Filomat 33:11 (2019), 3497–3508
https://doi.org/10.2298/FIL1911497K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Semicommutativity of Rings by the Way of Idempotents

Handan Kose[a], Burcu Ungor[b], Abdullah Harmanci[c]

[a]*Department of Mathematics, Kirsehir Ahi Evran University, Kirsehir, Turkey*
[b]*Department of Mathematics, Ankara University, Ankara, Turkey*
[c]*Department of Mathematics, Hacettepe University, Ankara, Turkey*

Abstract. In this paper, we focus on the semicommutative property of rings via idempotent elements. In this direction, we introduce a class of rings, so-called right e-semicommutative rings. The notion of right e-semicommutative rings generalizes those of semicommutative rings, e-symmetric rings and right e-reduced rings. We present examples of right e-semicommutative rings that are neither semicommutative nor e-symmetric nor right e-reduced. Some extensions of rings such as Dorroh extensions and some subrings of matrix rings are investigated in terms of right e-semicommutativity. We prove that if R is a right e-semicommutative clean ring, then the corner ring eRe is clean.

1. Introduction

Throughout this paper, all rings are associative with identity. Due to Bell [4], a ring R is called to satisfy *the Insertion-of-Factors-Property (IFP)* if $ab = 0$ implies $aRb = 0$ for any $a, b \in R$. In [14] Narbonne and in [18] Shin used the terms semicommutative and S I for the IFP, respectively. In this work, we say semicommutative for this notion. In ring theory, the notion of semicommutativity plays an important role and has generated wide interest. Semicommutativity and its generalizations have been studied by many authors. Some generalizations of semicommutative rings are given as central semicommutative rings [2] and nil-semicommutative rings [7]. A ring R is *central semicommutative* if $ab = 0$ implies that $aRb \subseteq C(R)$ for any $a, b \in R$ where $C(R)$ is the center of R. In [7], it is said that a ring R is *nil-semicommutative* if for every $a, b \in R$, ab being nilpotent implies that aRb is a nil subset of R. Every semicommutative ring is central semicommutative and nil-semicommutative.

A ring R is *symmetric* [11] if $abc = 0$ implies $acb = 0$ for all $a, b, c \in R$. In [13] symmetric rings are generalized to e-symmetric rings. The ring R is called *e-symmetric* for $e^2 = e \in R$ if $abc = 0$ implies $acbe = 0$ for all $a, b, c \in R$. Every symmetric ring is semicommutative.

Idempotent elements are important tools for studying the structure of a ring. In the light of aforementioned concepts, it is a reasonable question that what kind of properties does a ring gain when it satisfies semicommutativity by way of idempotent elements? This question is one of the motivations to deal with the semicommutative property using idempotents. Motivated by the works on semicommutativity and e-symmetricity, the goal of this paper is to extend the notion of semicommutativity via idempotent elements

2010 *Mathematics Subject Classification*. Primary 16U80; Secondary 16N40
Keywords. Semicommutative ring, e-semicommutative ring, symmetric ring, e-symmetric ring, idempotent
Received: 05 January 2019; Accepted: 15 July 2019
Communicated by Dragan S. Djordjević
Email addresses: handan.kose@ahievran.edu.tr (Handan Kose), bungor@science.ankara.edu.tr (Burcu Ungor), harmanci@hacettepe.edu.tr (Abdullah Harmanci)

of the rings, namely, e-semicommutativity. We present some characterizations of this notion in various ways. We prove that every e-symmetric ring is e-semicommutative, and give an example to show that the converse of this result need not be true. We also discuss properties of this class of rings and give some structure theorems. Furthermore, some applications of e-semicommutative rings are performed. In this direction, this concept is considered for some rings of matrices. On the other hand, as an application, we deal with the question: If R is a clean ring and $e^2 = e \in R$, is the ring eRe clean? This question was asked in [8]. Šter proved in [19] that a corner ring eRe may not be clean when R is a clean ring and e is a full idempotent of R. We show that if R is a right e-semicommutative and clean ring, then eRe is also clean.

In what follows, $\mathbb{Z}$ denotes the ring of integers and for a positive integer n, $\mathbb{Z}_n$ is the ring of integers modulo n. Also $U(R)$ and $\mathrm{nil}(R)$ stand for the group of units and the set of all nilpotent elements of R. We write $M_n(R)$ for the ring of all $n \times n$ matrices, $U_n(R)$ for the ring of all upper triangular matrices over R for a positive integer $n \geq 2$ and $D_n(R)$ is the ring of all matrices in $U_n(R)$ having main diagonal entries equal. For a ring R, $P(R)$ and $J(R)$ denote the prime radical and the Jacobson radical of R, respectively.

2. Properties of e-semicommutative rings

In [13] e-symmetric rings and e-reduced rings are introduced and investigated. Let R be a ring and $e^2 = e \in R$. Then R is called *e-symmetric* if whenever $abc = 0$, then $acbe = 0$ for every $a, b, c \in R$. The ring R is *right* (resp. *left*) *e-reduced* if $ae = 0$ (resp. $ea = 0$) for each nilpotent $a \in R$. By motivated these e-contexts, in that vein, in this section we will introduce and study the structures of right e-semicommutative rings generalizing e-symmetric rings and right e-reduced rings [13]. Throughout this paper, e denotes an idempotent element of a ring R which is under consideration.

Definition 2.1. Let R be a ring and e an idempotent of R. Then R is called *right* (resp. *left*) *e-semicommutative* if for any $a, b \in R$, $ab = 0$ implies $aRbe = 0$ (resp. $eaRb = 0$). The ring R is called *e-semicommutative* in case R is both right and left e-semicommutative.

The following example shows that the notion of e-semicommutativity is not left-right symmetric, that is, there are left e-semicommutative rings which are not right e-semicommutative and vice versa. Moreover, any right (left) e-semicommutative ring may not be semicommutative and the concept of right (left) e-semicommutativity depends on the idempotent.

Example 2.2. Let R be a semicommutative ring and $e = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \in U_2(R)$. Then $e^2 = e$. Let $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, $B = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \in U_2(R)$ with $AB = 0$. Then $ax = 0$, $ay + bz = 0$, $cz = 0$. Then $aRx = 0$ and $cRz = 0$. For any $C = \begin{bmatrix} u & v \\ 0 & r \end{bmatrix} \in U_2(R)$, we get $ACB = \begin{bmatrix} 0 & auy + avz + brz \\ 0 & 0 \end{bmatrix}$. Hence $eACB = 0$ for all $C \in U_2(R)$. So $U_2(R)$ is left e-semicommutative. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \in U_2(R)$. Then $Ae = 0$. For any $B = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in U_2(R)$, $(ABe)e = \begin{bmatrix} 0 & -a + b + c \\ 0 & 0 \end{bmatrix}$. We may find elements a, b and $c \in R$ such that $-a + b + c \neq 0$. Then $(ABe)e = \begin{bmatrix} 0 & -a + b + c \\ 0 & 0 \end{bmatrix} \neq 0$ for some $a, b, c \in R$. Hence $U_2(R)$ is not right e-semicommutative. This yields $U_2(R)$ is also not semicommutative. Now consider the idempotent $f = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \in U_2(R)$. Let $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, $B = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \in U_2(R)$ with $AB = 0$. As in above discussion, $ACB = \begin{bmatrix} 0 & auy + avz + brz \\ 0 & 0 \end{bmatrix}$, and so $ACBf = 0$ for every $C = \begin{bmatrix} u & v \\ 0 & r \end{bmatrix} \in U_2(R)$. Therefore $U_2(R)$ is right f-semicommutative. But $U_2(R)$ is not left f-semicommutative because for $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in U_2(R)$, we have $AB = 0$, but $fAfB \neq 0$.

Proposition 2.3. *Let R be a ring, $0 \neq e^2 = e \in R$ and $n > 1$ an integer and $E = (e_{ij}) \in U_n(R)$ where $e_{1j} = e$ and $e_{ij} = 0$ for $i \neq 1$. Then R is right e-semicommutative if and only if $U_n(R)$ is right E-semicommutative.*

Proof. First note that $E^2 = E \in U_n(R)$. Assume that R is right e-semicommutative. Let $A = (a_{ij}), B = (b_{ij}) \in U_n(R)$ with $AB = 0$. Then $a_{11}b_{11} = 0$. By assumption, $a_{11}Rb_{11}e = 0$. Let $C = (c_{ij}) \in U_n(R)$. Then $a_{11}c_{11}b_{11}e = 0$. This yields $ACBE = 0$. Thus $U_n(R)$ is right E-semicommutative.
Conversely, suppose that $U_n(R)$ is right E-semicommutative. Let $a, b \in R$ with $ab = 0$. Consider the matrices $A = (a_{ij}), B = (b_{ij}) \in U_n(R)$ where $a_{11} = a$, $b_{11} = b$ and other entries of A and B are zero. Then $AB = 0$. Hence $AU_n(R)BE = 0$. It follows that $aRbe = 0$. Therefore R is right e-semicommutative. □

An idempotent e is called *left* (resp. *right*)-*semicentral* if $(1-e)Re = 0$ (resp. $eR(1-e) = 0$). So e is left (resp. right)-semicentral if and only if $re = ere$ for all $r \in R$ (resp. $er = ere$ for all $r \in R$). We now characterize right e-semicommutativity of a ring R in terms of e being left-semicentral and semicommutativity of the corner ring eRe.

Theorem 2.4. *Let R be a ring and $e^2 = e \in R$. Then the following hold.*

(1) *R is right e-semicommutative if and only if e is left-semicentral and eRe is semicommutative.*
(2) *R is left e-semicommutative if and only if e is right-semicentral and eRe is semicommutative.*

Proof. (1) Assume that R is right e-semicommutative. Then $(1-e)e = 0$ implies $(1-e)Re = 0$. So e is left-semicentral. Let eae, $ebe \in eRe$ with $(eae)(ebe) = 0$. By assumption $eaeRebe = 0$. It implies that $(eae)(eRe)(ebe) = 0$. Hence eRe is semicommutative. Conversely, let $a, b \in R$ with $ab = 0$. Then $eabe = 0$. Since $re = ere$ for each $r \in R$, $eabe = 0$ implies $(eae)(eRe)(ebe) = 0$. This and $re = ere$ for each $r \in R$ yield $aRbe = 0$. This completes the proof.
(2) Similar to the proof of (1). □

We now present an e-semicommutative ring with $0 \neq e^2 = e \in R$, but not semicommutative.

Example 2.5. Let k be a field, $A = k < x, y, z >$ be the free algebra with indeterminates x, y, z over k where x commutes with both of y, z, and y, z are noncommuting. Consider the ideal $I =< x^2 - x, xy, y^2, xz >$ of A and the ring $R = A/I$. Let x, y, z coincide with their images in R for simplicity. Then $0 \neq x \in R$ is a central idempotent. We claim that xR is semicommutative and $(1-x)R$ is not semicommutative. On the one hand, let $a, b \in R$ with $xab = 0$. Since $xy = xz = 0$ and x is central, considering the form of the elements of R, we have $xaRb = 0$. Thus xR is semicommutative. On the other hand, $(1-x)yy = 0$ but $(1-x)yzy \neq 0$. Therefore $(1-x)R$ is not semicommutative. This yields that R is not semicommutative. By Theorem 2.4, R is x-semicommutative.

The concept of right e-semicommutativity is a generalization of that of α-semicommutativity introduced in [1]. A ring R with an endomorphism α is called *α-semicommutative* if for any $a, b \in R$,

(i) $ab = 0$ implies $aRb = 0$,
(ii) $ab = 0$ if and only if $a\alpha(b) = 0$.

Every α-semicommutative ring is semicommutative, and so it is right e-semicommutative. There are e-semicommutative rings but not α-semicommutative. For example, the ring $U_2(R)$ considered in Example 2.2 is right e-semicommutative but not semicommutative, and so not α-semicommutative. Another kind of α-semicommutativity was introduced in [3]. Let R be a ring and α be a nonzero non identity endomorphism of R. Then R is called *α-semicommutative* if whenever $ab = 0$ for $a, b \in R$, $aR\alpha(b) = 0$. Clearly, α-semicommutativity in the sense of [3] is a generalization of α-semicommutativity in the sense of [1]. We have the following relationship between right e-semicommutativity and α-semicommutativity in the sense of [3].

Proposition 2.6. *A ring R is right e-semicommutative if and only if*

(i) *$\alpha\colon R \to R$ defined by $\alpha(r) = re$ where $r \in R$ is an endomorphism,*

(ii) *R is α-semicommutative.*

Proof. Assume that R is right e-semicommutative. Let $a, b \in R$. Clearly, $\alpha(a+b) = \alpha(a) + \alpha(b)$. Since e is left-semicentral by Theorem 2.4(1), we have $\alpha(ab) = abe = aebe = \alpha(a)\alpha(b)$. Hence α is an endomorphism of R. Now assume that $a, b \in R$ with $ab = 0$. Then $aRbe = 0$, and so $aR\alpha(b) = 0$. Thus R is α-semicommutative. The converse is obvious. □

It is well known that every semicommutative ring is abelian, i.e., every idempotent is central.

Corollary 2.7. *Let R be a ring and $e^2 = e \in R$. If R is right e-semicommutative and $f^2 = f \in R$, then the following hold.*

(1) *eRe is an abelian ring.*
(2) *$afe = fae$ for any $a \in R$.*
(3) *If $f \in eRe$, then f is left-semicentral in R.*

Proof. (1) By Theorem 2.4(1), eRe is semicommutative. Then eRe is abelian.
(2) Let $a \in R$. Then $afe = (eae)(efe)e$ since e is left-semicentral and $fe = efe$ is an idempotent in eRe. By the abelianness of eRe, $(eae)(efe) = (efe)(eae)$. The idempotent e being left-semicentral implies $afe = (eae)(efe)e = (efe)(eae)e = fae$.
(3) Let $f = efe \in eRe$. For any $a \in R$, $af = (eae)(efe)(efe)e = (efe)(eae)(efe) = faf$. Hence f is left-semicentral in R. □

Obviously, R is a semicommutative ring if and only if R is a 1-semicommutative ring. The notion of an e-semicommutative ring is an extension of semicommutative rings as well as a generalization of e-reduced rings. Right e-reduced rings are e-symmetric by [13, Corollary 4.3]. We show that every e-symmetric ring is right e-semicommutative.

Proposition 2.8. *Every e-symmetric ring is right e-semicommutative.*

Proof. Assume that R is an e-symmetric ring. Let a, $b \in R$ with $ab = 0$. For any $r \in R$, we have $abr = 0$. By assumption $arbe = 0$. Hence R is right e-semicommutative. □

There are right e-semicommutative rings which are not e-symmetric as shown below.

Examples 2.9. (1) There are semicommutative rings which are not symmetric.
(2) Let R be a semicommutative ring which is not symmetric considered in (1). Then $U_2(R)$ is right e-semicommutative but not e-symmetric where $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in U_2(R)$.

Proof. (1) Let $Q_8 = \{1, x_{-1}, x_i, x_{-i}, x_j, x_{-j}, x_k, x_{-k}\}$ be the quaternion group and consider the group ring $R = \mathbb{Z}_2Q_8$. The elements of $\mathbb{Z}_2Q_8$ as $\mathbb{Z}_2$-linear combinations of $\{x_g : g \in Q_8\}$. By Marks [12, Example 7], R is a right duo ring, equivalently, $Rr \subseteq rR$ for all $r \in R$. Let a, $b \in R$ with $ab = 0$. Then $Rb \subseteq bR$. Hence $aRb \subseteq abR$. Since $ab = 0$, $aRb = 0$. Therefore, R is semicommutative and so R is e-semicommutative. But R is not symmetric by taking $a = 1 + x_j$, $b = 1 + x_i$ and $c = 1 + x_i + x_j + x_k$. Then $abc = 0$ but $bac \neq 0$. In fact $bac \neq 0$ as in [12, Example 7]. Hence R is not symmetric.
(2) Let R be a semicommutative ring which is not symmetric considered in (1). Let $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in U_2(R)$. Then $Ae = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} = eAe$. Hence e is left semicentral. Since $R \cong eU_2(R)e$, $eU_2(R)e$ is not symmetric. By [13, Theorem 2.2], $U_2(R)$ is not e-symmetric. Since e is left semicentral and $eU_2(R)e \cong R$, $U_2(R)$ is right e-semicommutative by Theorem 2.4(1). □

A ring R is *prime* if for any a, $b \in R$, $aRb = 0$ implies $a = 0$ or $b = 0$, and R is *semiprime* if for any $a \in R$, $aRa = 0$ implies $a = 0$.

Proposition 2.10. *For a semiprime ring R, the following are equivalent.*

(1) *R is right e-semicommutative.*
(2) *R is right e-reduced.*
(3) *R is e-symmetric.*

Proof. (1) ⇒ (2) Let $a^n = 0$ for $a \in R$. We may assume that n is even and $n = 2t$. Since e is left-semicentral, $(a^t e)R(a^t e) = 0$. By hypothesis, $a^t e = 0$. Again we may assume that $t = 2k$. Similarly, $a^k e = 0$. Continuing on this way, we may reach $a^2 e = 0$. Hence $(ae)R(ae) = 0$. By hypothesis again, $ae = 0$.
(2) ⇒ (3) is clear from [13, Corollary 4.3].
(3) ⇒ (1) is clear from Proposition 2.8. □

Let R be a ring and $e^2 = e, g^2 = g \in R$. By [17], e and g are called *isomorphic* if $Re \cong Rg$ as left R-modules, equivalently, $eR \cong gR$ as right R-modules (see [10, Proposition 21.20]).

Theorem 2.11. *Let R be a ring and $e^2 = e, g^2 = g \in R$. If e and g are isomorphic, then the following hold.*

(1) *If R is right e-semicommutative, then R is right g-semicommutative.*
(2) *If R is left e-semicommutative, then R is left g-semicommutative.*
(3) *If R is right e-semicommutative, then $eR = gR$.*

Proof. (1) Let $\alpha\colon Re \to Rg$ be the left R-module isomorphism and $g = \alpha(xe)$ for some $x \in R$ since α is an epimorphism. Then $eg = \alpha(exe) = \alpha(xe) = g$ since R is right e-semicommutative. To prove R is right g-semicommutative, let $a, b \in R$ with $ab = 0$. By hypothesis, $aRbe = 0$, so $aRbeg = aRbg$. Thus R is right g-semicommutative.
(2) Let $\sigma\colon eR \to gR$ be the right R-module isomorphism and $g = \sigma(ex)$ for some $x \in R$ by the surjectivity of σ. Multiplying the latter from the right by e we have $ge = g$ since R is left e-semicommutative. Let $a, b \in R$ with $ab = 0$. Then $eaRb = 0$, and so $geaRb = 0$, this implies $gaRb = 0$. Therefore R is left g-semicommutative.
(3) As in the proof of (1), the isomorphism $\alpha\colon Re \to Rg$ implies $eg = g$. On the other hand, the isomorphism $\alpha^{-1}\colon Rg \to Re$ implies $ge = e$. Hence $eR = gR$. □

Proposition 2.12. *Let R be a right e-semicommutative ring with $e^2 = e \in R$. Then the following hold.*

(1) *If P is a prime ideal of R, then $e \in P$ or $1 - e \in P$.*
(2) *$eR(1 - e) \subseteq P(R)$.*
(3) *If M is a maximal left ideal of R, then $e \in M$ or $1 - e \in M$.*
(4) *$eR(1 - e) \subseteq J(R)$.*
(5) *If $ReR = R$, then $e = 1$.*
(6) *$\bar{e} = e + J(R) \in R/J(R)$ is central in $R/J(R)$.*

Proof. Note that R being a right e-semicommutative ring implies $(1-e)Re = 0$. We use this property without refer in the proof.
(1) Let P be a prime ideal. Then $(1 - e)Re = 0$ implies $(1 - e)Re \subseteq P$. So $1 - e \in P$ or $e \in P$.
(2) Clear by (1).
(3) Let M be a maximal left ideal. Assume that $e \notin M$. We have $Re + M = R$. Then $1 = xe + m$ for some $x \in R$ and $m \in M$. Then $1 - e = (1 - e)(xe + m) = (1 - e)m \in M$.
(4) Clear by (3).
(5) Assume that $ReR = R$. There exist r_i, $s_j \in R$ such that $1 = \sum_{i,j} r_i e s_j$. By right e-semicommutativity of R, $1 = \sum_{i,j} e r_i e s_j$. Multiplying the latter from the left by $1 - e$, we have $1 - e = 0$.
(6) By (4), $eR(1 - e) \subseteq J(R)$, and $(1 - e)Re = 0 \subseteq J(R)$. Hence $ea - ae \in J(R)$ for each $a \in R$. □

In [6], a ring is called *right principally quasi-Baer* (or simply, *right p.q.-Baer)* if the right annihilator of a principal right ideal is generated (as a right ideal) by an idempotent. A left principally quasi-Baer ring is defined similarly.

Proposition 2.13. *Let R be a right e-semicommutative ring. Then the following hold.*

(1) *If R is a prime ring, then it is a right e-reduced ring.*
(2) *If R is a right principally quasi-Baer ring, then it is a right e-reduced ring.*
(3) *If R is a left principally quasi-Baer ring, then it is a left e-reduced ring.*

Proof. (1) Let $a \in R$ with $a^n = 0$ for some positive integer n. We may assume that $n = 2k$. By hypothesis $a^k a^k = 0$ implies $a^k R a^k e = 0$. Since e is left-semicentral and R is prime, $(ae)^k = a^k e = 0$. Again we may assume that k is even and $k = 2t$. Then $(ae)^t (ae)^t = 0$. By hypothesis, $(ae)^t R(ae)^t e = 0$. Hence $(ae)^t = 0$. In this way we may reach $ae = 0$.
(2) Let $a \in R$ with $a^n = 0$ for some positive integer n. There exists $f^2 = f \in R$ such that $a^{n-1}e \in r_R(aR) = fR$ where $r_R(aR)$ is the right annihilator of aR in R. Then $a^{n-1}f = 0$, $fa^{n-1}e = a^{n-1}e$. By Corollary 2.7(1), efe is an idempotent in eRe and eRe is abelian, $fa^{n-1}e = (efe)(ea^{n-1}e)e = (ea^{n-1}e)(efe)e = a^{n-1}fe = 0$. This and $fa^{n-1}e = a^{n-1}e$ imply $a^{n-1}e = 0$. Let $b = ae$. Since e is left-semicentral, $b^{n-1} = 0$. As it is proved we get $b^{n-2}e = 0$. Hence $a^{n-2}e = 0$. By reduction in this way we get $ae = 0$. Hence R is right e-reduced.
(3) Similar to (2). □

By [21], an element r of a ring R is called *left minimal* if Rr is a minimal left ideal of R, an idempotent $e \in R$ is called *left minimal idempotent* if e is a left minimal element in R. In [21], a ring R is called *left min-abel* if every left minimal idempotent element of R is left-semicentral. A ring R is said to be NI [12] if the set of all nilpotent elements nil(R) is an ideal of R. In [5], a ring R is called *2-primal* if nil(R) coincides with its prime radical $P(R)$. Clearly, a 2-primal ring is NI. A ring R is called *strongly left min-abel* [21] if for every left minimal idempotent element $e \in R$, $Re = eR$. For example, an abelian ring is strongly left min-abel.

Theorem 2.14. *If R is right e-semicommutative for each left minimal idempotent e of R, then R is left min-abel.*

Proof. Assume that R is right e-semicommutative for each left minimal idempotent e of R. By Theorem 2.4(1), e is left-semicentral. □

The converse statement of Theorem 2.14 need not be true in general as shown below.

Example 2.15. There are left min-abel rings which are not right e-semicommutative.

Proof. We consider the ring

$$R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{Z}) \mid a \equiv d(mod2), b \equiv c \equiv 0(mod2) \right\}.$$

The idempotents of R are zero and identity matrices. If $AE = 0$ for any idempotent E of R, then $ARE = AER = 0$. So the condition holds. However R is not right E-semicommutative where $E = I$ is the identity of R. Indeed, for $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \in R$, we have $AB = 0$ but $ACBI \neq 0$ where $C = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \in R$. □

One may ask whether or not an e-semicommutative abelian ring is I-finite, if it is possible. Recall that a ring R is called *I-finite* if it contains no infinite set of orthogonal idempotents (see [16]). Every semiperfect ring is I-finite. In the next example, we show that there are e-semicommutative abelian rings that are not I-finite.

Example 2.16. Let $\mathbb{Z}_{(2)}$ denote the ring of all rational numbers with odd denominators (when written in lowest terms) and R be the infinite direct product $\mathbb{Z}_{(2)} \times \mathbb{Z}_{(2)} \times \cdots$ of $\mathbb{Z}_{(2)}$. Let e_i denote the element of R having ith entry is 1, all other entries are 0. Then R is a commutative ring and the set $S = \{e_i \in R \mid i = 1, 2, \dots\}$ contains infinitely many orthogonal idempotents. Hence R is an abelian and e-semicommutative ring for any idempotent e that is not I-finite.

By [18, Theorem 1.5], it is known that every semicommutative ring is 2-primal. But there is no implication between being a 2-primal ring and being an e-semicommutative ring with $1 \neq e^2 = e \in R$ as shown below.

Example 2.17. (1) Consider the ring $R = A/I$ given in Example 2.5. Then R is x-semicommutative but not 2-primal. In fact, for $y \in \mathrm{nil}(R)$ and $z \in R$, $yz \notin \mathrm{nil}(R)$. Hence $\mathrm{nil}(R)$ is not an ideal of R, thus R is not NI. Therefore R is not 2-primal.

(2) The ring of all $n \times n$ upper triangular matrices over a 2-primal ring R are also 2-primal by [5], but it may not be right (left) e-semicommutative for some idempotent e by Example 2.2.

Lemma 2.18. *Let R be a ring with $e^2 = e \in R$. If R is both right e-semicommutative and right $(1-e)$-semicommutative, then*

(1) *R is semicommutative,*
(2) *e is a central idempotent.*

Proof. (1) Let $a, b \in R$ with $ab = 0$. Then $aRbe = 0$ and $aRb(1-e) = 0$. For any $r \in R$, $arbe = 0$ and $arb(1-e) = 0$. Hence $0 = arb(1-e) = arb - arbe$ and $arbe = 0$ imply $arb = 0$. Thus $aRb = 0$.
(2) Assume that R is a both right e-semicommutative and right $(1-e)$-semicommutative ring. Since R is right e-semicommutative, we have $(1-e)Re = 0$, and so $ere = re$ for any $r \in R$. Similarly, since R is right $(1-e)$-semicommutative, we have $eR(1-e) = 0$, and so $ere = er$ for any $r \in R$. Hence $er = re$ for all $r \in R$. Thus e is central in R. □

Theorem 2.19. *If R is a right e-semicommutative ring, then the following hold.*

(1) *$ae = 0$ implies $aRe = 0$ for all $a \in R$.*
(2) *$ea = 0$ implies $eRae = 0$ for all $a \in R$.*

Proof. (1) Let for any $a \in R$ with $ae = 0$. Since R is right e-semicommutative, $(aRe)e = aRe = 0$.
(2) Let for any $a \in R$ with $ea = 0$. Then $(eRa)e = 0$. □

There are rings satisfying (1) of the preceding theorem but not right e-semicommutative.

Example 2.20. The ring in Example 2.15 satisfies Theorem 2.19(1) but not e-semicommutative for some idempotent e.

Proposition 2.21. *Let R be a right e-semicommutative ring. Then for any $a \in \mathrm{nil}(R)$, ae and ea are nilpotent elements of R.*

Proof. Let $a \in \mathrm{nil}(R)$ and $e^2 = e \in R$. Then there exists a positive integer m such that $a^m = 0$. We have $a^{m-1}Rae = 0$ since R is right e-semicommutative. Then $a^{m-1}eae = 0$, and so $a^{m-2}aeae = 0$. Since R is right e-semicommutative, $a^{m-2}Raeaee = 0$. Similarly, $a^{m-2}eaeae = 0$. Continuing on this way, we get $ae \in \mathrm{nil}(R)$ and $ea \in \mathrm{nil}(R)$. □

We now give another characterization of a right e-semicommutative ring.

Proposition 2.22. *Let R be a ring. Then the following are equivalent.*

(1) *R is a right e-semicommutative ring.*
(2) *$AB = 0$ implies $ARBe = 0$ for any nonempty subsets A and B of R.*

Proof. (1) ⇒ (2) Assume that the condition (1) holds. Let $AB = 0$ for any nonempty subsets A and B of R. Then for any $a \in A$ and $b \in B$, we have $ab = 0$. Since R is right e-semicommutative $aRbe = 0$ for $e^2 = e \in R$. Thus we get $ARBe = \sum_{a \in A, b \in B} aRbe = 0$.
(2) ⇒ (1) Clear. □

Proposition 2.23. *Let I be an index set and $(R_i)_{i \in I}$ be a class of rings with $e_i^2 = e_i \in R_i$ and let $R = \prod_{i \in I} R_i$ be the direct product of $(R_i)_{i \in I}$ with $e^2 = e = (e_i) \in R$. Then R is right e-semicommutative if and only if R_i is right e_i-semicommutative for each $i \in I$.*

Proof. Necessity: Let $i \in I$ and $a_i, b_i \in R_i$ with $a_ib_i = 0$. Consider $a = (\overset{0}{\ldots}, a_i, \overset{0}{\ldots})$, $b = (\overset{0}{\ldots}, b_i, \overset{0}{\ldots}) \in R$. Then $ab = 0$. Hence $aRbe = 0$. So $a_iR_ib_ie_i = 0$. Thus R_i is right e_i-semicommutative.
Sufficiency: Let $a = (a_i)_{i\in I}$ and $b = (b_i)_{i\in I} \in R$ such that $ab = 0$. Then we have $a_ib_i = 0$ for each $i \in I$. Since R_i is right e_i-semicommutative, $a_iR_ib_ie_i = 0$ and $e_i^2 = e_i$ for each $i \in I$. Let $c = (c_i) \in R$. Then $a_ic_ib_ie_i = 0$ for all $i \in I$. Hence $acbe = 0$ for each $c \in R$. So R is right e-semicommutative. □

Corollary 2.24. *Let n be a positive integer, $I = \{1, 2, 3, \ldots, n\}$ and $(R_i)_{i\in I}$ be a class of rings with $e_i^2 = e_i \in R_i$ and let $R = \bigoplus_{i\in I} R_i$ be the direct sum of $(R_i)_{i\in I}$ with $e^2 = e = (e_i) \in R$. Then R is right e-semicommutative if and only if R_i is e_i-semicommutative for each $i \in I$.*

Proof. The proof of Proposition 2.23 works verbatim here. □

Lemma 2.25. *Let R be a ring with a subring S and $e^2 = e \in S$. If R is right e-semicommutative, then so is S.*

Proof. Let $s_1, s_2 \in S$ with $s_1s_2 = 0$. Then $s_1Rs_2e = 0$. Since $s_1Ss_2e \subseteq s_1Rs_2e$, $s_1Ss_2e = 0$. □

In [15], a ring is called *clean* if every element is the sum of a unit and an idempotent.

Proposition 2.26. *Let R be a right e-semicommutative ring and $a \in R$. If a is clean, then ae is clean.*

Proof. Since a is clean, there exist $f^2 = f \in R$ and $u \in U(R)$ such that $a = f + u$. Then $ae = fe + ue$. Since R is a right e-semicommutative ring, $(fe)^2 = fe$. On the other hand, we have $(ue{-}(1-e))(eu^{-1}e{-}(1-e)) = 1$. Since $ae = fe + ue$, we have $ae = (fe + (1-e)) + (ue{-}(1-e))$. So ae is clean. □

In general, if R is a clean ring, then eRe need not be clean. This is a question posed in [8]. But in the case of e-semicommutativity of R, we have the following result.

Theorem 2.27. *Let R be a right e-semicommutative ring. If R is clean, then eRe is clean.*

Proof. Let $eae \in eRe$. By hypothesis, a is clean, so ae is clean by Proposition 2.26. Since $ae = eae$ for each $eae \in eRe$, eRe is clean. □

Proposition 2.28. *If R is a right e-semicommutative ring, then $1 + ea - ae$ is clean for any $a \in R$.*

Proof. By Proposition 2.12(4), $ea - ae \in J(R)$ for each $a \in R$. Then $1 + ea - ae$ is invertible for each $a \in R$. □

Proposition 2.29. *Let R be a right e-semicommutative ring. Then $R(ae-1) + Ra = R$ and $(ea-1)R + aR = R$ for any a in R.*

Proof. By Proposition 2.12(4), $ae - ea$ is in $J(R)$. Then $1 - ae + ea$ and $1 + ae - ea$ are invertible. Thus, the conclusions are obtained. □

Let R be a ring and $e \in R$. In [20], e is called *op-idempotent* if $e^2 = -e$. Not every op-idempotent is idempotent in general. For example, let $R = M_2(\mathbb{Z})$ and $e = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Then $e^2 = -e$ and e is an op-idempotent, but not idempotent.

Proposition 2.30. *Let e be an op-idempotent in R and R be a right $(1+e)$-semicommutative ring. If $ReR = R$, then $e = -1$.*

Proof. Assume that R is a right $(1+e)$-semicommutative ring and $ReR = R$. Then $eR(1+e) = 0$. Hence $ReR(1+e) = 0$, which implies $e = -1$. □

We now study some kinds of extensions of e-semicommutative rings. Let R be a ring. The Dorroh extension $D(R, \mathbb{Z}) = \{(r, n) \mid r \in R, n \in \mathbb{Z}\}$ of a ring R is the ring defined by the direct sum $R \oplus \mathbb{Z}$ with the ring operations $(r_1, n_1) + (r_2, n_2) = (r_1 + r_2, n_1 + n_2)$ and $(r_1, n_1)(r_2, n_2) = (r_1r_2 + n_1r_2 + n_2r_1, n_1n_2)$, where $r_i \in R$ and $n_i \in \mathbb{Z}$ for $i = 1, 2$.

Proposition 2.31. *A ring R is right e-semicommutative if and only if $D(R,\mathbb{Z})$ is right $(e,0)$-semicommutative.*

Proof. For the forward implication, assume that R is right e-semicommutative. Let (r,n), $(s,m) \in D(R,\mathbb{Z})$. If $(r,n)(s,m) = 0$, then $rs + mr + ns = 0$ and $nm = 0$. $nm = 0$ implies $n = 0$ or $m = 0$. We divide in two cases:
Case I. $m = 0$. Then $rs + ns = 0$. So $(r + n1)s = 0$. By assumption $(r + n1)Rse = 0$. Let $(t,k) \in D(R,\mathbb{Z})$. Then $(r,n)(t,k)(s,0)(e,0) = ((r + n1)(t + k1)se, 0)$. Since $(r + n1)Rse = 0$, $(r,n)(t,k)(s,0)(e,0) = 0$.
Case II. $n = 0$. Then $(r,0)(s,m) = 0$ implies $r(s+m1) = 0$. By assumption, $rR(s+m1)e = 0$. Let $(t,k) \in D(R,\mathbb{Z})$. Then $(r,0)(t,k)(s,m)(e,0) = ((r(t + k1), 0)((s + m1)e, 0)$. Since $rR(s + m1)e = 0$, $(r,0)(t,k)(s,m)(e,0) = (r(t + k1), 0)((s + m1)e, 0) = 0$. Hence $D(R,\mathbb{Z})$ is right $(e,0)$-semicommutative.
For the reverse implication, suppose that $D(R,\mathbb{Z})$ is right $(e,0)$-semicommutative. Let a, $b \in R$. Set $ab = 0$. So $(a,0)$, $(b,0) \in D(R,\mathbb{Z})$ and $(a,0)(b,0) = 0$. By supposition $(a,0)(t,k)(b,0)(e,0) = 0$ for each $(t,k) \in D(R,\mathbb{Z})$. Since $(a,0)(t,k)(b,0)(e,0) = (a(t + k1)be, 0)$ and $(a,0)(t,k)(b,0)(e,0) = 0$, $a(t + k1)be = 0$ for all $t \in R$. By taking $k = 0$, we get $atbe = 0$ for all $t \in R$. Hence R is right e-semicommutative. □

Let R be a ring and $r \in R$ with $r^2 + r = 0$. Then $1 + r$ is an idempotent in R and $(r,1)$ is an idempotent in $D(R,\mathbb{Z})$. By the proof of Proposition 2.31, it is obvious the following proposition.

Proposition 2.32. *Let R be a ring and $r \in R$ with $r^2 + r = 0$. Then R is right $(1 + r)$-semicommutative if and only if $D(R,\mathbb{Z})$ is right $(r,1)$-semicommutative.*

3. *e*-semicommutativity of some subrings of matrix rings

The rings $L_{(s,t)}(R)$: Let R be a ring and $s,t \in C(R)$. Let $L_{(s,t)}(R) = \left\{\begin{bmatrix} a & 0 & 0 \\ sc & d & te \\ 0 & 0 & f \end{bmatrix} \in M_3(R) \mid a,c,d,e,f \in R\right\}$, where the operations are defined as those in $M_3(R)$. Then $L_{(s,t)}(R)$ is a subring of $M_3(R)$.

Lemma 3.1. *Let R be an integral domain.*

(1) *Let $E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in L_{(1,1)}(R)$. Then $L_{(1,1)}(R)$ is right E-semicommutative.*

(2) *Let $E = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in L_{(1,1)}(R)$. Then $L_{(1,1)}(R)$ is not right E-semicommutative.*

Proof. (1) Let $A = \begin{bmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{bmatrix}$, $B = \begin{bmatrix} x & 0 & 0 \\ y & z & t \\ 0 & 0 & u \end{bmatrix} \in L_{(1,1)}(R)$ with $AB = 0$. Then $dz = 0$. Let $C = \begin{bmatrix} g & 0 & 0 \\ h & i & k \\ 0 & 0 & l \end{bmatrix} \in L_{(1,1)}(R)$. Then $ACB = \begin{bmatrix} * & 0 & 0 \\ * & 0 & * \\ 0 & 0 & * \end{bmatrix}$. So $ACBE = 0$ for all $C \in L_{(1,1)}(R)$. Hence $L_{(1,1)}(R)$ is right E-semicommutative.
(2) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \in L_{(1,1)}(R)$. Then $AB = 0$. Let $C = \begin{bmatrix} a & 0 & 0 \\ c & d & g \\ 0 & 0 & f \end{bmatrix} \in L_{(1,1)}(R)$. Then $ACBE = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -d + g + f \\ 0 & 0 & 0 \end{bmatrix}$ is nonzero for $d = g = 0$ and $f = 1$. Hence $L_{(1,1)}(R)$ is not right E-semicommutative. □

The rings $H_{(s,t)}(R)$: Let R be a ring and $s,t \in C(R)$. Let

$$H_{(s,t)}(R) = \left\{\begin{bmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{bmatrix} \in M_3(R) \mid a,c,d,e,f \in R, a - d = sc, d - f = te\right\}.$$

Then $H_{(s,t)}(R)$ is a subring of $M_3(R)$.

Theorem 3.2. *Let R be a semicommutative ring. Then $H_{(1,1)}(R)$ is right E-semicommutative where* $E = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Proof. We claim that E is a left-semicentral idempotent. For if $A = \begin{bmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{bmatrix}$, then $AE = \begin{bmatrix} a & 0 & 0 \\ c+d & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $EAE = \begin{bmatrix} a & 0 & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Since $a - d = c$, $AE = EAE$. Hence E is a left-semicentral idempotent. It is easy to check that $EH_{(1,1)}(R)E$ is semicommutative. In fact, let $EAE = \begin{bmatrix} a & 0 & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $EBE = \begin{bmatrix} b & 0 & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in EH_{(1,1)}(R)E$ with $(EAE)(EBE) = 0$. Then $ab = 0$. Let $ECE = \begin{bmatrix} c & 0 & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in H_{(1,1)}(R)$. By hypothesis, $ab = 0$ implies, $acb = 0$. It follows that $(EAE)(ECE)(EBE) = 0$. By Theorem 2.4 (1), $H_{(1,1)}(R)$ is right E-semicommutative. □

Generalized matrix rings: Let R be a ring and s a central element of R. Then $\begin{bmatrix} R & R \\ R & R \end{bmatrix}$ becomes a ring denoted by $K_s(R)$ with addition defined componentwise and multiplication defined in [9] by

$$\begin{bmatrix} a_1 & x_1 \\ y_1 & b_1 \end{bmatrix} \begin{bmatrix} a_2 & x_2 \\ y_2 & b_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 + sx_1y_2 & a_1x_2 + x_1b_2 \\ y_1a_2 + b_1y_2 & sy_1x_2 + b_1b_2 \end{bmatrix}.$$

In [9], $K_s(R)$ is called a *generalized matrix ring over R*.

Lemma 3.3. *Let F be a field. Then the following hold.*

(1) *The set $Inv(K_0(F))$ of all invertible elements of $K_0(F)$ is*

$$Inv(K_0(F)) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in K_0(F) \mid a \neq 0, d \neq 0 \right\}.$$

(2) *$C(K_0(F))$ consists of all scalar matrices.*

Proof. (1) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{Inv}(K_0(F))$. There exists $B = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \in K_0(F)$ such that $AB = BA = I$, where I is the identity matrix. Then we have

$$ax = xa = 1, ay + bt = 0, cx + dz = 0, dt = td = 1.$$

So $x = a^{-1}$, $t = d^{-1}$, $z = -d^{-1}ca^{-1}$ and $y = -a^{-1}bd^{-1}$. Conversely, assume that a and d are nonzero with inverses $x = a^{-1}$ and $t = d^{-1}$, $y = -a^{-1}bd^{-1}$ and $z = -d^{-1}ca^{-1}$. Then $B = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ is the inverse of A in $K_0(F)$.
(2) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in C(K_0(F))$. By commuting A in turn with the matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ in $K_0(F)$ we reach at $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$. For the converse, any matrix A having a form as $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ commutes with every element of $K_0(F)$. □

Lemma 3.4. *Let F be a field and* $E = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *be an idempotent matrix in* $K_0(F)$. *Then E is the zero matrix, E is the identity matrix or* $E = \begin{bmatrix} 1 & b \\ c & 0 \end{bmatrix}$ *or* $E = \begin{bmatrix} 0 & b \\ c & 1 \end{bmatrix}$ *where* $b, c \in F$.

Proof. Let $E^2 = E = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $a^2 = a$, $(a+d)b = b$, $(a+d)c = c$ and $d^2 = d$. We divide in some cases:
(1) $b \neq 0$ or $c \neq 0$. Then $a+d = 1$ implies $a = 0$ and $d = 1$ or $a = 1$ and $d = 0$. Hence $E_1 = \begin{bmatrix} 1 & b \\ c & 0 \end{bmatrix}$ or $E_2 = \begin{bmatrix} 0 & b \\ c & 1 \end{bmatrix}$.
(2) $b = c = 0$. Then E is one the following matrices:
E is the zero matrix or E is the identity matrix or $E_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ or $E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. □

Theorem 3.5. *Let F be a field. Then the following hold.*

(1) $K_0(F)$ *is not right e-semicommutative for idempotents other than zero matrix in* $K_0(F)$.
(2) $K_0(F)$ *is not left e-semicommutative for idempotents other than zero matrix in* $K_0(F)$.

Proof. (1) The idempotent matrices E other than zero matrix and identity matrix do not satisfy the equality $AE = EAE$ for all A in $K_0(F)$. It is enough to check the equality $AE = EAE$ for $E = E_1$ and $E = E_2$.
(i) Let $E_1 = \begin{bmatrix} 1 & b \\ c & 0 \end{bmatrix}$ and $A = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$. Then $AE_1 = \begin{bmatrix} x & xb \\ z+tc & 0 \end{bmatrix}$ and $E_1AE_1 = \begin{bmatrix} x & xb \\ cx & 0 \end{bmatrix}$. So $AE_1 \neq E_1AE_1$.
(ii) Let $E_2 = \begin{bmatrix} 0 & b \\ c & 1 \end{bmatrix}$ and $A = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$. Then $AE_2 = \begin{bmatrix} 0 & xb+y \\ tc & t \end{bmatrix}$ and $E_2AE_2 = \begin{bmatrix} 0 & bt \\ tc & t \end{bmatrix}$. This yields $AE_2 \neq E_2AE_2$.
Hence E_1 and E_2 are not left-semicentral idempotents. By Theorem 2.4(1), $K_0(F)$ is not right e-semicommutative for idempotents other than zero.
(2) Also it is enough to check the equality $EA = EAE$ for $E = E_1$ and $E = E_2$ as in the proof of (1).
(i) Let $E_1 = \begin{bmatrix} 1 & b \\ c & 0 \end{bmatrix}$ and $A = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$. Then $E_1A = \begin{bmatrix} x & y+bt \\ cx & 0 \end{bmatrix}$ and $E_1AE_1 = \begin{bmatrix} x & xb \\ cx & 0 \end{bmatrix}$. Hence $E_1A \neq E_1AE_1$.
(ii) Let $E_2 = \begin{bmatrix} 0 & b \\ c & 1 \end{bmatrix}$ and $A = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$. Then $E_2A = \begin{bmatrix} 0 & bt \\ cx+z & t \end{bmatrix}$ and $E_2AE_2 = \begin{bmatrix} 0 & bt \\ tc & t \end{bmatrix}$. Hence $E_2A \neq E_2AE_2$.
Thus E_1 and E_2 are not right-semicentral idempotents. By Theorem 2.4(2), $K_0(F)$ is not left e-semicommutative for idempotents other than zero. □

Acknowledgement: The authors would like to express their sincere thanks to the referee for his/her helpful suggestions and comments which have greatly improved the presentation of this paper.

References

[1] N. Agayev, A. Harmanci, S. Halicioglu, On abelian rings, Turk. J. Math. 34 (2010) 456–474.
[2] N. Agayev, T. Ozen, A. Harmanci, On a class of semicommutative rings, Kyungpook Math. J. 51 (2011) 283–291.
[3] M. Baser, A. Harmanci, T. K. Kwak, Generalized semicommutative rings and their extensions, Bull. Korean Math. Soc. 45(2) (2008) 285–297.
[4] H. E. Bell, Near-rings in which each element is a power of itself, Bull. Austral. Math. Soc. 2 (1970) 363–368.
[5] G. F. Birkenmeier, H. E. Heatherly, E. K. Lee, Completely prime ideals and associated radicals, in: S. K. Jain, S. T. Rizvi, editors, Proceedings of the Biennial Ohio State-Denison Conference (1992), World Scientific, Singapore (1993) 102–129.
[6] G. F. Birkenmeier, J. Y. Kim, J. K. Park, Principally quasi-Baer rings, Comm. Algebra 29(5) (2001) 639–660.
[7] W. Chen, On nil-semicommutative rings, Thai J. Math. 9(1) (2011) 39–47.
[8] J. Han, W. K. Nicholson, Extensions of clean rings, Comm. Algebra 29(6) (2001) 2589–2595.
[9] P. A. Krylov, On the isomorphism of generalized matrix rings, Algebra Logika 47(4) (2008) 456–463.
[10] T. Y. Lam, A First Course in Noncommutative Rings, Graduate Texts in Mathematics, vol. 131, Springer-Verlag, New York, 2001.
[11] J. Lambek, On the representation of modules by sheaves of factor modules, Canad. Math. Bull. 14 (1971) 359–368.
[12] G. Marks, Reversible and symmetric rings, J. Pure Appl. Algebra 174(3) (2002) 311–318.
[13] F. Meng, J. Wei, e-symmetric rings, Comm. Contemp. Math. 20(3) (2018) 8 pp.

[14] L. Motais de Narbonne, Anneaux semi-commutatifs et unis riels anneaux dont les id aux principaux sont idempotents, Proceedings of the 106th National Congress of Learned Societies (Perpignan, 1981), Bib. Nat., Paris (1982) 71–73.
[15] W. K. Nicholson, Lifting idempotents and exchange rings, Trans. Amer. Math. Soc. 229 (1977) 269–278.
[16] W. K. Nicholson, M. F. Yousif, Quasi-Frobenius Rings, Cambridge University Press, 2003.
[17] M. Osima, Notes on basic rings, Math. J. Okayama Univ. 2 (1953) 103–110.
[18] G. Shin, Prime ideals and sheaf representation of a pseudo symmetric ring, Trans. Amer. Math. Soc. 184 (1973) 43–60.
[19] J. Šter, Corner rings of a clean ring need not be clean, Comm. Algebra 40(5) (2012) 1595–1604.
[20] S. Wang, On op-idempotents, Kyungpook Math. J. 45(2) (2005) 171–175.
[21] J. C. Wei, Certain rings whose simple singular modules are nil-injective, Turk. J. Math. 32 (2008) 393–408.