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Existence and Uniqueness of the Solution to Granuloma Model in Visceral Leishmaniasi

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Abstract. In this paper we study a granuloma model in visceral leishmaniasis, which contains eleven coupled reaction diffusion equations. The existence and uniqueness of the model in the local solution is proved by using the Banach Fixed Point Theorem and theory of parabolic equation. Then, the existence and uniqueness of the global solution are obtained by using the extension method.

1. Introduction

The free boundary problem of some cells growth is under consideration because it can fit the mechanism of disease and predict its dynamic development, which provides a scientific explanation and theoretical basis for effective prevention and control of diseases. Therefore, it is necessary to systematically establish a mathematical model of some cells growth and conduct a rigorous mathematical analysis.

With consideration of the free boundary problem of tumor cell growth, Friedman and Reitich [10] obtained the existence and uniqueness of the global solution, the stable solution and the asymptotic behavior of the solution to model. From then on, a series of studies on the free boundary problem modeling some cells growth were conducted with abundant theoretical results developed in [5-8,16-18].

In specific, we consider the granuloma model of leishmaniasis. Leishmaniasis is a parasitic disease caused by infection with an obligate intra-cellular protozoa called Leishmania, the disease parasite. Two common forms of leishmaniasis are cutaneous leishmaniasis (CL) and visceral leishmaniasis (VL). VL is the more serious of the two and the primary symptoms of VL include abdominal pain, fever, shivering and weight loss [9,11,13]. Furthermore, the characteristic of VL is the formation of granulomas in the liver or the spleen. Granulomas are inflammatory foci containing infected cells, which are formed as immune cells migrate towards the source of infection, surround the infected cells and try to kill or control them. That is, following initial infection, there is a recruitment of new macrophages from the blood. These macrophages play a critical role both in the pathogenesis, and in the fight against leishmaniasis [4,12].

A mathematical model describing the evolution of visceral leishmaniasis was proposed by Siewe. N et al. [14] in 2016, which was represented by the system of ordinary differential equations. In this paper,

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however, we focus on a partial differential equations model which was also proposed by Siewe. N et al. [15]. To be specific, the model consists of a free boundary problem of a system of partial differential equations in three dimensions in the form of

$$\frac{\partial M_1}{\partial t} + \nabla\cdot(\vec{u}M_1) - \delta_1\nabla^2 M_1 = k_2M_2\frac{I_4}{I_4+c_4} - k_1M_1\frac{I_2}{I_2+c_2} - k_{11}M_1\frac{P_1^2}{P_1^2+(\frac{NM_1}{150})^2} - \mu_1M_1, \quad 0\le r\le R(t),\ t>0, \tag{1}$$

$$\frac{\partial M_2}{\partial t} + \nabla\cdot(\vec{u}M_2) - \delta_2\nabla^2 M_2 = k_1M_1\frac{I_2}{I_2+c_4} - k_2M_2\frac{I_4}{I_4+c_4} - k_{12}M_2\frac{P_2^2}{P_2^2+(\frac{NM_2}{150})^2} - \mu_2M_2, \quad 0\le r\le R(t),\ t>0, \tag{2}$$

$$\begin{aligned}\frac{\partial P_1}{\partial t} + \nabla\cdot(\vec{u}P_1) - \delta_3\nabla^2 P_1 = {} & \max\{\alpha_1P_1(1-\frac{P_1}{\frac{N_*M_1}{150}}),0\} + k_2P_2\frac{I_4}{I_4+c_4} \\ & + k_{12}\frac{NM_2}{150}\frac{P_2^2}{P_2^2+(\frac{NM_2}{150})^2}\theta + k_{11}\frac{NM_1}{150}\frac{P_1^2}{P_1^2+(\frac{NM_1}{150})^2}\theta \\ & - k_{11}\frac{NM_1}{150}\frac{P_1^2}{P_1^2+(\frac{NM_1}{150})^2} - k_1P_1\frac{I_2}{I_2+c_2} - \mu_3P_1(1+k_5\frac{E}{E+c_5} \\ & + k_4\frac{I_4}{I_4+c_4}) - \mu_1N_1P_1, \qquad 0\le r\le R(t),\ t>0,\end{aligned} \tag{3}$$

$$\begin{aligned}\frac{\partial P_2}{\partial t} + \nabla\cdot(\vec{u}P_2) - \delta_4\nabla^2 P_2 = {} & \max\{\alpha_2P_2(1-\frac{P_2}{\frac{N_*M_2}{150}}),0\} + k_1P_1\frac{I_2}{I_2+c_2} \\ & + k_{12}\frac{NM_2}{150}\frac{P_2^2}{P_2^2+(\frac{NM_2}{150})^2}(1-\theta) + k_{11}\frac{NM_1}{150}\frac{P_1^2}{P_1^2+(\frac{NM_1}{150})^2}(1-\theta) \\ & - k_{11}\frac{NM_2}{150}\frac{P_2^2}{P_2^2+(\frac{NM_2}{150})^2} - k_2P_2\frac{I_4}{I_4+c_4} - \mu_4P_2(1+k_5\frac{E}{E+c_5} \\ & + k_4\frac{I_4}{I_4+c_4}) - \mu_2N_2P_2, \qquad 0\le r\le R(t),\ t>0,\end{aligned} \tag{4}$$

$$\frac{\partial D}{\partial t} + \nabla\cdot(\vec{u}D) - \delta_5\nabla^2 D = \lambda_1\frac{P_1+P_2}{\frac{N(M_1+M_2)}{150}+P_1+P_2} - \mu_5D,\ 0\le r\le R(t),\ t>0, \tag{5}$$

$$\frac{\partial Q}{\partial t} + \nabla\cdot(\vec{u}Q) - \delta_6\nabla^2 Q = k_6Q\frac{I_1}{I_1+c_1} + \lambda_2\frac{M_1}{M_1+K_M}\frac{I_3}{I_3+c_3} - \mu_6Q,\ 0\le r\le R(t),\ t>0, \tag{6}$$

$$\frac{\partial I_1}{\partial t} - \delta_7\nabla^2 I_1 = k_7Q - \mu_7I_1,\ 0\le r\le R(t),\ t>0, \tag{7}$$

$$\frac{\partial I_2}{\partial t} - \delta_8\nabla^2 I_2 = k_8M_2 - \mu_8I_2,\ 0\le r\le R(t),\ t>0, \tag{8}$$

$$\frac{\partial I_3}{\partial t} - \delta_9\nabla^2 I_3 = k_9M_1 - \mu_9I_3,\ 0\le r\le R(t),\ t>0, \tag{9}$$

$$\frac{\partial I_4}{\partial t} - \delta_{10}\nabla^2 I_4 = \lambda_3Q - \mu_{10}I_4,\ 0\le r\le R(t),\ t>0, \tag{10}$$

$$\frac{\partial E}{\partial t} - \delta_{11}\nabla^2 E = \lambda_4(M_1+M_2)(1+k_3\frac{I_4}{I_4+c_4}) - \mu_{11}E,\ 0\le r\le R(t),\ t>0, \tag{11}$$

$$\frac{dR(t)}{dt} = u(R(t), t), \; t > 0, \tag{12}$$

$$u(0, t) = 0, \tag{13}$$

$$\nabla \cdot \vec{u} = h(M_1, M_2, P_1, P_2, D, Q, I_1, I_2, I_3), \tag{14}$$

$$\frac{\partial M_1}{\partial r}(0, t) = 0, \; \frac{\partial M_1}{\partial r} + \beta k_{10}\frac{I_4}{I_4 + c_4}(M_1 - M_0) = 0 \; on \; r = R(t), \; M_1(r, 0) = M_{10}(r), \tag{15}$$

$$\frac{\partial M_2}{\partial r}(0, t) = 0, \; \frac{\partial M_2}{\partial r}(R(t), t) = 0, \;, \; M_2(r, 0) = M_{20}(r), \tag{16}$$

$$\frac{\partial P_1}{\partial r}(0, t) = 0, \; \frac{\partial P_1}{\partial r}(R(t), t) = 0, \; P_1(r, 0) = P_{10}(r), \tag{17}$$

$$\frac{\partial P_2}{\partial r}(0, t) = 0, \; \frac{\partial P_2}{\partial r}(R(t), t) = 0, \; P_2(r, 0) = P_{20}(r), \tag{18}$$

$$\frac{\partial D}{\partial r}(0, t) = 0, \; \frac{\partial D}{\partial r} + \beta_0 D = 0 \; on \; r = R(t), \; D(r, 0) = D_0(r), \tag{19}$$

$$\frac{\partial Q}{\partial r}(0, t) = 0, \; \frac{\partial Q}{\partial r} + \beta\lambda_0\frac{D}{k_D + D}\frac{I_3}{I_3 + c_3}(Q - Q_0) = 0 \; on \; r = R(t), \; Q(r, 0) = Q_0(r), \tag{20}$$

$$\frac{\partial I_1}{\partial r}(0, t) = 0, \; \frac{\partial I_1}{\partial r}(R(t), t) = 0, \; I_1(r, 0) = I_{10}(r), \tag{21}$$

$$\frac{\partial I_2}{\partial r}(0, t) = 0, \; \frac{\partial I_2}{\partial r}(R(t), t) = 0, \; I_2(r, 0) = I_{20}(r), \tag{22}$$

$$\frac{\partial I_3}{\partial r}(0, t) = 0, \; \frac{\partial I_3}{\partial r}(R(t), t) = 0, \; I_3(r, 0) = I_{30}(r), \tag{23}$$

$$\frac{\partial I_4}{\partial r}(0, t) = 0, \; \frac{\partial I_4}{\partial r}(R(t), t) = 0, \; I_4(r, 0) = I_{40}(r), \tag{24}$$

$$\frac{\partial E}{\partial r}(0, t) = 0, \; \frac{\partial E}{\partial r}(R(t), t) = 0, \; E(r, 0) = E_0(r), \tag{25}$$

$$R(0) = R_0, \tag{26}$$

where

$$M_1 + M_2 + D + Q = 1, \delta_1 = \delta_2 = \delta_5 = \delta_6,$$

$$\begin{aligned} h(M_1, M_2, P_1, P_2, D, Q, I_1, I_2, I_3) = &-(k_{11}M_1\frac{P_1^2}{P_1^2 + (\frac{NM_1}{150})^2} + k_{12}M_2\frac{P_2^2}{P_2^2 + (\frac{NM_2}{150})^2}) + k_6 Q\frac{I_1}{I_1 + c_1} \\ &+ \lambda_1\frac{P_1 + P_2}{\frac{N(M_1+M_2)}{150} + P_1 + P_2} + \lambda_2\frac{M_1}{M_1 + K_M}\frac{I_3}{I_3 + c_3} \\ &- (\mu_1 M_1 + \mu_2 M_2 + \mu_5 D + \mu_6 Q), \end{aligned}$$

with M_1, M_2, P_1, P_2, D, Q representing the density of pro-inflammatory (classically activated) macrophages, the density of anti-inflammatory (alternatively activated) macrophages, the density of Leishmania in macrophages M_1, the density of Leishmania in macrophages M_2, the density of dendritic cells, the density of CD4$^+$T cells and I_1, I_2, I_3, I_4, E denoting the concentration of IL-2, the concentration of IL-10, the concentration of IL-12, the concentration of IFN-γ, and the concentration of Nitric Oxide respectively. Moreover,u is the component of the velocity $\vec{u}$ in the radial direction and $\delta_m (m = 1, 2, \ldots, 11)$ is the diffusion coefficient. $k_i (i = 1, 2, \ldots, 12)$, $\lambda_j (j = 0, 1, \ldots, 4)$, μ_m, $c_n (n = 1, 2, 3, 4)$, N, N_*, β, β_0, K_M, K_D, M_0, Q_0 are positive constants.

Though theres simulations, the theoretical results are still undiscovered. So we will make a rigorous mathematical analysis to the model.

We assume the following conditions on initial functions:

(A1)$M_{10}(r), M_{20}(r), P_{10}(r), P_{20}(r), D_0(r), Q_0(r), I_{10}(r), I_{20}(r), I_{30}(r), I_{40}(r), E_0(r) \in D_p(0,1)$;

(A2) $M_{10}(r), M_{20}(r), P_{10}(r), P_{20}(r), D_0(r), Q_0(r), I_{10}(r), I_{20}(r), I_{30}(r), I_{40}(r), E_0(r) \geq 0$.

The main result of this paper is as follows:

Theorem 1.1 *Under the conditions (A1) and (A2), there exists a constant $C(T)$ depending on time T, such that problem (1)–(26) has a unique global solution $(M_1, M_2, P_1, P_2, D, Q, I_1, I_2, I_3, I_4, E)$. Moreover, for any $T > 0$, there exists*

$$R(t) \in C^1[0, T] \quad for \;\; 0 \leq t \leq T;$$

$$0 \leq M_1, M_2, P_1, P_2, D, Q, I_1, I_2, I_3, I_4, E \leq C(T) \quad for \;\; 0 \leq t \leq T.$$

The structure of this paper is as follows. In Section 2 we transform the problem (1)–(26) into an equivalent problem defined on a fixed domain. Section 3 is devoted to presenting some preliminary lemmas that will be used in later analysis. In Section 4 we prove local existence of solutions by using the reformulated version of the problem presented in Section 2. The last section, Section 5, aims at presenting the proof of global existence and uniqueness.

2. Preliminary Lemmas

In this section some preliminary lemmas are shown. We first introduce some notations:

(1) Denote $\Omega_T = \{(z, \tau)| : 0 < z < 1,\ 0 < \tau < T\}$. $\bar{\Omega}_T$ is the closure of Ω_T. Define

$$W_p^{2,1}(\Omega_T) = \{u \in L^p(\Omega_T) : \partial_x^m \partial_t^k \in L^p(\Omega_T) \;\; \text{for} \;\; |m| + 2k \leq 2\},$$

and set

$$\|u\|_{W_p^{2,1}(\Omega_T)} = \sum_{|m|+2k\leq 2} \|\partial_x^m \partial_t^k\|_p,$$

where $\|\cdot\|_p$ represents the L^p norm.

(2) For a number $p > \frac{5}{2}$, we denote by $D_p(0,1)$ the trace space of $W_p^{2,1}(\Omega_T)$ at $t = 0$, i.e. , $\varphi \in D_p(0,1)$ if and only if there exists $u \in W_p^{2,1}(\Omega_T)$ such that $u(\cdot, 0) = \varphi$. The norm in $D_p(0,1)$ is defined as follows:

$$\|\varphi\|_{D_p(0,1)} = \inf\{T^{-\frac{1}{p}} \|u\|_{W_p^{2,1}(\Omega_T)} : u \in W_p^{2,1}(\Omega_T), u(\cdot, 0) = \varphi\}.$$

Since for $p > \frac{5}{2}$, $W_p^{2,1}(\Omega_T)$ is continuously embedded in $C(\bar{\Omega}_T)$ [1], the above definition makes sense. Besides, it is clear that if $\varphi \in W^{2,p}(0,1)$, then $\varphi \in D_p(0,1)$ and $\|\varphi\|_{D_p(0,1)} \leq \|\varphi\|_{W^{2,p}(0,1)}$.

Lemma 2.1 [3] *Let D be a positive constant and $a(z,\tau)$, $b(z,\tau)$ be bounded continuous functions defined on $\bar{\Omega}_T$ $(T > 0)$, $f(z,\tau) \in L^p(\Omega_T), \varphi(z,\tau) \in C^1[0,T]$ and $C_0 \in D_p(0,1)$ for $1 < p < \infty$. Let $Bu = \alpha\frac{\partial u}{\partial n} + \beta(z,\tau)$ where (1)$\alpha = 0, \beta = 1$, (2)$\alpha = 1, \beta \geq 0$, then the initial value problem*

$$\frac{\partial c}{\partial \tau} = D\frac{\partial^2 c}{\partial z^2} + a(z,\tau)\frac{\partial c}{\partial z} + b(z,\tau)c + f(z,\tau) \quad for \;\; 0 < z \leq 1,\;\; 0 < \tau \leq T,$$

$$z = 0, 1 : Bc = \varphi \quad for \;\; 0 < \tau \leq T,$$

$$c(z, 0) = c_0(z) \quad for \;\; 0 \leq z \leq 1,$$

has a unique solution $c(z,\tau) \in W_p^{2,1}(\Omega_T)$. Moreover, there exists a positive constant $C_p(T)$ depending only on p, T, $\|a\|_\infty$, $\|b\|_\infty$, such that

$$\|c\|_{W_p^{2,1}(\Omega_T)} \leq C_p(T)\Big(\|\varphi\|_{W^{1,p}(0,T)} + \|c_0\|_{D_p(0,1)} + \|f\|_p\Big)$$

$C_p(T)$ are bounded for all T in a bounded set.

3. Reformulation of the Problem

Making change of variables

$$z=\frac{r}{R(t)},\quad \tau=\int_0^t\frac{ds}{R^2(s)},\quad \eta(\tau)=R(t),\quad M_1'(z,\tau)=M_1(r,t),\quad M_2'(z,\tau)=M_2(r,t),$$
$$P_1'(z,\tau)=P_1(r,t),\quad P_2'(z,\tau)=P_2(r,t),\quad D'(z,\tau)=D(r,t),\quad Q'(z,\tau)=Q(r,t),\quad I_1'(z,\tau)=I_1(r,t),$$
$$I_2'(z,\tau)=I_2(r,t),\quad I_3'(z,\tau)=I_3(r,t),\quad I_4'(z,\tau)=I_4(r,t),\quad E'(z,\tau)=E(r,t),\quad u'(z,\tau)=R(t)u(r,t),\tag{27}$$

the free boundary problem (1)–(26) is transformed into the following initial-boundary value problem on the fixed domain $\{(z,\tau):0\le z\le 1,\tau\ge 0\}$:

$$\frac{\partial M_1'}{\partial\tau}=\delta_1\frac{1}{z^2}\frac{\partial}{\partial z}(z^2\frac{\partial M_1'}{\partial z})-v(z,\tau)\frac{\partial M_1'}{\partial z}-\eta^2(\tau)(k_1\frac{I_2'}{I_2'+c_2}+k_{11}\frac{P_1'^2}{P_1'^2+(\frac{NM_1'}{150})^2}+\mu_1+h')M_1'+g_1,\tag{28}$$
$$0\le z\le 1,\ \tau>0,$$

$$\frac{\partial M_1'}{\partial z}(0,\tau)=0,\ \frac{\partial M_1'}{\partial z}+\beta k_{10}\frac{I_4'}{I_4'+c_4}(M_1'-M_0)=0\ \ on\ \ z=1,\ M_1'(z,0)=M_{10}(z),\tag{29}$$

$$\frac{\partial M_2'}{\partial\tau}=\delta_2\frac{1}{z^2}\frac{\partial}{\partial z}(z^2\frac{\partial M_2'}{\partial z})-v(z,\tau)\frac{\partial M_2'}{\partial z}-\eta^2(\tau)(k_1\frac{I_4'}{I_4'+c_2}+k_{11}\frac{P_2'^2}{P_2'^2+(\frac{NM_2'}{150})^2}+\mu_2+h')M_2'+g_2,\tag{30}$$
$$0\le z\le 1,\ \tau>0,$$

$$\frac{\partial M_2'}{\partial z}(0,\tau)=0,\ \frac{\partial M_2'}{\partial z}(1,\tau)=0,\ M_2'(z,0)=M_{20}(z),\tag{31}$$

$$\begin{aligned}\frac{\partial P_1'}{\partial\tau}=&\delta_3\frac{1}{z^2}\frac{\partial}{\partial z}(z^2\frac{\partial P_1'}{\partial z})-v(z,\tau)\frac{\partial P_1'}{\partial z}+\eta^2(\tau)(\max\{\alpha_1P_1'(1-\frac{P_1'}{\frac{N_*M_1'}{150}}),0\}\\&+k_{11}\frac{NM_1'}{150}\frac{P_1'^2}{P_1'^2+(\frac{NM_1'}{150})^2}\theta-k_{11}\frac{NM_1'}{150}\frac{P_1'^2}{P_1'^2+(\frac{NM_1'}{150})^2}-k_1P_1'\frac{I_2'}{I_2'+c_2}\\&-\mu_3P_1'(1+k_5\frac{E'}{E'+c_5}+k_4\frac{I_4'}{I_4'+c_4})-\mu_1N_1P_1'-h'P_1')+g_3,\quad 0\le z\le 1,\ \tau>0,\end{aligned}\tag{32}$$

$$\frac{\partial P_1'}{\partial z}(0,\tau)=0,\ \frac{\partial P_1'}{\partial z}(1,\tau)=0,\ ,\ P_1'(z,0)=P_{10}(z),\tag{33}$$

$$\begin{aligned}\frac{\partial P_2'}{\partial\tau}=&\delta_4\frac{1}{z^2}\frac{\partial}{\partial z}(z^2\frac{\partial P_2'}{\partial z})-v(z,\tau)\frac{\partial P_2'}{\partial z}+\eta^2(\tau)(\max\{\alpha_2P_2'(1-\frac{P_2'}{\frac{N_*M_2'}{150}}),0\}\\&+k_{12}\frac{NM_2'}{150}\frac{P_2'^2}{P_2'^2+(\frac{NM_1'}{150})^2}(1-\theta)-k_{11}\frac{NM_2'}{150}\frac{P_2'^2}{P_2'^2+(\frac{NM_2'}{150})^2}-k_2P_2'\frac{I_4'}{I_4'+c_4}\\&-\mu_4P_2'(1+k_5\frac{E'}{E'+c_5}+k_4\frac{I_4'}{I_4'+c_4})-\mu_2N_2P_2'-h'P_2')+g_4,\quad 0\le z\le 1,\ \tau>0,\end{aligned}\tag{34}$$

$$\frac{\partial P_2'}{\partial z}(0,\tau)=0,\ \frac{\partial P_2'}{\partial z}(1,\tau)=0,\ ,\ P_2'(z,0)=P_{20}(z),\tag{35}$$

$$\frac{\partial D'}{\partial\tau}=\delta_5\frac{1}{z^2}\frac{\partial}{\partial z}(z^2\frac{\partial D'}{\partial z})-v(z,\tau)\frac{\partial D'}{\partial z}-\eta^2(\tau)(\mu_5+h')D'+g_5,\quad 0\le z\le 1,\ \tau>0,\tag{36}$$

$$\frac{\partial D'}{\partial z}(0,\tau)=0,\ \frac{\partial D'}{\partial z}+\beta_0D'=0\ \ on\ \ z=1,\ D'(z,0)=D_0(z),\tag{37}$$

$$\frac{\partial Q'}{\partial \tau} = \delta_6 \frac{1}{z^2}\frac{\partial}{\partial z}(z^2 \frac{\partial Q'}{\partial z}) - v(z,\tau)\frac{\partial Q'}{\partial z} - \eta^2(\tau)(k_6 \frac{I_1'}{I_1' + c_1}) - \mu_6 - h')Q' + g_6, \quad 0 \le z \le 1,\ \tau > 0, \tag{38}$$

$$\frac{\partial Q'}{\partial z}(0,\tau) = 0,\ \frac{\partial Q'}{\partial z} + \beta_0\lambda_0 \frac{D'}{k_D + D'}\frac{I_3'}{I_3' + c_3}(Q' - Q_0) = 0 \ on\ z = 1,\ Q'(z,0) = Q_0(z), \tag{39}$$

$$\frac{\partial I_1'}{\partial \tau} = \delta_7 \frac{1}{z^2}\frac{\partial}{\partial z}(z^2 \frac{\partial I_1'}{\partial z}) + u'(1,\tau)z\frac{\partial I_1'}{\partial z} - \eta^2(\tau)\mu_7 I_1' + g_7, \quad 0 \le z \le 1,\ \tau > 0, \tag{40}$$

$$\frac{\partial I_1'}{\partial z}(0,\tau) = 0,\ \frac{\partial I_1'}{\partial z}(1,\tau) = 0,\ ,\ I_1'(z,0) = I_{10}(z), \tag{41}$$

$$\frac{\partial I_2'}{\partial \tau} = \delta_8 \frac{1}{z^2}\frac{\partial}{\partial z}(z^2 \frac{\partial I_2'}{\partial z}) + u'(1,\tau)z\frac{\partial I_2'}{\partial z} - \eta^2(\tau)\mu_8 I_2' + g_8, \quad 0 \le z \le 1,\ \tau > 0, \tag{42}$$

$$\frac{\partial I_2'}{\partial z}(0,\tau) = 0,\ \frac{\partial I_2'}{\partial z}(1,\tau) = 0,\ ,\ I_2'(z,0) = I_{20}(z), \tag{43}$$

$$\frac{\partial I_3'}{\partial \tau} = \delta_9 \frac{1}{z^2}\frac{\partial}{\partial z}(z^2 \frac{\partial I_3'}{\partial z}) + u'(1,\tau)z\frac{\partial I_3'}{\partial z} - \eta^2(\tau)\mu_9 I_3' + g_9, \quad 0 \le z \le 1,\ \tau > 0, \tag{44}$$

$$\frac{\partial I_3'}{\partial z}(0,\tau) = 0,\ \frac{\partial I_3'}{\partial z}(1,\tau) = 0,\ ,\ I_3'(z,0) = I_{30}(z), \tag{45}$$

$$\frac{\partial I_4'}{\partial \tau} = \delta_{10} \frac{1}{z^2}\frac{\partial}{\partial z}(z^2 \frac{\partial I_4'}{\partial z}) + u'(1,\tau)z\frac{\partial I_4'}{\partial z} - \eta^2(\tau)\mu_{10} I_4' + g_{10}, \quad 0 \le z \le 1,\ \tau > 0, \tag{46}$$

$$\frac{\partial I_4'}{\partial z}(0,\tau) = 0,\ \frac{\partial I_4'}{\partial z}(1,\tau) = 0,\ I_4'(z,0) = I_{40}(z), \tag{47}$$

$$\frac{\partial E'}{\partial \tau} = \delta_{11} \frac{1}{z^2}\frac{\partial}{\partial z}(z^2 \frac{\partial E'}{\partial z}) + u'(1,\tau)z\frac{\partial E'}{\partial z} - \eta^2(\tau)\mu_{11} E' + g_{11}, \quad 0 \le z \le 1,\ \tau > 0, \tag{48}$$

$$\frac{\partial E'}{\partial z}(0,\tau) = 0,\ \frac{\partial E'}{\partial z}(1,\tau) = 0,\ E'(z,0) = E_0(z), \tag{49}$$

$$\frac{d\eta(\tau)}{d\tau} = \eta(\tau)u'(1,\tau),\ \tau > 0, \tag{50}$$

$$\eta(0) = R_0, \tag{51}$$

$$v(z,\tau) = u'(1,\tau) - zu'(1,\tau), \quad 0 \le z \le 1,\ \tau > 0, \tag{52}$$

$$\frac{1}{z^2}\frac{\partial}{\partial z}(z^2 u') = \eta^2(\tau)h(M_1', M_2', P_1', P_2', D', Q', I_1', I_2', I_3') = \eta^2 h', \tag{53}$$

$$u'(0,\tau) = 0,\ \tau > 0, \tag{54}$$

where

$$g_1 = \eta^2(\tau)k_2 M_2' \frac{I_4'}{I_4' + c_4},\ g_2 = \eta^2(\tau)k_1 M_1' \frac{I_2'}{I_2' + c_4},$$

$$g_3 = \eta^2(\tau)(k_2 P_2' \frac{I_4'}{I_4' + c_4} + k_{12}\frac{NM_2'}{150}\frac{P_2'^2}{P_2'^2 + (\frac{NM_2'}{150})^2}\theta),$$

$$g_4 = \eta^2(\tau)(k_1 P_1' \frac{I_2'}{I_2' + c_2} + k_{11}\frac{NM_1'}{150}\frac{P_1'^2}{P_1'^2 + (\frac{NM_1'}{150})^2}(1 - \theta)),$$

$$g_5 = \eta^2 \lambda_1 \frac{P_1' + P_2'}{\frac{N(M_1' + M_2')}{150} + P_1' + P_2'}, g_6 = \eta^2 \lambda_2 \frac{M_1'}{M_1' + K_M}\frac{I_3'}{I_3' + c_3},\ g_7 = \eta^2 k_7 Q',$$

$$g_8 = \eta^2 k_8 M_2',\ g_9 = \eta^2 k_9 M_1',\ g_{10} = \eta^2 \lambda_3 Q',\ g_{11} = \eta^2 \lambda_4 (M_1' + M_2')(1 + k_3 \frac{I_4'}{I_4' + c_4}),$$

and

$$M_1' + M_2' + D' + Q' = 1,$$

$$\begin{aligned} h(M_1', M_2', P_1', P_2', D', Q', I_1', I_2', I_3') = &-(k_{11}M_1' \frac{P_1'^2}{P_1'^2 + (\frac{NM_1'}{150})^2} + k_{12}M_2' \frac{P_2'^2}{P_2'^2 + (\frac{NM_2'}{150})^2}) + k_6 Q' \frac{I_1'}{I_1' + c_1} \\ &+ \lambda_1 \frac{P_1' + P_2'}{\frac{N(M_1'+M_2')}{150} + P_1' + P_2'} + \lambda_2 \frac{M_1'}{M_1' + K_M} \frac{I_3'}{I_3' + c_3} \\ &- (\mu_1 M_1' + \mu_2 M_2' + \mu_5 D' + \mu_6 Q'), \end{aligned}$$

We summarize the above result in the following lemma:

Lemma 3.1 *Under the variable transformation (27), the free boundary problem (1)–(26) is equivalent to the initial-boundary value problem (28)–(54).*

4. Local Existence

We now prove local solvability of (1)–(26) by using its reformulated version (28)– (54). Denote

$$\begin{aligned} U &= (M_1', M_2', P_1', P_2', D', Q', I_1', I_2', I_3', I_4', E'), \\ \tilde{U} &= (\tilde{M}_1', \tilde{M}_2', \tilde{P}_1', \tilde{P}_2', \tilde{D}', \tilde{Q}', \tilde{I}_1', \tilde{I}_2', \tilde{I}_3', \tilde{I}_4', \tilde{E}'), \\ U_1 &= (M_{11}', M_{21}', P_{11}', P_{21}', D_1', Q_1', I_{11}', I_{21}', I_{31}', I_{41}', E_1'), \\ U_2 &= (M_{12}', M_{22}', P_{12}', P_{22}', D_2', Q_2', I_{12}', I_{22}', I_{32}', I_{42}', E_2'). \end{aligned}$$

For given $T > 0$ and positive constant W, we introduce a metric space (X_T, d) as follows: X_T consists of vectorvalued functions $(U, \eta) = (U, \eta(\tau))(0 \le z \le 1, 0 \le \tau \le T)$ satisfying:

(i) $\eta(\tau) \in C[0, T], \eta(0) = R_0, \frac{1}{2}R_0 \le \eta(\tau) \le 2R_0$,

(ii) $M_1', M_2', P_1', P_2', D', Q', I_1', I_2', I_3', I_4', E' \in C(\bar{\Omega}_T), 0 \le M_1', M_2', P_1', P_2', D', Q', I_1', I_2', I_3', I_4', E' \le W$,

(iii) $| h(M_1', M_2', P_1', P_2', D', Q', I_1', I_2', I_3') | \le A$, i.e $| h' | \le A$.

The metric d in X_T is defined as

$$d((U_1, \eta_1), (U_2, \eta_2)) = \|U_1 - U_2\|_\infty + \|\eta_1 - \eta_2\|_\infty.$$

It is obvious that X_T is a complete metric space.

Integrating (53) yields

$$u'(z, \tau) = \frac{\eta^2(\tau)}{z^2} \int_0^z h' s^2 ds. \tag{55}$$

Substituting (55) into (50), we obtain

$$\frac{d\eta(\tau)}{d\tau} = \eta^3(\tau) \int_0^1 h' s^2 ds. \tag{56}$$

We define a mapping $F : X_T \to X_T$ (for small $T > 0$) in the following way: given any $(U, \eta(\tau)) \in X_T$ and consider the following problem on $(\tilde{U}, \eta(\tilde{\tau}))$:

$$\frac{d\tilde{\eta}(\tau)}{d\tau} = \tilde{\eta}(\tau) u'(1, \tau), \ \tau > 0, \tag{57}$$

$$\tilde{\eta}(0) = R_0, \tag{58}$$

$$\frac{\partial \tilde{M}_1'}{\partial \tau} = \delta_1 \frac{1}{z^2}\frac{\partial}{\partial z}(z^2 \frac{\partial \tilde{M}_1'}{\partial z}) - v(z,\tau)\frac{\partial \tilde{M}_1'}{\partial z} - \eta^2(\tau)(k_1 \frac{I_2'}{I_2' + c_2} + k_{11}\frac{P_1'^2}{P_1'^2 + (\frac{NM_1'}{150})^2} + \mu_1 + h')\tilde{M}_1' + g_1, \quad 0 \le z \le 1,\ \tau > 0, \tag{59}$$

$$\frac{\partial \tilde{M}_1'}{\partial z}(0,\tau) = 0,\ \frac{\partial \tilde{M}_1'}{\partial z} + \beta k_{10}\frac{I_4'}{I_4' + c_4}(\tilde{M}_1' - M_0) = 0 \ \ on \ \ z = 1,\ \tilde{M_1}'(z,0) = M_{10}(z), \tag{60}$$

$$\frac{\partial \tilde{M}_2'}{\partial \tau} = \delta_2 \frac{1}{z^2}\frac{\partial}{\partial z}(z^2 \frac{\partial \tilde{M}_2'}{\partial z}) - v(z,\tau)\frac{\partial \tilde{M}_2'}{\partial z} - \eta^2(\tau)(k_1 \frac{I_4'}{I_4' + c_2} + k_{11}\frac{P_2'^2}{P_2'^2 + (\frac{NM_2'}{150})^2} + \mu_2 + h')\tilde{M}_2' + g_2, \quad 0 \le z \le 1,\ \tau > 0, \tag{61}$$

$$\frac{\partial \tilde{M}_2'}{\partial z}(0,\tau) = 0,\ \frac{\partial \tilde{M}_2'}{\partial z}(1,\tau) = 0,\ \tilde{M}_2'(z,0) = M_{20}(z), \tag{62}$$

$$\begin{aligned} \frac{\partial \tilde{P}_1'}{\partial \tau} =& \delta_3 \frac{1}{z^2}\frac{\partial}{\partial z}(z^2 \frac{\partial \tilde{P}_1'}{\partial z}) - v(z,\tau)\frac{\partial \tilde{P}_1'}{\partial z} + \eta^2(\tau)(\max\{\alpha_1 \tilde{P}_1'(1 - \frac{P_1'}{\frac{N_* M_1'}{150}}), 0\} \\ &+ k_{11}\frac{NM_1'}{150}\frac{P_1'\tilde{P}_1'}{P_1'^2 + (\frac{NM_1'}{150})^2}\theta - k_{11}\frac{NM_1'}{150}\frac{P_1'\tilde{P}_1'}{P_1'^2 + (\frac{NM_1'}{150})^2} - k_1 \tilde{P}_1' \frac{I_2'}{I_2' + c_2} \\ &- \mu_3 \tilde{P}_1'(1 + k_5 \frac{E'}{E' + c_5} + k_4 \frac{I_4'}{I_4' + c_4}) - \mu_1 N_1 \tilde{P}_1' - h'\tilde{P}_1') + g_3, \quad 0 \le z \le 1,\ \tau > 0, \end{aligned} \tag{63}$$

$$\frac{\partial P_1'}{\partial z}(0,\tau) = 0,\ \frac{\partial P_1'}{\partial z}(1,\tau) = 0, \ ,\ P_1'(z,0) = P_{10}(z), \tag{64}$$

$$\begin{aligned} \frac{\partial \tilde{P}_2'}{\partial \tau} =& \delta_4 \frac{1}{z^2}\frac{\partial}{\partial z}(z^2 \frac{\partial \tilde{P}_2'}{\partial z}) - v(z,\tau)\frac{\partial \tilde{P}_2'}{\partial z} + \eta^2(\tau)(\max\{\alpha_2 \tilde{P}_2'(1 - \frac{P_2'}{\frac{N_* M_2'}{150}}), 0\} \\ &+ k_{12}\frac{NM_2'}{150}\frac{P_2'\tilde{P}_2'}{P_2'^2 + (\frac{NM_1'}{150})^2}(1-\theta) - k_{11}\frac{NM_2'}{150}\frac{P_2'\tilde{P}_2'}{P_2'^2 + (\frac{NM_2'}{150})^2} - k_2 \tilde{P}_2' \frac{I_4'}{I_4' + c_4} \\ &- \mu_4 \tilde{P}_2'(1 + k_5 \frac{E'}{E' + c_5} + k_4 \frac{I_4'}{I_4' + c_4}) - \mu_2 N_2 \tilde{P}_2' - h'\tilde{P}_2') + g_4, \quad 0 \le z \le 1,\ \tau > 0, \end{aligned} \tag{65}$$

$$\frac{\partial \tilde{P}_2'}{\partial z}(0,\tau) = 0,\ \frac{\partial \tilde{P}_2'}{\partial z}(1,\tau) = 0, \ ,\ \tilde{P}_2'(z,0) = P_{20}(z), \tag{66}$$

$$\frac{\partial \tilde{D}'}{\partial \tau} = \delta_5 \frac{1}{z^2}\frac{\partial}{\partial z}(z^2 \frac{\partial \tilde{D}'}{\partial z}) - v(z,\tau)\frac{\partial \tilde{D}'}{\partial z} - \eta^2(\tau)(\mu_5 + h')\tilde{D}' + g_5, \quad 0 \le z \le 1,\ \tau > 0, \tag{67}$$

$$\frac{\partial \tilde{D}'}{\partial z}(0,\tau) = 0,\ \frac{\partial \tilde{D}'}{\partial z} + \beta_0 \tilde{D}' = 0 \ \ on \ \ z = 1,\ \tilde{D}'(z,0) = D_0(z), \tag{68}$$

$$\frac{\partial \tilde{Q}'}{\partial \tau} = \delta_6 \frac{1}{z^2}\frac{\partial}{\partial z}(z^2 \frac{\partial \tilde{Q}'}{\partial z}) - v(z,\tau)\frac{\partial \tilde{Q}'}{\partial z} - \eta^2(\tau)(k_6 \frac{I_1'}{I_1' + c_1}) - \mu_6 - h')\tilde{Q}' + g_6, \quad 0 \le z \le 1,\ \tau > 0, \tag{69}$$

$$\frac{\partial \tilde{Q}'}{\partial z}(0,\tau) = 0,\ \frac{\partial \tilde{Q}'}{\partial z} + \beta_0 \lambda_0 \frac{D'}{k_D + D'}\frac{I_3'}{I_3' + c_3}(\tilde{Q}' - Q_0) = 0 \ on \ z = 1,\ \tilde{Q}'(z,0) = Q_0(z), \tag{70}$$

$$\frac{\partial \tilde{I}_1'}{\partial \tau} = \delta_7 \frac{1}{z^2}\frac{\partial}{\partial z}(z^2 \frac{\partial \tilde{I}_1'}{\partial z}) + u'(1,\tau) z \frac{\partial \tilde{I}_1'}{\partial z} - \eta^2(\tau)\mu_7 \tilde{I}_1' + g_7, \quad 0 \le z \le 1,\ \tau > 0, \tag{71}$$

$$\frac{\partial \tilde{I}'_1}{\partial z}(0,\tau)=0,\ \frac{\partial \tilde{I}'_1}{\partial z}(1,\tau)=0,\ ,\ \tilde{I}'_1(z,0)=I_{10}(z), \tag{72}$$

$$\frac{\partial \tilde{I}'_2}{\partial \tau}=\delta_8\frac{1}{z^2}\frac{\partial}{\partial z}(z^2\frac{\partial \tilde{I}'_2}{\partial z})+u'(1,\tau)z\frac{\partial \tilde{I}'_2}{\partial z}-\eta^2(\tau)\mu_8\tilde{I}'_2+g_8,\quad 0\le z\le 1,\ \tau>0, \tag{73}$$

$$\frac{\partial \tilde{I}'_2}{\partial z}(0,\tau)=0,\ \frac{\partial \tilde{I}'_2}{\partial z}(1,\tau)=0,\ ,\ \tilde{I}'_2(z,0)=I_{20}(z), \tag{74}$$

$$\frac{\partial \tilde{I}'_3}{\partial \tau}=\delta_9\frac{1}{z^2}\frac{\partial}{\partial z}(z^2\frac{\partial \tilde{I}'_3}{\partial z})+u'(1,\tau)z\frac{\partial \tilde{I}'_3}{\partial z}-\eta^2(\tau)\mu_9\tilde{I}'_3+g_9,\quad 0\le z\le 1,\ \tau>0, \tag{75}$$

$$\frac{\partial \tilde{I}'_3}{\partial z}(0,\tau)=0,\ \frac{\partial \tilde{I}'_3}{\partial z}(1,\tau)=0,\ ,\ \tilde{I}'_3(z,0)=I_{30}(z), \tag{76}$$

$$\frac{\partial \tilde{I}'_4}{\partial \tau}=\delta_{10}\frac{1}{z^2}\frac{\partial}{\partial z}(z^2\frac{\partial \tilde{I}'_4}{\partial z})+u'(1,\tau)z\frac{\partial \tilde{I}'_4}{\partial z}-\eta^2(\tau)\mu_{10}\tilde{I}'_4+g_{10},\quad 0\le z\le 1,\ \tau>0, \tag{77}$$

$$\frac{\partial \tilde{I}'_4}{\partial z}(0,\tau)=0,\ \frac{\partial \tilde{I}'_4}{\partial z}(1,\tau)=0,\ \tilde{I}'_4(z,0)=I_{40}(z), \tag{78}$$

$$\frac{\partial \tilde{E}'}{\partial \tau}=\delta_{11}\frac{1}{z^2}\frac{\partial}{\partial z}(z^2\frac{\partial \tilde{E}'}{\partial z})+u'(1,\tau)z\frac{\partial \tilde{E}'}{\partial z}-\eta^2(\tau)\mu_{11}\tilde{E}'+g_{11},\quad 0\le z\le 1,\ \tau>0, \tag{79}$$

$$\frac{\partial \tilde{E}'}{\partial z}(0,\tau)=0,\ \frac{\partial \tilde{E}'}{\partial z}(1,\tau)=0,\ \tilde{E}'(z,0)=E_0(z). \tag{80}$$

With this problem solved, we define $F:(U,\eta)\to(\tilde{U},\tilde{\eta})$.

(b1) Clearly, for any $T>0$, there exists a unique solution $\tilde{\eta}(\tau)\in C^1[0,T]$ of (57)-(58) thanks to Picard Theorem of Existence and Uniqueness:

$$\tilde{\eta}(\tau)=R_0\mathrm{e}^{\int_0^\tau u'(1,s)ds}. \tag{81}$$

According to conditions (i) and (iii), we have $|\ u'(1,\tau)\ |\le \frac{4}{3}AR_0^2$, which implies that

$$R_0\mathrm{e}^{-\frac{4}{3}AR_0^2}\le\tilde{\eta}(\tau)\le R_0\mathrm{e}^{\frac{4}{3}AR_0^2}\quad\text{for } 0<\tau\le T.$$

It follows that if $T>0$ is sufficiently small such that $\mathrm{e}^{\frac{4}{3}AR_0^2}\le 2$, then $\frac{1}{2}R_0\le\tilde{\eta}(\tau)\le 2R_0$, so that $\tilde{\eta}(\tau)$ satisfies Condition (i).

(b2) By Lemma 3.1 we see that (59)–(60) has a unique solution $\tilde{M}'_1\in W_p^{2,1}(\Omega_T)$ satisfying

$$\|\tilde{M}'_1\|_{W_p^{2,1}(\Omega_T)}\le C(T)(\|M_{10}(z)\|_{D_P(0,L)}+\|\beta k_{10}\frac{I'_4}{I'_4+c_4}M_0\|_{W^{1,p}(0,T)}+\|\eta^2(\tau)k_2M'_2\frac{I'_4}{I'_4+c_4}\|_{L^p})\le C(T)K,$$

where K is a positive constant depends on $M_{10}(z)$ and W.

(b3) Similarly, by Lemma 3.1, (61)–(80) have unique solutions $\tilde{M}'_2,\tilde{P}'_1,\tilde{P}'_2,\tilde{D}',\tilde{Q}',\tilde{I}'_1,\tilde{I}'_2,\tilde{I}'_3,\tilde{I}'_4,\tilde{E}'$ $\in W_p^{2,1}(\Omega_T)$ satisfying

$$\|(\tilde{M}'_2,\tilde{P}'_1,\tilde{P}'_2,\tilde{D}',\tilde{Q}',\tilde{I}'_1,\tilde{I}'_2,\tilde{I}'_3,\tilde{I}'_4,\tilde{E}')\|_{W_p^{2,1}(\Omega_T)}\le C(T)K.$$

Combining (b2) and (b3), we conclude that if $K>0$ and $T>0$ sufficiently small, then $C(T)$ is bounded such that $C(T)K\le W$ and $\|\ \tilde{M}'_2,\tilde{P}'_1,\tilde{P}'_2,\tilde{D}',\tilde{Q}',\tilde{I}'_1,\tilde{I}'_2,\tilde{I}'_3,\tilde{I}'_4,\tilde{E}'\ \|_{W_p^{2,1}(\Omega_T)}\le W$. By the embedding $W_p^{2,1}(\Omega_T)\subset C^{\lambda,\frac{\lambda}{2}}(\bar{\Omega}_T)$ when $p>\frac{5}{2}$ and $\lambda=2-\frac{5}{p}$, we have $\|\tilde{M}'_2,\tilde{P}'_1,\tilde{P}'_2,\tilde{D}',\tilde{Q}',\tilde{I}'_1,\tilde{I}'_2,\tilde{I}'_3,\tilde{I}'_4,$ $\tilde{E}'\|_\infty\le W$, so $\tilde{M}'_2,\tilde{P}'_1,\tilde{P}'_2,\tilde{D}',\tilde{Q}',\tilde{I}'_1,\tilde{I}'_2,\tilde{I}'_3,\tilde{I}'_4,\tilde{E}'$ satisfies condition (ii) and

$$\|(\frac{\partial\tilde{M}'_1}{\partial z},\frac{\partial\tilde{M}'_2}{\partial z},\frac{\partial\tilde{P}'_1}{\partial z},\frac{\partial\tilde{P}'_2}{\partial z},\frac{\partial\tilde{D}'}{\partial z},\frac{\partial\tilde{Q}'}{\partial z},\frac{\partial\tilde{I}'_1}{\partial z},\frac{\partial\tilde{I}'_2}{\partial z},\frac{\partial\tilde{I}'_3}{\partial z},\frac{\partial\tilde{I}'_4}{\partial z},\frac{\partial\tilde{E}'}{\partial z})\|_\infty\le C(T).$$

From the above assertions we see that the mapping F is well-defined, and it maps X_T into itself for small T. In the sequel we prove that F is a contraction mapping if T is further small.

Let $U_i, \eta_i \in X_T (i = 1, 2)$, the denote

$$(\tilde{U}_i, \tilde{\eta}_i) = F(U_i, \eta_i),$$
$$d((\tilde{U}_1, \tilde{\eta}_1), (\tilde{U}_2, \tilde{\eta}_2)) = \|\tilde{U}_1 - \tilde{U}_2\|_\infty + \|\tilde{\eta}_1 - \tilde{\eta}_2\|_\infty.$$

(B1) By direct calculations we deduce

$$| u_1'(z,\tau) - u_2'(z,\tau) | \le C(T)d,$$

and consequently from (81) we have

$$\|\tilde{\eta}_1 - \tilde{\eta}_2\|_\infty \le TC(T)d.$$

(B2) Denoting $\tilde{M}_1^* = \tilde{M}_{11}' - \tilde{M}_{12}'$, we have

$$\frac{\partial \tilde{M}_1^*}{\partial \tau} = \delta_1 \frac{1}{z^2}\frac{\partial}{\partial z}(z^2 \frac{\partial \tilde{M}_1^*}{\partial z}) - v(z,\tau)\frac{\partial \tilde{M}_1^*}{\partial z} - \eta_1^2(\tau)(k_1 \frac{I_{21}'}{I_{21}' + c_2} + k_{11}\frac{P_{11}'^2}{P_{11}'^2 + (\frac{NM_{11}'}{150})^2} + \mu_1 + h_1')\tilde{M}_1^* + f_{M_1},$$

$$\frac{\partial \tilde{M}_1^*}{\partial z}|_{z=0} = 0,\ \frac{\partial \tilde{M}_1^*}{\partial z} + \eta_1 \beta k_{10}\frac{I_{41}'}{I_{41}' + c_4}\tilde{M}_1^* = \beta\tilde{k_4}(\eta_1 \frac{I_{41}'}{I_{41}' + c_4} - \eta_2 \frac{I_{42}'}{I_{42}' + c_4})(M_0 - \tilde{M}_{12}')\ on\ z = 1,\ \tilde{M}_1^*(z,0) = 0,$$

where

$$\begin{aligned} f_{M_1} =& (v_1 - v_2)\frac{\partial \tilde{M}_{12}'}{\partial z} + k_2(\eta_1 M_{21}' \frac{I_{41}'}{I_{41}' + c_4} - \eta_2 M_{22}' \frac{I_{42}'}{I_{42}' + c_4}) - k_{11}\tilde{M}_{12}'(\eta_1 \frac{P_{11}'^2}{P_{11}'^2 + (\frac{NM_{11}'}{150})^2} \\ &- \eta_2 \frac{P_{12}'^2}{P_{12}'^2 + (\frac{NM_{12}'}{150})^2}) - \mu_1 \tilde{M}_{12}'(\eta_1^2 - \eta_2^2) - \tilde{M}_{12}'(\eta_1^2 h_1' - \eta_2^2 h_2') - k_1 \tilde{M}_{12}'(\eta_1^2 \frac{I_{21}'}{I_{21}' + c_2} - \eta_2^2 \frac{I_{22}'}{I_{22}' + c_2}). \end{aligned}$$

Thus

$$\|f_{M_1}\|_\infty \le C(T)Wd,$$

$$\begin{aligned} \|\beta\tilde{k_4}(\eta_1 \frac{I_{41}'}{I_{41}' + c_4} - \eta_2 \frac{I_{42}'}{I_{42}' + c_4})(M_0 - \tilde{M}_{12}')\|_\infty &\le \beta\tilde{k_4}\|\eta_1 \frac{I_{41}'}{I_{41}' + c_4} - \eta_2 \frac{I_{42}'}{I_{42}' + c_4}\|_\infty \|M_0 - \tilde{M}_{12}'\|_\infty \\ &\le C(T)Wd. \end{aligned}$$

It follows from the maximum principle [2] that

$$\|\tilde{M}_1^*\|_\infty \le (\|f_{M_1}\|_\infty + \|\beta\tilde{k_4}(\eta_1 \frac{I_{41}'}{I_{41}' + c_4} - \eta_2 \frac{I_{42}'}{I_{42}' + c_4})(M_0 - \tilde{M}_{12}')\|_\infty) \le C(T)Wd.$$

(B3) Similarly, denoting

$$\begin{aligned} &\tilde{M}_2^* = \tilde{M}_{21}' - \tilde{M}_{22}', \quad \tilde{P}_1^* = \tilde{P}_{11}' - \tilde{P}_{12}', \quad \tilde{P}_2^* = \tilde{P}_{21}' - \tilde{P}_{22}', \quad \tilde{D}^* = \tilde{D}_1' - \tilde{D}_2', \quad \tilde{Q}^* = \tilde{Q}_1' - \tilde{Q}_2', \\ &\tilde{I}_1^* = \tilde{I}_{11}' - \tilde{I}_{12}', \quad \tilde{I}_2^* = \tilde{I}_{21}' - \tilde{I}_{22}', \quad \tilde{I}_3^* = \tilde{I}_{31}' - \tilde{I}_{32}', \quad \tilde{I}_4^* = \tilde{I}_{41}' - \tilde{I}_{42}', \quad \tilde{E}^* = \tilde{E}_1' - \tilde{E}_2', \end{aligned}$$

we get

$$\begin{aligned} &\|\tilde{M}_2^*\|_\infty \le TC(T)Wd, \quad \|\tilde{P}_1^*\|_\infty \le TC(T)Wd, \quad \|\tilde{P}_2^*\|_\infty \le TC(T)Wd, \quad \|\tilde{D}^*\|_\infty \le TC(T)Wd, \\ &\|\tilde{Q}^*\|_\infty \le TC(T)Wd, \quad \|\tilde{I}_1^*\|_\infty \le TC(T)Wd, \quad \|\tilde{I}_2^*|_\infty \le TC(T)Wd, \quad \|\tilde{I}_3^*\|_\infty \le TC(T)Wd, \\ &\|\tilde{I}_4^*\|_\infty \le TC(T)Wd, \quad \|\tilde{E}^*\|_\infty \le TC(T)Wd. \end{aligned}$$

From (B2) and (B3), we have

$$\|\tilde{U_1} - \tilde{U_2}\|_\infty \leq C(T)Wd.$$

Combining with (B1), we conclude that

$$d((\tilde{U_1}, \tilde{\eta_1}), (\tilde{U_2}, \tilde{\eta_2})) = \|\tilde{U_1} - \tilde{U_2}\|_\infty + \|\tilde{\eta_1} - \tilde{\eta_2}\|_\infty \leq C(T)Wd.$$

If T is further small such that $C(T)W < 1$, F will be a contraction mapping. In view of the contraction mapping theorem F has a unique fixed point (U, η) in X_T, which is the unique local solution of (57)–(80).

Back to the original problem (1)–(26), we get the following result:

Theorem 4.1 *Under the conditions (A1) and (A2), problem (1)–(26) has a unique solution for* $0 \leq t \leq T$.

5. Global Existence

Note that by (56) we cannot expect that the solution to (28)–(54) exists for all $\tau \geq 0$. However, since the variables t and τ can be rewritten as $t = \int_0^\tau \eta^2(s)ds$ and $\tau = \int_0^t \frac{1}{R^2(S)}ds$, we cannot conclude that the solution to (1)-(26) does not exist for all $t \geq 0$. Hence, in order to get global solutions to (1)–(26), we must study this problem directly. For this purpose we now establish the following preliminary lemmas:

Lemma 5.1 *The solution to (1)–(26) satisfies*

$$M_1, M_2, P_1, P_2, D, Q, I_1, I_2, I_3, I_4, E \geq 0.$$

Proof. Clearly, g_1 is monotone increasing in $M_2', Q', I_1', I_2', I_3', I_4', E'$, g_2 is monotone increasing in $M_1', Q', I_1', I_2', I_3', I_4', E'$, g_6 is monotone increasing in $M_1', M_2', I_1', I_2', I_3', I_4', E'$, g_7 is monotone increasing in $M_1', M_2', Q', I_2', I_3', I_4', E'$, g_8 is monotone increasing in $M_1', M_2', Q', I_1', I_3', I_4', E'$, g_9 is monotone increasing in $M_1', M_2', Q', I_1', I_2', I_4', E'$, g_{10} is monotone increasing in $M_1', M_2', Q', I_1', I_2', I_3', E'$, g_{11} is monotone increasing in $M_1', M_2', Q', I_1', I_2', I_3', I_4'$. Hence, (28), (30), (38), (40), (42), (44), (46) and (48) generate a quasi-monotone increasing system and $(0, 0, 0, 0, 0, 0, 0, 0)$ is a lower solution of (28)–(31) and (38)–(39), so that

$$M_1', M_2', Q', I_1', I_2', I_3', I_4', E' \geq 0$$

It follows that g_3 is monotone increasing in P_2', D', g_4 is monotone increasing in P_1', D', g_3 is monotone increasing in P_1', P_2', then (32),(34) and (36) are a quasi-monotone increasing system and $(0, 0, 0)$ is a lower solution of (32)–(37), therefore

$$P_1', P_2', D' \geq 0.$$

Consequently

$$M_1', M_2', P_1', P_2', D', Q', I_1', I_2', I_3', I_4', E' \geq 0.$$

Thanks to Lemma 2.1, we have

$$M_1, M_2, P_1, P_2, D, Q, I_1, I_2, I_3, I_4, E \geq 0.$$

□

Next, it follows from $M_1 + M_2 + D + Q = 1$ that $0 \leq M_1, M_2, D, Q \leq 1$. Denoting

$$A = \max\{h | 0 \leq M_1, M_2, D, Q \leq 1\},$$

and (14) can be written as

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u) = h,$$

by differentiating it with respect to r, we can deduce

$$u(r,t) = \frac{1}{r^2}\int_0^r s^2 h\, ds.$$

then

$$|u(r,t)| \le \frac{1}{3}Ar,$$

i.e.

$$-\frac{1}{3}Ar \le u(r,t) \le \frac{1}{3}Ar,\ 0 \le r \le R(t),\ 0 \le t \le T,$$

and substituting it into (12) we get

$$-\frac{1}{3}AR(t) \le \dot{R}(t) \le \frac{1}{3}AR(t),\ 0 \le t \le T.$$

Consequently

$$R_0 e^{-\frac{1}{3}At} \le R(t) \le R_0 e^{\frac{1}{3}At},\ 0 \le t \le T,$$

which implies that

$$R_0 e^{-\frac{1}{3}AT} \le R(t) \le R_0 e^{\frac{1}{3}AT},$$

i.e.

$$R_0 e^{-\frac{1}{3}AT} \le \eta(\tau) \le R_0 e^{\frac{1}{3}AT}.$$

Lemma 5.2 *For any $1 < p < \infty$, there exists a positive constant $C(T)$ depending on time T such that*

$$\|(M_1, M_2, P_1, P_2, D, Q, I_1, I_2, I_3, I_4, E)\|_\infty \le C(T).$$

Proof. Since $0 \le M_1', M_2', D', Q' \le 1$ and $R_0 e^{-\frac{1}{3}AT} \le \eta(\tau) \le R_0 e^{\frac{1}{3}AT}$, we have $g_i \in L^p(\Omega_T)(i = 1, 2, 5, 6, \dots, 11)$, so that problems (28)–(31) and (38)–(39) have unique solution

$$M_1', M_2', D', Q', I_1', I_2', I_3', I_4', E' \in W_p^{2,1}\Omega_T,$$

and from Lemma 3.1,

$$\|(M_1', M_2', D', Q', I_1', I_2', I_3', I_4', E')\|_{W_p^{2,1}\Omega_T} \le C(T).$$

Applying $W_p^{2,1}(\Omega_T) \subset C^{\lambda,\frac{\lambda}{2}}(\bar{\Omega}_T)$ when $p > \frac{5}{2}$ and $\lambda = 2 - \frac{5}{p}$, we get

$$\|M_1', M_2', D', Q', I_1', I_2', I_3', I_4', E'\|_\infty \le C(T).$$

By Lemma 2.1, we also have

$$\|(M_1, M_2, D, Q, I_1, I_2, I_3, I_4, E)\|_\infty \le C(T).$$

With additional consideration of (32) and (34), we obtain

$$\begin{aligned}\frac{\partial(P_1' + P_2')}{\partial\tau} \le & \delta\nabla^2(P_1' + P_2') - v(z,\tau)\frac{\partial(P_1' + P_2')}{\partial z} - c(P_1' + P_2') \\ & + C(T)\max\{\alpha_1 P_1'(1 - \frac{P_1'}{\frac{N_* M_1'}{150}}), 0\} + C(T)\max\{\alpha_2 P_2'(1 - \frac{P_2'}{\frac{N_* M_2'}{150}}), 0\},\end{aligned} \tag{82}$$

where $\delta = \delta_3 = \delta_4$ and c is a positive constant.

Multiplying both sides of (82) by $(P_1'+P_2')^k$ and integrating on Ω_T, we get

$$\begin{aligned}\int_0^T\int_0^1 \frac{\partial(P_1'+P_2')}{\partial\tau}(P_1'+P_2')^k dzd\tau \leq &\delta\int_0^T\int_0^1 \nabla^2(P_1'+P_2')(P_1'+P_2')^k dzd\tau\\ &-\int_0^T\int_0^1 v(z,\tau)\frac{\partial(P_1'+P_2')}{\partial z}(P_1'+P_2')^k dzd\tau\\ &-c\int_0^T\int_0^1 (P_1'+P_2')^{k+1}dzd\tau\\ &+C(T)\int_0^T\int_0^1 \max\{\alpha_1 P_1'(1-\frac{P_1'}{\frac{N_*M_1'}{150}}),0\}(P_1'+P_2')^k dzd\tau\\ &+C(T)\int_0^T\int_0^1 \max\{\alpha_2 P_2'(1-\frac{P_2'}{\frac{N_*M_2'}{150}}),0\}(P_1'+P_2')^k dzd\tau\\ :=&J_1+J_2+J_3+J_4+J_5.\end{aligned}$$

We can also know that

$$\begin{aligned}\int_0^T\int_0^1 \frac{\partial(P_1'+P_2')}{\partial\tau}(P_1'+P_2')^k dzd\tau &= \frac{1}{k+1}\int_0^1\frac{d}{d\tau}\int_0^T (P_1'+P_2')^{k+1}dzd\tau\\ &=\frac{1}{k+1}\frac{d}{d\tau}\int_0^T\int_0^1 (P_1'+P_2')^{k+1}dzd\tau\end{aligned}\tag{83}$$

and

$$J_1=\int_0^T\int_0^1 \nabla^2(P_1'+P_2')(P_1'+P_2')^k dzd\tau = -k\int_0^T\int_0^1 |\nabla(P_1'+P_2')|^2(P_1'+P_2')^{k-1}dzd\tau \leq 0 \tag{84}$$

We assume that $\frac{\partial(P_1'+P_2')}{\partial z}\geq 0$ on $\Omega_{T_1}\subset\bar{\Omega}_T$ and $\frac{\partial(P_1'+P_2')}{\partial z}<0$ on $\bar{\Omega}_T\setminus\Omega_{T_1}$.
Set

$$\begin{aligned}J_2=&-\int_{\Omega_{T_1}} v(z,\tau)\frac{\partial(P_1'+P_2')}{\partial z}(P_1'+P_2')^k dzd\tau-\int_{\bar{\Omega}_T\setminus\Omega_{T_1}} v(z,\tau)\frac{\partial(P_1'+P_2')}{\partial z}(P_1'+P_2')^k dzd\tau\\ :=&J_{21}+J_{22}.\end{aligned}$$

By $|v(z,\tau)|\leq C(T)$ and Young's inequality with ε we compute

$$\begin{aligned}J_{21}=&\int_{\Omega_{T_1}}(-v(z,\tau))\frac{\partial(P_1'+P_2')}{\partial z}(P_1'+P_2')^k dzd\tau\\ \leq&\|v(z,\tau)\|_{L^\infty}\int_{\Omega_{T_1}}\frac{\partial(P_1'+P_2')}{\partial z}(P_1'+P_2')^k dzd\tau\\ \leq&C(T)\int_{\Omega_{T_1}}\frac{\partial(P_1'+P_2')}{\partial z}(P_1'+P_2')^k dzd\tau\\ \leq&C(T)\frac{\varepsilon}{2}\int_{\Omega_{T_1}}|\nabla(P_1'+P_2')|^2(P_1'+P_2')^{k-1}dzd\tau+\frac{C(T)}{2\varepsilon}\int_{\Omega_{T_1}}(P_1'+P_2')^{k+1}dzd\tau,\end{aligned}\tag{85}$$

and

$$\begin{aligned} J_{22} &= \int_{\bar{\Omega}_T \setminus \Omega_{T_1}} v(z,\tau)(-\frac{\partial(P_1'+P_2')}{\partial z})(P_1'+P_2')^k dzd\tau \\ &\leq \|v(z,\tau)\|_{L^\infty} \int_{\bar{\Omega}_T \setminus \Omega_{T_1}} (-\frac{\partial(P_1'+P_2')}{\partial z})(P_1'+P_2')^k dzd\tau \\ &\leq C(T)\frac{\varepsilon}{2} \int_{\bar{\Omega}_T \setminus \Omega_{T_1}} |\nabla(P_1'+P_2')|^2 (P_1'+P_2')^{k-1} dzd\tau + \frac{C(T)}{2\varepsilon} \int_{\bar{\Omega}_T \setminus \Omega_{T_1}} (P_1'+P_2')^{k+1} dzd\tau. \end{aligned} \tag{86}$$

Therefore

$$\begin{aligned} J_2 &= J_{21} + J_{22} \\ &\leq C(T)\varepsilon \int_0^T \int_0^1 |\nabla(P_1'+P_2')|^2 (P_1'+P_2')^{k-1} dzd\tau + \frac{C(T)}{\varepsilon} \int_0^T \int_0^1 (P_1'+P_2')^{k+1} dzd\tau. \end{aligned} \tag{87}$$

Combining (84), (86) and let $\varepsilon = \varepsilon_0$ be sufficiently small such that $-\delta k + \varepsilon C(T) < 0$ yields

$$J_1 + J_2 + J_3 \leq \frac{C(T)}{\varepsilon} \int_0^T \int_0^1 (P_1'+P_2')^{k+1} dzd\tau.$$

Let $J_4 = J_{41} + J_{42}$, if $\{\max \alpha_1 P_1'(1 - \frac{P_1'}{\frac{N_* M_1'}{150}}), 0\} = \alpha_1 P_1'(1 - \frac{P_1'}{\frac{N_* M_1'}{150}})$ on $\Omega_{T_2} \subset \bar{\Omega}_T$, then

$$\begin{aligned} J_{41} &= C(T) \int_{\Omega_{T_2}} \max\{\alpha_1 P_1'(1 - \frac{P_1'}{\frac{N_* M_1'}{150}}), 0\}(P_1'+P_2')^k dzd\tau \\ &\leq C(T) \int_{\Omega_{T_2}} \alpha_1 P_1'(P_1'+P_2')^k dzd\tau \\ &\leq C(T) \int_{\Omega_{T_2}} (P_1'+P_2')^{k+1} dzd\tau. \end{aligned}$$

And if $\max\{\alpha_1 P_1'(1 - \frac{P_1'}{\frac{N_* M_1'}{150}}), 0\} = 0$ on $\bar{\Omega}_T \setminus \Omega_{T_2}$, then

$$J_{42} = C(T) \int_{\bar{\Omega}_T \setminus \Omega_{T_2}} \max\{\alpha_1 P_1'(1 - \frac{P_1'}{\frac{N_* M_1'}{150}}), 0\}(P_1'+P_2')^k dzd\tau = 0.$$

Therefore

$$J_4 = J_{41} + J_{42} \leq C(T) \int_0^T \int_0^1 (P_1'+P_2')^{k+1} dzd\tau.$$

Similarly, we can also find that

$$J_5 \leq C(T) \int_0^T \int_0^1 (P_1'+P_2')^{k+1} dzd\tau.$$

Hence

$$\int_0^T \int_0^1 \frac{\partial(P_1'+P_2')}{\partial \tau}(P_1'+P_2')^k dzd\tau \leq C(T) \int_0^T \int_0^1 (P_1'+P_2')^{k+1} dzd\tau. \tag{88}$$

Using similar method, substituting $\rho(z,\tau) = \int_0^T \int_0^1 (P_1'+P_2')^{k+1} dzd\tau$ into (89) and combining (83), we infer that

$$\frac{d\rho(z,\tau)}{d\tau} \leq C(T)\rho(z,\tau),$$

which leads to $\rho(z,\tau) \leq C(T)$ due to the Gronwall inequality.

From the above assertions we see that $\|P_1' + P_2'\|_{L^p(\Omega_T)} \leq C(T)$. According to Lemma 5.1 we obtain

$$\|P_1'\|_{L^p(\Omega_T)} \leq C(T), \quad \|P_2'\|_{L^p(\Omega_T)} \leq C(T).$$

Further using Lemma 3.1 yields

$$\|P_1'\|_{W_p^{2,1}(\Omega_T)} \leq C(T), \quad \|P_2'\|_{W_p^{2,1}(\Omega_T)} \leq C(T).$$

Besides, applying the embedding $W_p^{2,1}(\Omega_T) \subset C^{\lambda,\frac{\lambda}{2}}(\bar{\Omega}_T)$ when $p > \frac{5}{2}$ and $\lambda = 2 - \frac{5}{p}$, we have

$$\|P_1'\|_\infty \leq C(T), \quad \|P_2'\|_\infty \leq C(T).$$

Moreover, using Lemma 2.1, we get

$$\|P_1\|_\infty \leq C(T), \quad \|P_2\|_\infty \leq C(T).$$

Now we conclude that

$$\|(M_1, M_2, P_1, P_2, D, Q, I_1, I_2, I_3, I_4, E)\|_\infty \leq C(T).$$

□

By Theorem 4.1 and Lemma 5.2, we complete the proof of Theorem 1.

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