(a^h abb^h)(bb^h a^h a) = bb^h a^h abb^h a^h a = b(ab)^h ab(ab)^h a = b(ab)^h a = bb^h a^h a,$$

i.e. $bb^h a^h a \in (a^h abb^h)\{2\}$.

Since $ab \in \mathrm{comm}(a) \cap \mathrm{comm}(b)$, we have $ab, (ab)^h \in \mathrm{comm}(a^h) \cap \mathrm{comm}(b^h)$, so

$$a^h abb^h bb^h a^h a = a^h abb^h a^h a = aba^h(ab)^h a = (ab)(ab)^h aa^h = (ab)b^h a^h aa^h = abb^h a^h = ab(ab)^h$$

and

$$bb^ha^haa^habb^h = bb^ha^habb^h = b(ab)^habb^h = b^hb(ab)^hab = b^hbb^ha^hab = b^ha^hab = (ab)^hab,$$

i.e. $bb^ha^ha \in (a^habb^h)\{5\}$.

Using $(ab)^h = b^ha^h$, we get

$$(ab)^h = b^ha^h = b^ha^haa^h = (ab)^haa^h, \text{ and } (ab)^h = b^ha^h = b^hbb^ha^h = bb^h(ab)^h.$$

Then we have

$$\begin{aligned}(a^hb^h)^2ab(ab)^h - a^habb^ha^ha &= a^hb^ha^hb^hab(ab)^h - a^hab(ab)^ha \\ &= a^h(ab)^hb^hab(ab)^h - ab(ab)^haa^h = a^h(ab)^habb^h(ab)^h - ab(ab)^h \\ &= a^h(ab)^ha(ab)^h - ab(ab)^h = (ab)^haa^h(ab)^h - ab(ab)^h \\ &= ((ab)^h)^2 - ab(ab)^hab(ab)^h = ((ab)^h)^2 - (ab)^h((ab)^2(ab)^h) \\ &= ((ab)^h)^2 - (ab)^h((ab)^h)^h \in N(R).\end{aligned}$$

By Lemma 1.4, we have

$$\begin{aligned}(a^habb^h)^2 - (a^habb^h)(bb^ha^ha) &= (a^hb^h)^2(ab)^2 - a^habb^ha^ha \\ &= (a^hb^h)^2((ab)^2 - ab(ab)^h) + ((a^hb^h)^2ab(ab)^h - a^habb^ha^ha) \\ &= (a^hb^h)^2((ab)^2 - ab(ab)^h) + ((ab)^h)^2 - (ab)^h((ab)^h)^h \in N(R),\end{aligned}$$

i.e. $bb^ha^ha \in (a^habb^h)\{7\}$. Then $a^habb^h \in R^h$, and $(a^habb^h)^h = bb^ha^ha$. Moreover, $(ab)^h = b^ha^h = b^hbb^ha^haa^h = b^h(a^habb^h)^ha^h$. So the the statement (1) holds.

Applying Theorem 2.5, we note that the statements (1) − (8) are equivalent.

$\Leftarrow$: If $ab, a^habb^h \in R^h$, $(a^habb^h)^h = bb^ha^ha$ and $(ab)^h = b^h(a^habb^h)^ha^h$, then $(ab)^h = b^hbb^ha^haa^h = b^ha^h$. □

Removing the conditions $a^hab, abb^h \in R^h$ in Theorem 2.4 and Corollary 2.3, we have the following equivalent characterizations of the mixed reverse order law $(a^habb^h)^h = bb^ha^ha$.

Theorem 2.7. *If $a, b \in R^h$, $a^hab \in comm(b)$ and $abb^h \in comm(a)$, then the following statements are equivalent:*
(1) $a^habb^h \in R^h$ and $(a^habb^h)^h = bb^ha^ha$;
(2) $a^habb^h \in R^h$ and $b^h(a^habb^h)^ha^h = b^ha^h$;
(3) $bb^ha^ha \in (a^habb^h)\{2\}$.

Proof. (1) ⇒ (2): Using the assumption $(a^habb^h)^h = bb^ha^ha$, we deduce $b^h(a^habb^h)^ha^h = b^hbb^ha^haa^h = b^ha^h$.

(2) ⇒ (3): By $b^h(a^habb^h)^ha^h = b^ha^h$ we have that $(bb^ha^ha)(a^habb^h)(bb^ha^ha) = bb^ha^habb^ha^ha = bb^h(a^habb^h)^ha^habb^h(a^habb^h)^ha^ha = bb^h(a^habb^h)^ha^ha = bb^ha^ha$, i.e. $bb^ha^ha \in (a^habb^h)\{2\}$.

(3) ⇒ (1): Since $a^hab \in \text{comm}(b)$ and $abb^h \in \text{comm}(a)$, we get that $a^hab \in \text{comm}(b^h)$ and $abb^h \in \text{comm}(a^h)$, so $(a^habb^h)(bb^ha^ha) = b^hba^habb^ha^ha = (bb^ha^ha)(a^habb^h)$, i.e. $bb^ha^ha \in (a^habb^h)\{5\}$. Note that

$$\begin{aligned}(a^habb^h)^2 - (a^habb^h)(bb^ha^ha) &= (a^habb^h)(a^habb^h) - a^habb^ha^ha \\ &= a^ha^haabb^hbb^h - a^ha^haabb^h = a^habb^h - a^habb^h = 0.\end{aligned}$$

i.e. $bb^ha^ha \in (a^habb^h)\{7\}$. By the assumption $bb^ha^ha \in (a^habb^h)\{2\}$, we claim that the statement (1) holds. □

Finally, we discuss equivalent characterizations of the reverse order law $(ab)^h = b^ha^h$ under the conditions $a^hab \in \text{comm}(b)$ and $abb^h \in \text{comm}(a)$.

Theorem 2.8. *If $a, b \in R^h$, $a^hab \in comm(b)$ and $abb^h \in comm(a)$, then the following statements are equivalent:*
(1) $ab \in R^h$ and $(ab)^h = b^ha^h$;
(2) $ab, a^habb^h \in R^h$, $(ab)^h = b^h(a^habb^h)^ha^h$ and $(a^habb^h)^h = bb^ha^ha$;
(3) $a^hab^ha^h = b^ha^h$ and $b^ha^h \in (ab)\{7\}$;
(4) $b^ha^hbb^h = b^ha^h$ and $b^ha^h \in (ab)\{7\}$.

Proof. (1) ⇒ (2): Using the assumption $(ab)^h = b^h a^h$, we conclude that
$(bb^h a^h a)(a^h abb^h)(bb^h a^h a) = bb^h a^h abb^h a^h a = b(ab)^h ab(ab)^h a = b(ab)^h a = bb^h a^h a$,
i.e. $bb^h a^h a \in (a^h abb^h)\{2\}$. Then we have $a^h abb^h \in R^h$ and $(a^h abb^h)^h = bb^h a^h a$ by Theorem 2.7. Moreover, $b^h(a^h abb^h)^h a^h = b^h bb^h a^h aa^h = b^h a^h = (ab)^h$.

(2) ⇒ (1): It is obvious.

(1) ⇒ (3): Using the assumption $(ab)^h = b^h a^h$, together with Definition 1.1, we get $b^h a^h \in (ab)\{7\}$. Since $a^h ab \in \mathrm{comm}(b)$ and $abb^h \in \mathrm{comm}(a)$, we have $a^h ab^h a^h = a^h abb^h b^h a^h = b^h a^h abb^h a^h = (ab)^h ab(ab)^h = (ab)^h = b^h a^h$.

(3) ⇒ (1): Suppose $a^h ab^h a^h = b^h a^h$ and $b^h a^h \in (ab)\{7\}$, we only need to prove that $b^h a^h \in (ab)\{2, 5\}$.

Since $a^h ab \in \mathrm{comm}(b)$ and $abb^h \in \mathrm{comm}(a)$, we have $(ab)(b^h a^h) = a^h abb^h = b^h a^h ab = (b^h a^h)(ab)$, i.e. $b^h a^h \in (ab)\{5\}$. In addition, $(b^h a^h)(ab)(b^h a^h) = b^h(a^h ab)b^h a^h = a^h abb^h b^h a^h = a^h ab^h a^h = b^h a^h$, i.e. $b^h a^h \in (ab)\{2\}$.

(1) ⇒ (4): Using the assumption $(ab)^h = b^h a^h$ and Definition 1.1, we know $b^h a^h \in (ab)\{7\}$. Since $a^h ab \in \mathrm{comm}(b)$ and $abb^h \in \mathrm{comm}(a)$, we have $b^h a^h bb^h = b^h a^h a^h abb^h = b^h a^h b^h a^h ab = (ab)^h (ab)^h ab = (ab)^h = b^h a^h$.

(4) ⇒ (1): It is similar to the proof of (3) ⇒ (1). □

Acknowledgments. We are grateful to the referees for their careful reading and valuable suggestions on this paper.

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