



Entwined Hom-Modules and Frobenius Properties

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Abstract. Entwined Hom-modules were introduced by Karacuha in [13], which can be viewed as a generalization of Doi-Hom Hopf modules and entwined modules. In this paper, the sufficient and necessary conditions for the forgetful functor $F : \widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C \rightarrow \widetilde{\mathcal{H}}(\mathcal{M}_k)_A$ and its adjoint $G : \widetilde{\mathcal{H}}(\mathcal{M}_k)_A \rightarrow \widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C$ form a Frobenius pair are obtained, one is that $A \otimes C$ and the $C^* \otimes A$ are isomorphic as $(A; C^{*op} \# A)$ -bimodules, where (A, C, ψ) is a Hom-entwining structure. Then we can describe the isomorphism by using a generalized type of integral. As an application, a Maschke type theorem for entwined Hom-modules is given.

1. Introduction

Makhlouf and Silvestrov in [18] introduced Hom-algebras and Hom-coalgebras, which can be viewed as generalizations of ordinary algebras and coalgebras in the following sense: the associativity of the multiplication is replaced by the Hom-associativity, and the Hom-coassociativity can be considered in a similar way. Later, they described the structures of Hom-bialgebras and Hom-Hopf algebras, and extended some important theories from ordinary Hopf algebras to Hom-Hopf algebras in [19] and [20]. Recently, many more properties and structures of Hom-Hopf algebras have been developed, see as [6–11, 14, 22] and the references cited therein.

Caenepeel and Goyvaerts in [3] investigated Hom-bialgebras and Hom-Hopf algebras from the point of view of monoidal categories, in a natural way, they called them monoidal Hom-bialgebras and monoidal Hom-Hopf algebras respectively, which are slightly different from the above Hom-bialgebras and Hom-Hopf algebras. In [16], Makhlouf and Panaite defined Yetter-Drinfeld modules over Hom-bialgebras, and obtained that Yetter-Drinfeld modules over a Hom-bialgebra with bijective structure map provide solutions of the Hom-Yang-Baxter equation. Liu and Shen [15] also studied Yetter-Drinfeld modules over monoidal Hom-bialgebras, they called them Hom-Yetter-Drinfeld modules, and showed that the category of Hom-Yetter-Drinfeld modules is a braided monoidal categories. Chen and Zhang [8] introduced the category of Hom-Yetter-Drinfeld modules, which is differs from that of [15], and indicated that it is a full

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monoidal subcategory of the left center of left Hom-module category. In [10], we defined the category of Doi Hom-Hopf modules and proved that the category of Hom-Yetter-Drinfeld modules is a subcategory of our category of Doi Hom-Hopf modules.

Entwining modules were introduced in [1], which have arisen from noncommutative geometry, are modules of an algebra and comodules of a coalgebra such that the action and the coaction satisfy a certain compatibility condition. The most interesting example is that Doi-Hopf modules are the special cases of entwined modules, but, the formalism for entwined modules is more transparent than the one for Doi-Hopf modules. Many results for Doi-Hopf modules can be generalized to entwined modules.

As a generalization of entwining modules in a Hopf algebra setting, entwined Hom-modules were introduced by Karacuca [13]. In [10] and [12], some properties of Doi Hom-Hopf modules are discussed. It turns out that many pairs of adjoint functors are special cases, for example the functor forgetting action or coaction, extension and restriction of scalars and coscalars. In this paper, as a generalization of [5], we focus our attention on the functor F , which is from the category of entwined Hom-modules to the category of right (A, β) -modules forgetting the (C, γ) -coaction. This functor has a right adjoint $G = C \otimes \bullet$. A natural question that arises is following: when is G also a left adjoint of F ? This is the motivation of this paper.

In this paper, we give the notion of a entwined Hom-module and prove that the functor F from the category of entwined Hom-modules to the category of right (A, β) -Hom-modules has a right adjoint. And then we obtain the main result of this paper in Sec.4, that is, one of the equivalent conditions for the forgetful functor $F : \widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C \rightarrow \widetilde{\mathcal{H}}(\mathcal{M}_k)_A$ and its adjoint $G : \widetilde{\mathcal{H}}(\mathcal{M}_k)_A \rightarrow \widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C$ form a Frobenius pair is : $A \otimes C$ and the $C^* \otimes A$ are isomorphic as $(A; C^{*op} \# A)$ -bimodules. At the end of the paper, we give a Maschke type theorem for entwined Hom-modules.

2. Preliminaries

Throughout this paper, we work over a commutative ring k , we recall from [3] some information about Hom-structures which are needed in what follows.

Let \mathcal{C} be a category. We introduce a new category $\widetilde{\mathcal{H}}(\mathcal{C})$ as follows: objects are couples (M, μ) , with $M \in \mathcal{C}$ and $\mu \in \text{Aut}_{\mathcal{C}}(M)$. A morphism $f : (M, \mu) \rightarrow (N, \nu)$ is a morphism $f : M \rightarrow N$ in \mathcal{C} such that $\nu \circ f = f \circ \mu$.

Let \mathcal{M}_k denote the category of k -modules. $\mathcal{H}(\mathcal{M}_k)$ will be called the Hom-category associated to \mathcal{M}_k . If $(M, \mu) \in \mathcal{M}_k$, then $\mu : M \rightarrow M$ is obviously a morphism in $\mathcal{H}(\mathcal{M}_k)$. It is easy to show that $\widetilde{\mathcal{H}}(\mathcal{M}_k) = (\mathcal{H}(\mathcal{M}_k), \otimes, (I, I), \widetilde{a}, \widetilde{l}, \widetilde{r})$ is a monoidal category by Proposition 2.1 of [3]: the tensor product of (M, μ) and (N, ν) in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ is given by the formula $(M, \mu) \otimes (N, \nu) = (M \otimes N, \mu \otimes \nu)$.

Assume that $(M, \mu), (N, \nu), (P, \pi) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$. The associativity and unit constraints are given by the following formulas

$$\begin{aligned} \widetilde{a}_{M,N,P}((m \otimes n) \otimes p) &= \mu(m) \otimes (n \otimes \pi^{-1}(p)), \\ \widetilde{l}_M(x \otimes m) &= \widetilde{r}_M(m \otimes x) = x\mu(m). \end{aligned}$$

Let's now recall the definition of the monoidal Hom-algebra, monoidal Hom-coalgebra, monoidal Hom-bialgebra and monoidal Hom-Hopf algebra.

Definition 2.1. A monoidal Hom-algebra is an object $(A, \alpha) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with a k -linear map $m_A : A \otimes A \rightarrow A$ and an element $1_A \in A$ such that

$$\alpha(ab) = \alpha(a)\alpha(b), \quad \alpha(1_A) = 1_A, \quad \alpha(a)(bc) = (ab)\alpha(c), \quad a1_A = 1_Aa = \alpha(a),$$

for all $a, b, c \in A$. Here we use the notation $m_A(a \otimes b) = ab$.

Definition 2.2. A monoidal Hom-coalgebra is an object $(C, \gamma) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with k -linear maps $\Delta : C \rightarrow C \otimes C$, $\Delta(c) = c_{(1)} \otimes c_{(2)}$ (summation implicitly understood) and $\gamma : C \rightarrow C$ such that

$$\Delta(\gamma(c)) = \gamma(c_{(1)}) \otimes \gamma(c_{(2)}), \quad \varepsilon(\gamma(c)) = \varepsilon(c),$$

and

$$\gamma^{-1}(c_{(1)}) \otimes c_{(2)(1)} \otimes c_{(2)(2)} = c_{(1)(1)} \otimes c_{(1)(2)} \otimes \gamma^{-1}(c_{(2)}), \quad \varepsilon(c_{(1)})c_{(2)} = \varepsilon(c_{(2)})c_{(1)} = \gamma^{-1}(c)$$

for all $c \in C$.

Definition 2.3. A monoidal Hom-bialgebra $H = (H, \alpha, m, \eta, \Delta, \varepsilon)$ is a bialgebra in the monoidal category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$. This means that (H, α, m, η) is a monoidal Hom-algebra, $(H, \alpha, \Delta, \varepsilon)$ is a monoidal Hom-coalgebra such that Δ and ε are morphisms of algebras, that is, for any $a, b \in H$,

$$\Delta(ab) = a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}, \quad \Delta(1_H) = 1_H \otimes 1_H, \quad \varepsilon(ab) = \varepsilon(a)\varepsilon(b), \quad \varepsilon(1_H) = 1_H.$$

Definition 2.4. A monoidal Hom-bialgebra (H, α) is called monoidal Hom-Hopf algebra if there exists a morphism (called antipode) $S : H \rightarrow H$ in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$, (i.e., $S\alpha = \alpha S$), such that

$$S * I = I * S = \eta\varepsilon.$$

Note that the antipode of monoidal Hom-Hopf algebras has almost all the properties of antipode of Hopf algebras such as $\varepsilon S = \varepsilon$.

Definition 2.5. Let (A, α) be a monoidal Hom-algebra. A right (A, α) -Hom-module is an object $(M, \mu) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ consists of a k -module and a linear map $\mu : M \rightarrow M$ together with a morphism $\psi : M \otimes A \rightarrow M$, $\psi(m \otimes a) = m \cdot a$, in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ such that

$$(m \cdot a) \cdot \alpha(b) = \mu(m) \cdot (ab), \quad m \cdot 1_A = \mu(m),$$

for all $a, b \in A$ and $m \in M$. The fact that $\psi \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ means that

$$\mu(m \cdot a) = \mu(m) \cdot \alpha(a).$$

A morphism $f : (M, \mu) \rightarrow (N, \nu)$ in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ is called right A -linear if it preserves the A -action, that is, $f(m \cdot a) = f(m) \cdot a$. $\widetilde{\mathcal{H}}(\mathcal{M}_k)_A$ will denote the category of right (A, α) -Hom-modules and A -linear morphisms.

Definition 2.6. Let (C, γ) be a monoidal Hom-coalgebra. A right (C, γ) -Hom-comodule is an object $(M, \mu) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with a k -linear map $\rho_M : M \rightarrow M \otimes C$ notation $\rho_M(m) = m_{[0]} \otimes m_{[1]}$ in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ such that

$$m_{[0][0]} \otimes (m_{[0][1]} \otimes \gamma^{-1}(m_{[1]})) = \mu^{-1}(m_{[0]}) \otimes \Delta_C(m_{[1]}); \quad m_{[0]}\varepsilon(m_{[1]}) = \mu^{-1}(m),$$

for all $m \in M$. The fact that $\rho_M \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ means that

$$\rho_M(\mu(m)) = \mu(m_{[0]}) \otimes \gamma(m_{[1]}).$$

The category of right (C, γ) -Hom-comodules will be denoted by $\widetilde{\mathcal{H}}(\mathcal{M}_k)^C$.

Theorem 2.7. (Rafael Theorem) Let $L : C \rightarrow \mathcal{D}$ be the left adjoint functor of $R : \mathcal{D} \rightarrow C$. Then L is a separable functor if and only if the unit η of the adjunction (L, R) has a natural retraction, i.e., there is a natural transformation $\nu : RL \rightarrow id_C$ such that $\nu \circ \eta = id$.

Definition 2.8. A pair of adjoint functors (F, G) is called a Frobenius pair if G is at the same time a right and left adjoint of F .

The following result can be found in any book on category theory: G is a left adjoint of F if and only if there exist natural transformations $v \in V = \underline{\underline{Nat}}(GF, 1_C)$ and $\zeta \in W = \underline{\underline{Nat}}(1_{\mathcal{D}}, FG)$ such that

$$F(v_M) \circ \zeta_{F(M)} = I_{F(M)}, \tag{2.1}$$

$$v_{G(N)} \circ G(\zeta_N) = I_{G(N)}. \tag{2.2}$$

3. Adjoint functor

Now we introduce the notion of the right-right Hom-entwining structure, following Karacuha [13].

Definition 3.1. A right-right Hom-entwining structure is a triple $[(A, \beta), (C, \gamma)]_\psi$ (we write it (A, C, ψ) for short), where (A, β) is a monoidal Hom-algebra and (C, γ) is a monoidal Hom-coalgebra with a k -linear map $\psi : C \otimes A \rightarrow A \otimes C$ satisfying the following conditions for all $a, b \in A, c \in C$:

$$(ab)_\psi \otimes \gamma(c)^\psi = a_\psi b_\psi \otimes \gamma(c^{\psi\varphi}), \tag{3.1}$$

$$1_\psi \otimes c^\psi = 1_A \otimes c, \tag{3.2}$$

$$\beta^{-1}(a_\psi) \otimes c_{(1)}^\psi \otimes c_{(2)}^\psi = \beta^{-1}(a)_{\psi\varphi} \otimes c_{(1)}^\varphi \otimes c_{(2)}^\psi, \tag{3.3}$$

$$a_\psi \varepsilon(c^\psi) = a\varepsilon(c), \tag{3.4}$$

where $\psi(c \otimes a) = a_\psi \otimes c^\psi, a \in A, c \in C$. It is said that (C, γ) and (A, β) are entwined by ψ . $\psi \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ has the relation $\beta(a)_\psi \otimes \gamma(c)^\psi = \beta(a_\psi) \otimes \gamma(c^\psi)$.

Over a Hom-entwining structure (A, C, ψ) , a right-right entwined Hom-module $(M, \mu) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ is both a right (C, γ) -Hom-comodule with coaction $\rho_M : M \rightarrow M \otimes C, m \mapsto m_{[0]} \otimes m_{[1]}$, and a right (A, β) -Hom-module with action $\mu : M \otimes A \rightarrow M, m \otimes a \mapsto m \cdot a$ satisfying

$$\begin{aligned} \rho_M(m \cdot a) &= \mu(m_{[0]}) \cdot \psi(m_{[1]} \otimes \beta^{-1}(a)) \\ &= m_{[0]} \cdot \beta^{-1}(a)_\psi \otimes \gamma(m_{[1]}^\psi), \end{aligned} \tag{3.5}$$

for all $a \in A$ and $m \in M$. Let $\widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C$ denote the category of (A, C, ψ) -entwined Hom-modules together with the morphisms.

Example 3.2. Let (H, α) be a monoidal Hom-Hopf algebra. Define $\psi : H \otimes H \rightarrow H \otimes H$ with $\psi(l \otimes h) = \alpha(h_{(1)}) \otimes \alpha^{-1}(l)h_{(2)}$. It is easy to verify that (H, H, ψ) form a Hom-entwining structure. Then the objects of $\widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_H^H$ are right Hopf (H, α) -Hom-comodule. In fact, by Eq.(3.5), for all $m \in M$ and $h \in H$, we have

$$\begin{aligned} \rho_M(m \cdot h) &= \mu(m_{[0]}) \cdot \psi(m_{[1]} \otimes \alpha^{-1}(h)) \\ &= \mu(m_{[0]}) \cdot (\alpha^{-1}(h)_\psi \otimes m_{[1]}^\psi) \\ &= m_{[0]} \cdot h_{(1)} \otimes m_{[1]}h_{(2)}. \end{aligned}$$

Example 3.3. Let (H, α) be a monoidal Hom-Hopf algebra, (A, β) a right (H, α) -Hom comodule algebra and (C, γ) a right (H, α) -Hom module coalgebra. Then (C, A, ψ) has a Hom-entwining structure with $\psi : C \otimes A \rightarrow A \otimes C$ by $\psi(c \otimes a) = \beta(a_{[0]}) \otimes \gamma^{-1}(c) \cdot a_{[1]}$ for any $a \in A$ and $c \in C$, and hence $\widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C$ is a entwined Hom-modules.

In particular, for any $(M, \mu) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C$, we have

$$\rho_M(m \cdot a) = m_{[0]} \cdot a_{[0]} \otimes m_{[1]}a_{[1]}.$$

In this case, we say that this category is a category of right Doi Hom-Hopf modules and denote it by $\widetilde{\mathcal{H}}(\mathcal{M}_k)(H)_A^C$.

Example 3.4. Let (H, α) be a monoidal Hom-Hopf algebra. Define $\phi : H \otimes H \rightarrow H \otimes H$ given by $\phi(g \otimes h) = \alpha^2(h_{(2)(1)}) \otimes S(h_{(1)})(\alpha^{-2}(g)h_{(2)(2)})$ for all $h, g \in H$, and hence $\widetilde{\mathcal{H}}(\mathcal{M}_k)(\phi)_H^H$ is a entwined Hom-modules. In fact, for any $(M, \mu) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)(\phi)_H^H$, the compatible condition gives

$$\rho_M(m \cdot h) = \mu(m_{[0]}) \cdot \psi(m_{[1]} \otimes \alpha^{-1}(h)) = m_{[0]} \cdot \alpha(h_{(2)(1)}) \otimes S(h_{(1)})(\alpha^{-1}(m_{[1]})h_{(2)(2)}).$$

Note that the category $\widetilde{\mathcal{H}}(\mathcal{M}_k)(\phi)_H^H$ is a category of Hom-right-right Yetter-Drinfeld modules see [10] for more details.

Proposition 3.5. The forgetful functor $F : \widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C \rightarrow \widetilde{\mathcal{H}}(\mathcal{M}_k)_A$ has a right adjoint $G : \widetilde{\mathcal{H}}(\mathcal{M}_k)_A \rightarrow \widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C$. G is defined by

$$G(M) = M \otimes C,$$

with structure maps

$$\begin{aligned} (m \otimes c) \cdot a &= m \cdot \beta^{-1}(a)_\psi \otimes \gamma(c^\psi), \\ \rho_{G(M)}(m \otimes c) &= (\mu^{-1}(m) \otimes c_{(1)}) \otimes \gamma(c_{(2)}), \end{aligned}$$

for all $a \in A$ and $m \in M, c \in C$.

Proof. First show that $G(M)$ is an object of $\widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C$. It is routine to check that $G(M)$ is a right (C, γ) -Hom-comodule and a right (A, β) -Hom-module. Now we only prove the compatibility condition. For all $a \in A, m \in M$ and $c \in C$,

$$\begin{aligned} \rho_{G(M)}((m \otimes c) \cdot a) &= \rho_{G(M)}(m \cdot \beta^{-1}(a)_\psi \otimes \gamma(c^\psi)) \\ &= (\mu^{-1}(m) \cdot \beta^{-1}(\beta^{-1}(a)_\psi) \otimes \gamma(c^\psi)_{(1)}) \otimes \gamma(\gamma(c^\psi)_{(2)}) \\ &= m \cdot \beta^{-1}(a)_\psi \otimes (\gamma(c^\psi)_{(1)} \otimes \gamma(c^\psi)_{(2)}) \\ &\stackrel{(3.3)}{=} m \cdot \beta(\beta^{-2}(a)_{\psi\psi}) \otimes (\gamma(c_{(1)}^\psi) \otimes \gamma(c_{(2)}^\psi)) \\ &= (\mu^{-1}(m) \cdot \beta^{-2}(a)_{\psi\psi} \otimes \gamma(c_{(1)}^\psi) \otimes \gamma(\gamma(c_{(2)}^\psi))) \\ &= (\mu^{-1}(m) \otimes c_{(1)}) \cdot \beta^{-1}(a)_\psi \otimes \gamma(\gamma(c_{(2)}^\psi)) \\ &= \rho_{G(M)}(m \otimes c) \cdot a. \end{aligned}$$

This is exactly what we want to show.

For an A -linear map $\varphi : (M, \mu) \rightarrow (N, \nu)$, let

$$G(\varphi) = \varphi \otimes id_C : M \otimes C \rightarrow N \otimes C.$$

Standard computations show that $G(\varphi)$ is morphisms of right (A, β) -Hom-modules and right (C, γ) -Hom-comodules.

Next we describe the unit η and the counit δ of the adjunction. The unit is described by the coaction: for $(M, \mu) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C$, we define $\eta_M : M \rightarrow M \otimes C$ as follows: for all $m \in M$,

$$\eta_M(m) = m_{[0]} \otimes m_{[1]}.$$

Then $\eta_M \in \widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C$. In fact, for any $m \in M$, we have

$$\begin{aligned} \eta_M(m \cdot a) &= (m \cdot a)_{[0]} \otimes (m \cdot a)_{[1]} \\ &= m_{[0]} \cdot \beta^{-1}(a)_\psi \otimes \gamma(m_{[1]}^\psi) \\ &= (m_{[0]} \otimes m_{[1]}) \cdot a = \eta_M(m) \cdot a, \end{aligned}$$

and

$$\begin{aligned} \rho_{M \otimes C} \circ \eta_M(m) &= \rho_{M \otimes C}(m_{[0]} \otimes m_{[1]}) \\ &= (\mu^{-1}(m_{[0]}) \otimes m_{1}) \otimes \gamma(m_{[1](2)}) \\ &= (m_{[0][0]} \otimes m_{[0][1]}) \otimes m_{[1]} \\ &= (\eta_M \otimes id_C)(m_{[0]} \otimes m_{[1]}) \\ &= (\eta_M \otimes id_C) \circ \rho_M(m). \end{aligned}$$

For any $(N, \nu) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)_A$, let $\delta_N : N \otimes C \rightarrow N$, for all $n \in N$ and $c \in C$,

$$\delta_N(n \otimes c) = \varepsilon(c)\nu(n).$$

We can observe that δ_N is (A, β) -linear. In fact, for any $n \in N$, we have

$$\begin{aligned} \delta_N((n \otimes c) \cdot a) &= \delta_N(n \cdot \beta^{-1}(a)_\psi \otimes \gamma(c^\psi)) \\ &= \varepsilon(\gamma(c^\psi))v(n \cdot \beta^{-1}(a)_\psi) \\ &= \varepsilon(c)v(n) \cdot a = \delta_N(n \otimes c) \cdot a. \end{aligned}$$

This is what we need to show. We can check that η and δ defined above are all natural transformations and satisfied

$$\begin{aligned} G(\delta_N) \circ \eta_{G(N)} &= I_{G(N)}, \\ \delta_{F(M)} \circ F(\eta_M) &= I_{F(M)}, \end{aligned}$$

for all $(M, \mu) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C$ and $(N, \nu) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)_A$. □

4. The functor forgetting the coaction

Let V_1 be the k -module consisting of all k -linear maps $\theta : (C, \gamma) \otimes (C, \gamma) \rightarrow (A, \beta)$ such that

$$\beta(\beta^{-1}(a)_{\varphi_\psi})\theta(d^\psi \otimes c^\varphi) = \theta(d \otimes c)a, \tag{4.1}$$

$$\theta(\gamma^{-1}(d) \otimes c_{(1)}) \otimes \gamma(c_{(2)}) = \theta(d_{(2)} \otimes \gamma^{-1}(c))_\psi \otimes d_{(1)}^\psi. \tag{4.2}$$

Proposition 4.1. *The map $\Psi : V \rightarrow V_1$ given by $\Psi(v) = \theta$ with*

$$\theta(c \otimes d) = \widetilde{r}_A(id_A \otimes \varepsilon_C)v_{A \otimes C}((1_A \otimes c) \otimes \gamma(d)), \tag{4.3}$$

is an isomorphism of k -modules. The inverse $\Psi^{-1}(v) = \theta$ is defined as follows $v_M : M \otimes C \rightarrow M$, which is given by

$$v_M(m \otimes c) = \mu(m_{[0]})\theta(m_{[1]} \otimes \gamma^{-1}(c)). \tag{4.4}$$

Proof. According to the naturality of v , we have

$$\theta(c \otimes d) = (\varepsilon_C \otimes id_A)v_{C \otimes A}((c \otimes 1_A) \otimes \gamma(d)) = (id_A \otimes \varepsilon_C)v_{A \otimes C}((1_A \otimes c) \otimes \gamma(d)).$$

Then it is easily checked that $GF(A \otimes C) = (A \otimes C) \otimes C \in {}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C$, the left (A, β) -action is induced by the multiplication in (A, β) and $v_{A \otimes C}$ is a morphism in ${}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C$. Hence $v_{A \otimes C}$ and $(id_A \otimes \varepsilon_C)v_{A \otimes C}$ are left and right (A, β) -linear, and

$$\begin{aligned} \theta(c \otimes d)a &= (id_A \otimes \varepsilon_C)v_{A \otimes C}((1_A \otimes c) \otimes \gamma(d))a \\ &= (id_A \otimes \varepsilon_C)v_{A \otimes C}((1_A \otimes c) \otimes \gamma(d))a \\ &= (id_A \otimes \varepsilon_C)v_{A \otimes C}(\beta^{-1}(a)1_A \otimes [\gamma(c^\psi) \otimes \gamma(d^\psi)]) \\ &= a_{\psi\psi}(id_A \otimes \varepsilon_C)v_{A \otimes C}([1_A \otimes c^\psi \otimes d^\psi]) \\ &= a_{\psi\psi}(id_A \otimes \varepsilon_C)v_{A \otimes C}([1_A \otimes c^\psi] \otimes \gamma(d^\psi)) \\ &= \beta(\beta^{-1}(a)_{\psi\psi})\theta(c^\psi \otimes d^\psi), \end{aligned}$$

which gives (4.1).

To prove (4.2), first we check at once that $GF(C \otimes A) = C \otimes (C \otimes A) \in {}_C\widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C$, and its left (C, γ) -coaction of $C \otimes A$ is given by

$$c \otimes a \mapsto \gamma(c_{(1)}) \otimes (c_{(2)} \otimes \beta^{-1}(a)).$$

We also get $v_{C \otimes A} : (C \otimes A) \otimes C \rightarrow C \otimes A$ is a morphism in ${}_C\widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C$. Thus we conclude that $v_{C \otimes A}$ is left and right (C, γ) -colinear. Take $c, d \in C$, and let

$$v_{C \otimes A}((c \otimes 1_A) \otimes d) = \sum_i p_i \otimes b_i \in C \otimes A.$$

As $v_{C \otimes A}$ is left (C, γ) -colinear, and applying ε_C to the second factor, we obtain

$$\gamma^2(c_{(1)}) \otimes \theta(c_{(2)} \otimes \gamma^{-1}(d)) = \sum_i c_i \otimes a_i.$$

Since $v_{C \otimes A}$ is also right (C, γ) -colinear,

$$v_{C \otimes A}(\gamma^{-1}(c) \otimes 1_A \otimes \gamma(d_{(1)})) \otimes \gamma^2(d_{(2)}) = c_{i(1)} \otimes \beta^{-1}(a_{i\psi}) \otimes \gamma^{-1}(c_{i(2)}^\psi),$$

and applying ε_C to the second factor, we find

$$\theta(\gamma^{-1}(c) \otimes d_{(1)}) \otimes \gamma^2(d_{(2)}) = a_{i\psi} \otimes \gamma^{-2}(c_i^\psi).$$

Hence, (4.2) holds. This proves that there is a well-defined map $\Psi : V \rightarrow V_1$.

To show that the map Ψ^{-1} defined by (4.4) is well-defined, we need to prove $v_M \in \widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C$, i.e., v_M is right (A, β) -linear and right (C, γ) -colinear, and v is a natural transformation. The proof is similar to [10], we leave it to the reader.

Given any morphism $f : M \rightarrow N$ in $\widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C$, one easily checks that for all $m \in M$ and $c \in C$, we have

$$v_N(f(m) \otimes C) = f(\mu(m_{[0]}))\theta(m_{[1]} \otimes \gamma^{-1}(c)) = f(\mu(m_{[0]})\theta(m_{[1]} \otimes \gamma^{-1}(c))) = f(v_M(m \otimes c)),$$

i.e., v is natural. The verification that Ψ and Ψ^{-1} are inverses of each other is left to the reader.

Now we give a description of $W = \underline{\text{Nat}}(1_{\mathcal{M}_A}, FG)$. Let

$$W_1 = \{z \in A \otimes C \mid az = za, (\beta \otimes \gamma)(z) = z, \text{ for all } a \in A\}, \tag{4.5}$$

i.e., $z = \sum_l a_l \otimes c_l \in W_1$ if and only if

$$\sum_l \beta^{-1}(a) a_l \otimes \gamma(c_l) = \sum_l a_l \beta^{-1}(a) \otimes \gamma(c_l^\psi), \quad (\beta \otimes \gamma)(z) = z. \tag{4.6}$$

Proposition 4.2. *Let (A, C, ψ) be a right-right Hom-entwining structure. Then there is an isomorphism of k -modules $\Phi : W \rightarrow W_1$ given by*

$$\Phi(\zeta) = \zeta(1_A). \tag{4.7}$$

The inverse of Φ is $\Phi^{-1}(\sum_l a_l \otimes c_l) = \zeta$, with $\zeta_N : N \rightarrow N \otimes C$ given by

$$\zeta_N(n) = \sum_l v^{-1}(n) a_l \otimes \gamma(c_l),$$

for any $(N, v) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C$ and $n \in N$.

Proof. The proof is based on the fact that ζ_A is left and right (A, β) -linear, we leave it to the reader. □

Theorem 4.3. *Let $F : \widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C \rightarrow \widetilde{\mathcal{H}}(\mathcal{M}_k)_A$ be the forgetful functor, and $G : \widetilde{\mathcal{H}}(\mathcal{M}_k)_A \rightarrow \widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C$ its adjoint. Then F is separable if and only if there exists $\theta \in V_1$ such that*

$$\theta \circ \Delta_C = \varepsilon_C,$$

and G is separable if and only if there exists $z = \sum_l a_l \otimes c_l \in W_1$ such that

$$\sum_l \varepsilon_C(c_l) a_l = 1_A.$$

Proof. This follows immediately from Propositions 4.1 and 4.2. □

Next we will show that (F, G) is a Frobenius pair if and only if there exist $\theta \in V_1$ and $z \in W_1$, which satisfy different normalizing conditions.

Theorem 4.4. Let $F : \widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C \rightarrow \widetilde{\mathcal{H}}(\mathcal{M}_k)_A$ be the forgetful functor, and $G : \widetilde{\mathcal{H}}(\mathcal{M}_k)_A \rightarrow \widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C$ its adjoint. Then F is separable if and only if there exist $\theta \in V_1$ and $z = \sum_1 a_l \otimes c_l \in W_1$ such that the following normalizing condition holds, for all $d \in C$,

$$\varepsilon_C(d)1_A = \sum_l \beta(a_l)\theta(c_l \otimes \gamma^{-2}(d)) \tag{4.8}$$

$$= a_{l\psi}\theta(\gamma^{-1}(d)^\psi \otimes \gamma^{-1}(c_l)). \tag{4.9}$$

Proof. Suppose that (F, G) is a Frobenius pair. Then there exist $v \in V$ and $\zeta \in W$ such that (2.1-2.2) hold. Let $\theta = \Psi(v) \in V_1$ and $z = \sum_1 a_l \otimes c_l = \Phi(\zeta) \in W_1$. Then (2.1) can be rewritten as

$$v_M(\sum_l \mu^{-1}(m)a_l \otimes \gamma(c_l)) = (m_{[0]} \cdot a_{l\psi}) \cdot \theta(\gamma(m_{[1]})^\psi \otimes c_l) = m,$$

for any $m \in M \in \widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C$. Taking $M = C \otimes A, m = d \otimes 1_A$, then we have

$$\begin{aligned} d \otimes 1_A &= v_{C \otimes A}(\sum_l \mu^{-1}(d \otimes 1_A)a_l \otimes \gamma(c_l)) \\ &= ((d \otimes 1_A)_{[0]} \cdot a_{l\psi}) \cdot \theta(\gamma((d \otimes 1_A)_{[1]})^\psi \otimes c_l) \\ &= ((d_{(1)} \otimes 1_A) \cdot a_{l\psi}) \cdot \theta(d_{(2)}^\psi \otimes c_l) \\ &= \gamma^2(d_{(1)}) \otimes [a_{l\psi}\theta(\gamma^{-1}(d_{(2)}^\psi) \otimes \gamma^{-1}(c_l))], \end{aligned}$$

thus

$$\begin{aligned} \varepsilon_C(d)1_A &= \widetilde{1}_A(\varepsilon_C(\gamma^2(d_{(1)})) \otimes [a_{l\psi}\theta(\gamma^{-1}(d_{(2)}^\psi) \otimes \gamma^{-1}(c_l))]) \\ &= \beta(a_{l\psi})\theta(\varepsilon_C(d_{(1)})d_{(2)}^\psi \otimes c_l) \\ &= \beta(a_{l\psi})\theta(\gamma^{-1}(d)^\psi \otimes c_l). \end{aligned}$$

One obtains (4.9).

For all $n \in N \in \widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A$ and $c \in C$, one has

$$\begin{aligned} v_{G(N)}(G(\zeta_N)(n \otimes d)) &= v_{G(N)}(v^{-1}(n)a_l \otimes \gamma(c_l) \otimes d) \\ &= v^{-1}(n)a_l \otimes \gamma^2(c_{l(1)})\theta(\gamma^2(c_{l(2)}) \otimes \gamma^{-1}(d)) \\ &= n(a_l\theta(c_{l(2)} \otimes \gamma^{-3}(d))_\psi) \otimes \gamma^3(c_{l(1)})^\psi \\ &= n(a_l\theta(\gamma^{-1}(c_{l(1)}) \otimes \gamma^{-2}(d_{(1)})) \otimes \gamma(d_{(2)})) \\ &= n \otimes d, \end{aligned}$$

and (2.2) can be written as

$$n(a_l\theta(\gamma^{-1}(c_{l1}) \otimes \gamma^{-2}(d_{(1)})) \otimes \gamma(d_{(2)})) = n \otimes d.$$

Taking $N = A$ and $n = 1_A$, we obtain

$$\beta(a_l)\theta(c_{l1} \otimes \gamma^{-1}(d_{(1)})) \otimes \gamma(d_{(2)}) = 1_A \otimes d.$$

Applying ε_C to the second factor yields (4.8). □

In [10], it is shown that if (H, A, C) is a Doi Hom-Hopf datum, (A, β) is faithfully flat as a k -module, and (C, γ) is projective as a k -module, then (C, γ) is finitely generated.

The next proposition illustrates that the assumption that (C, γ) is projective is superfluous.

Proposition 4.5. Let (A, C, ψ) be a right-right Hom-entwining structure. If (F, G) is a Frobenius pair, then $A \otimes C$ is finitely generated and projective as a left (A, β) -Hom-module.

Proof. Let θ and $z = \sum_l a_l \otimes c_l$ be as in Theorem 4.4. Then for any $d \in C$,

$$\begin{aligned} 1_A \otimes d &= \psi(d \otimes 1_A) \\ &= \psi(\gamma(d_1) \otimes \varepsilon(d_2) 1_A) \\ &= \psi(\gamma(d_1) \otimes a_{i\psi} \theta(\gamma^{-1}(d_2)^\psi \otimes \gamma^{-1}(c_l))) \\ &= a_{i\psi} \theta(\gamma^{-1}(d_2)^\psi \otimes \gamma^{-1}(c_l))_\psi \otimes \gamma(d_1)^{\psi\psi} \\ &= a_{i\psi} \theta(\gamma^{-1}(d_2)^\psi \otimes \gamma^{-1}(c_l))_\psi \otimes \gamma(d_1)^{\psi\psi} \\ &= a_{i\psi} \theta(\gamma^{-2}(d)^\psi \otimes c_{l1}) \otimes \gamma(c_{l2}). \end{aligned}$$

Write $c_{ll} \otimes c_{l2} = \sum_{j=1}^{m_l} c_{lj} \otimes c'_{lj}$ and for all l, j , we consider the map

$$\sigma_{lj} : A \otimes C \rightarrow A, \quad \sigma_{lj}(a \otimes d) = \beta^{-1}(a)[a_{i\psi} \theta(\gamma^{-2}(d)^\psi \otimes c_{lj})].$$

Then for all $a \in A$ and $d \in C$,

$$a \otimes d = \sigma_{lj}(a \otimes d)(1 \otimes c'_{lj}),$$

so $\{\sigma_{lj}, 1 \otimes c'_{lj} \mid l = 1, \dots, n, j = 1, \dots, m_l\}$ is a finite dual basis for $A \otimes C$ as a left (A, β) -Hom-module. □

In some situations, one can conclude that (C, γ) is finitely generated and projective as a k -module.

Corollary 4.6. *Let (A, C, ψ) be a right-right Hom-entwining structure and (E, G) a Frobenius pair.*

- 1) *If (A, β) is faithfully flat as a k -module, then (C, γ) is finitely generated as a k -module.*
- 2) *If (A, β) is commutative and faithfully flat as a k -module, then (C, γ) is finitely generated projective as a k -module.*
- 3) *If k is a field, then (C, γ) is finite dimensional as a k -vector space.*
- 4) *If $A = k$, then (C, γ) is finitely generated projective as a k -module.*

Assume that (C, γ) is finitely generated and projective as a k -module, and let $\{d_i, d_i^* \mid i = 1, \dots, m\}$ be a finite dual basis for (C, γ) . Then $C^* \otimes A$ can be made into an object of ${}_A \widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C$ as follows: for all $a, b, b' \in A, c^* \in C^*$,

$$b(c^* \otimes a) = \sum_i \langle c^*, d_i^\psi \rangle \gamma^*(d_i^*) \otimes \beta^{-1}(b_\psi) a, \tag{4.10}$$

$$(c^* \otimes a) b' = \gamma^*(c^*) \otimes a \beta^{-1}(b'), \tag{4.11}$$

$$\rho^r(c^* \otimes a) = \sum_i \gamma^{*-1}(d_i^*) * \gamma^{*-2}(c^*) \otimes \beta^{-1}(a)_\psi \otimes \gamma(d_i^\psi). \tag{4.12}$$

This can be checked directly. The map $\lambda : C \otimes A \otimes C \rightarrow A$ induces $\bar{\phi} : A \otimes C \rightarrow C^* \otimes A$. This is the map we need. At some place it is convenient to use $C^* \otimes A$ as the image space. Note that $\bar{\phi}$ is given by

$$\bar{\phi}(a \otimes c) = \gamma^*(d_i^*) \otimes \beta^{-1}(a)_\psi \theta(\gamma^{-1}(d_i^\psi) \otimes \gamma^{-2}(c)). \tag{4.13}$$

It turns out that $\bar{\phi}$ is a morphism in ${}_A \widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C$. Let V_2 be the k -module consisting of all left (A, β) -linear, right (A, β) -linear, (C, γ) -colinear maps $\bar{\phi} : A \otimes C \rightarrow C^* \otimes A$. Then we have the following result.

Proposition 4.7. *Let (A, C, ψ) be a right-right Hom-entwining structure, and assume that (C, γ) is finitely generated projective as a k -module. Then*

$$V \cong V_1 \cong V_2.$$

The isomorphism is $\alpha_1 : V_1 \rightarrow V_2$, with $\alpha_1(\theta) = \bar{\phi}$, which given by (4.13). The inverse of α_1 is

$$\alpha_1^{-1}(\bar{\phi})(d \otimes c) = \bar{\phi}(1_A \otimes c) d.$$

Proof. First show that $\bar{\phi} \in V_2$. For all $a, b \in A$ and $c \in C$, we have

$$\begin{aligned} b\bar{\phi}(a \otimes c) &= b(\gamma^*(d_i^*) \otimes [\beta^{-1}(a)_\psi \theta(\gamma^{-1}(d_i^\psi) \otimes \gamma^{-2}(c))]) \\ &= \sum \langle \gamma^*(d_i^*), d_j^\psi \rangle \gamma^*(d_j^*) \otimes \beta^{-1}(b)_\psi (\beta^{-1}(a)_\varphi \theta(\gamma^{-1}(d_i^\varphi) \otimes \gamma^{-2}(c))) \\ &= \sum \gamma^*(d_i^*) \otimes \beta^{-1}(b)_\psi (\beta^{-1}(a)_\varphi \theta(\gamma^{-1}(\gamma^{-1}(d_i^\psi)^\varphi) \otimes \gamma^{-2}(c))) \\ &= \sum \gamma^*(d_i^*) \otimes (\beta^{-1}(\beta^{-1}(b)_\psi) \beta^{-1}(a)_\varphi) \theta(\gamma^{-1}(d_i^\psi)^\varphi \otimes \gamma^{-1}(c)) \\ &= \sum \gamma^*(d_i^*) \otimes \beta^{-1}(\beta^{-1}(b)a)_\varphi \theta(\gamma^{-1}(d_i^\psi) \otimes \gamma^{-1}(c)) \\ &= \bar{\phi}(\beta^{-1}(b)a \otimes \gamma(c)) \\ &= \bar{\phi}(b \cdot (a \otimes c)), \end{aligned}$$

which proves that $\bar{\phi}$ is left (A, β) -linear. And it is also right (A, β) -linear, because

$$\begin{aligned} \bar{\phi}(a \otimes c)b &= \sum \gamma^{*2}(d_i^*) \otimes [\beta^{-1}(a)_\psi \theta(\gamma^{-1}(d_i^\psi) \otimes \gamma^{-2}(c))] \beta^{-1}(b) \\ &= \sum \gamma^{*2}(d_i^*) \otimes \beta(\beta^{-1}(a)_\psi) [\theta(\gamma^{-1}(d_i^\psi) \otimes \gamma^{-2}(c)) \beta^{-2}(b)] \\ &= \sum \gamma^{*2}(d_i^*) \otimes [\beta^{-1}(a)_\psi \beta(\beta^{-3}(b)_{\varphi\nu})] \theta(\gamma^{-1}(d_i^\psi)^\nu \otimes \gamma^{-2}(c)^\varphi) \\ &= \sum \gamma^{*2}(d_i^*) \otimes [\beta^{-1}(a)_\psi \beta(\beta^{-3}(b)_{\varphi\nu})] \theta(d_i^{\psi\nu} \otimes \gamma^{-2}(c)^\varphi) \\ &= \sum \gamma^{*2}(d_i^*) \otimes [(\beta^{-1}(a)\beta^{-2}(b)_{\varphi\psi})] \theta(\gamma^{-1}(\gamma(d_i)^\psi) \otimes \gamma^{-1}(c)^\varphi) \\ &= \sum \gamma^*(d_i^*) \otimes [\beta^{-1}(a(\beta^{-1}(b)_{\varphi}))_\psi] \theta(\gamma^{-1}(d_i^\psi) \otimes \gamma^{-1}(c)^\varphi) \\ &= \bar{\phi}(a(\beta^{-1}(b)_\psi) \otimes \gamma(c^\psi)) = \bar{\phi}((a \otimes c)b). \end{aligned}$$

Notice that the dual basis for (C, γ) satisfies the following equality

$$\sum \Delta(d_i) \otimes d_i^* = \sum d_i \otimes d_j \otimes d_i^* * d_j^*.$$

Using this equality one can compute

$$\begin{aligned} &\rho^r(\bar{\phi}(a \otimes c)) \\ &= \sum \rho^r(\gamma^*(d_i^*) \otimes \beta^{-1}(a)_\psi \theta(\gamma^{-1}(d_i^\psi) \otimes \gamma^{-2}(c))) \\ &= \sum (\gamma^{*-1}(d_i^*) \otimes \beta^{-1}(\beta^{-1}(a)_\varphi) \theta(\gamma^{-1}(d_{i(2)}^\varphi) \otimes \gamma^{-2}(c)))_\psi \otimes \gamma(d_{i(1)}^\psi) \\ &= \sum (\gamma^{*-1}(d_i^*) \otimes \beta^{-1}(\beta^{-1}(a)_\varphi)_\nu \theta(\gamma^{-2}(d_{i(2)}^\varphi) \otimes \gamma^{-3}(c))_\psi) \otimes \gamma^2(\gamma^{-1}(d_{i(1)})^{\nu\psi}) \\ &= \sum (\gamma^{*-1}(d_i^*) \otimes \beta^{-1}(\beta^{-1}(a)_{\varphi\nu}) \theta(\gamma^{-2}(d_{i(2)}^\varphi) \otimes \gamma^{-3}(c))_\psi) \otimes \gamma^2(\gamma^{-1}(d_{i(1)}^\nu)^\psi) \\ &= \sum (\gamma^{*-1}(d_i^*) \otimes \beta^{-2}(a_\varphi) \theta(\gamma^{-2}(d_{i(2)}^\varphi) \otimes \gamma^{-3}(c))_\psi) \otimes \gamma^2(\gamma^{-2}(d_{i(1)}^\varphi)^\psi) \\ &= \sum (\gamma^{*-1}(d_i^*) \otimes \beta^{-2}(a_\varphi) \theta(\gamma^{-3}(d_{i(1)}^\varphi) \otimes \gamma^{-2}(c_{(1)}))) \otimes \gamma(c_{(2)}) \\ &= \sum (\gamma^*(d_i^*) \otimes \beta^{-2}(a)_\varphi \theta(\gamma^{-1}(d_i^\varphi) \otimes \gamma^{-2}(c_{(1)}))) \otimes \gamma(c_{(2)}) \\ &= \sum \bar{\phi}(\beta^{-1}(a) \otimes c_{(1)}) \otimes \gamma(c_{(2)}). \end{aligned}$$

This proves that $\bar{\phi}$ is right C -colinear.

Conversely, given $\phi \in V_2$, one needs to show that $\theta = \alpha_1^{-1}(\bar{\phi}) \in V_1$. Now it is more convenient to work with $\text{Hom}(C, A)$ rather than $C^* \otimes A$. For $f \in \text{Hom}(C, A)$, $b \in A$, we have

$$b_\varphi f(\gamma^{-1}(c^\varphi)) = (b \cdot f)(c),$$

and

$$f(\gamma^{-1}(c))b = (f \cdot b)(c).$$

Take any $c, d \in C, a \in A$ and compute

$$\begin{aligned} \theta(c \otimes d)a &= (\bar{\phi}(1_A \otimes d)c)a \\ &= (\bar{\phi}(1_A \otimes d)a)\gamma(c) \\ &= \sum \bar{\phi}(\beta(\beta^{-1}(a)_\psi) \otimes \gamma(d^\psi))\gamma(c) \\ &= \sum (\beta(\beta^{-1}(a)_\psi)\bar{\phi}(1_A \otimes d^\psi))\gamma(c) \\ &= \sum \beta(\beta^{-1}(a)_\psi)_\varphi (\bar{\phi}(1_A \otimes d^\psi)\gamma^{-1}(\gamma(c)^\varphi)) \\ &= \sum \beta(\beta^{-1}(a)_\psi)_\varphi \bar{\phi}(c^\varphi \otimes d^\psi), \end{aligned}$$

thus (4.1) holds. Before proving (4.2), we write $\rho^r(f) = f_{[0]} \otimes f_{[1]}$ for $f = c^* \otimes a \in \text{Hom}(C, A) \cong C^* \otimes A$, and then for all $c \in C$, we have

$$\begin{aligned} f_{[0]}(c) \otimes f_{[1]} &= \sum (\gamma^{*-1}(d_i^*) * \gamma^{*-2}(c^*) \otimes \beta^{-1}(a)_\psi)(c) \otimes \gamma(d_i^\psi) \\ &= \sum \langle \gamma^{*-1}(d_i^*), c_1 \rangle \langle \gamma^{*-2}(c^*), c_2 \rangle \beta(\beta^{-1}(a)_\psi) \otimes \gamma(d_i^\psi) \\ &= \sum \langle c^*, \gamma^2(c_2) \rangle a_\psi \otimes \gamma^2(c_1)^\psi \\ &= \langle c^*, \gamma^2(c_2) \rangle \psi(\gamma^2(c_1) \otimes a) = \psi(\gamma^2(c_1) \otimes f(\gamma(c_2))). \end{aligned}$$

Explicitly, we have

$$\begin{aligned} \sum \theta(d_{(2)} \otimes \gamma^{-1}(c))_\psi \otimes \gamma(d_{(1)})^\psi &= \psi(\gamma(d_{(1)}) \otimes \bar{\phi}(1_A \otimes \gamma^{-1}(c))(d_{(2)})) \\ &= \psi(1_A \otimes \gamma^{-1}(c))_0(\gamma^{-1}(d)) \otimes \psi(1_A \otimes \gamma^{-1}(c))_{(1)} \\ &= \psi(1_A \otimes \gamma^{-1}(c_{(1)}))(\gamma^{-1}(d)) \otimes c_{(2)} \\ &= \theta(\gamma^{-1}(d) \otimes \gamma^{-1}(c_{(1)})) \otimes c_{(2)}. \end{aligned}$$

It remains to be shown that α_1 and α_1^{-1} are inverse of each other. First take $\theta \in V_1$, for all $c, d \in C$, we have

$$\begin{aligned} \alpha_1^{-1}(\alpha(\theta))(d \otimes c) &= \alpha_1(\theta)(1_A \otimes c)(d) \\ &= (\gamma^*(d_i^*) \otimes 1_A \theta(\gamma^{-1}(d_i) \otimes \gamma^{-2}(c)))(d) \\ &= \langle \gamma^*(d_i^*), d \rangle \theta(\gamma(d_i) \otimes c) = \theta(d \otimes c). \end{aligned}$$

Finally, for $\bar{\phi} \in V_2, a \in A, c, d \in C$:

$$\begin{aligned} &\alpha_1(\alpha^{-1}(\bar{\phi}))(a \otimes c)(d) \\ &= \sum (\gamma^*(d_i^*) \otimes \beta^{-1}(a)_\psi \alpha_1^{-1}(\bar{\phi})(\gamma^{-1}(d_i^\psi) \otimes \gamma^{-2}(c)))(d) \\ &= \sum \langle d_i^*, \gamma^{-1}(d_i) \rangle \beta(\beta^{-1}(a)_\psi \alpha_1^{-1}(\bar{\phi})(\gamma^{-1}(d_i^\psi) \otimes \gamma^{-2}(c))) \\ &= \sum \beta(\beta^{-1}(a)_\psi) \alpha_1^{-1}(\bar{\phi})(\gamma^{-1}(d)^\psi \otimes \gamma^{-1}(c)) \\ &= \sum \beta(\beta^{-1}(a)_\psi) \alpha_1^{-1}(\bar{\phi})(1_A \otimes \gamma^{-1}(c)) \gamma^{-1}(d)^\psi \\ &= a \cdot \bar{\phi}(1_A \otimes \gamma^{-1}(c))d = \bar{\phi}(a \otimes c)d. \end{aligned}$$

□

Now we give an alternative description for W_2 .

Proposition 4.8. *Let (C, γ) be finitely generated and projective as a k -module. Then*

$$W \cong W_1 \cong W_2 = \text{Hom}_{AA}^{kA}(C^* \otimes A, A \otimes C).$$

The isomorphism $\beta_1 : W_1 \rightarrow W_2$ is given by $\beta_1(z) = \phi$ with

$$\phi(c^* \otimes a) = \sum_l a_l \beta^{-1}(a)_\psi \otimes \gamma(c_{l(1)})^\psi \langle \gamma^{*-2}(c^*), c_{l(2)} \rangle$$

and the inverse of β_1 is given by

$$\beta_1^{-1}(\phi) = \phi(\varepsilon \otimes 1).$$

Proof. We need to show that $\beta_1(z) = \phi$ is left and right (A, β) -linear and right (C, γ) -colinear. For all $c^* \in C^*$ and $a, b \in A$,

$$\begin{aligned} \phi((c^* \otimes a)b) &= \phi(\gamma^*(c^*) \otimes a\beta^{-1}(b)) \\ &= \sum_l a_l \beta^{-1}(a\beta^{-1}(b))_\psi \otimes \gamma(c_{l(1)})^\psi \langle \gamma^{*-1}(c^*), c_{l(2)} \rangle \\ &= \sum_l a_l \beta^{-1}(a)_\psi \beta^{-2}(b)_\psi \otimes \gamma(c_{l(1)})^{\psi\psi} \langle \gamma^{*-1}(c^*), c_{l(2)} \rangle \\ &= \sum_l [\beta^{-1}(a_l)\beta^{-1}(a)_\psi] \beta^{-1}(b)_\psi \otimes \gamma(c_{l(1)})^{\psi\psi} \langle \gamma^{*-1}(c^*), c_{l(2)} \rangle \\ &= \sum_l [\beta^{-1}(a_l)\beta^{-1}(a)_\psi] \beta^{-1}(b)_\psi \otimes \gamma(c_{l(1)})^{\psi\psi} \langle \gamma^{*-1}(c^*), c_{l(2)} \rangle \\ &= \sum_l [\beta^{-1}(a_l)\beta^{-1}(a)_\psi \otimes c_{l(1)}^\psi \langle \gamma^{*-1}(c^*), c_{l(2)} \rangle] \cdot b \\ &= \sum_l [a_l \beta^{-1}(a)_\psi \otimes \gamma(c_{l(1)})^\psi \langle \gamma^{*-2}(c^*), \gamma(c_{l(2)}) \rangle] \cdot b \\ &= \sum_l [a_l \beta^{-1}(a)_\psi \otimes \gamma(c_{l(1)})^\psi \langle \gamma^{*-2}(c^*), c_{l(2)} \rangle] \cdot b \\ &= \phi(c^* \otimes a)b, \end{aligned}$$

which proves that ϕ is right (A, β) -linear. The proof of left (A, β) -linearity goes as follows:

$$\begin{aligned} \phi(b(c^* \otimes a)) &= \phi(\langle c^*, d_i^\psi \rangle \gamma^*(d_i^*) \otimes \beta^{-1}(b_\psi)a) \\ &= \sum_l \langle c^*, d_i^\psi \rangle a_l \beta^{-1}(\beta^{-1}(b_\psi)a)_\psi \otimes \gamma(c_{l(1)})^\psi \langle \gamma^{*-1}(d_i^*), c_{l(2)} \rangle \\ &= \sum_l \langle c^*, d_i^\psi \rangle a_l \beta^{-2}(b_\psi)_\psi \beta^{-1}(a)_\psi \otimes \gamma(c_{l(1)})^{\psi\psi} \langle \gamma^{*-1}(d_i^*), c_{l(2)} \rangle \\ &= \sum_l \langle c^*, [\gamma(c_l)^\psi]_{(2)} \rangle a_l \beta^{-2}(b_\psi)_\psi \beta^{-1}(a)_\psi \otimes [\gamma(c_l)^\psi]_{(1)}^\psi \\ &= \sum_l \langle c^*, [\gamma(c_l)^\psi]_{(2)} \rangle [\beta^{-1}(a_l)\beta^{-2}(b_\psi)] a_\psi \otimes [\gamma(c_l)^\psi]_{(1)}^\psi \\ &\stackrel{(4.8)}{=} \sum_l \langle c^*, \gamma(c_{l(2)}) \rangle [\beta^{-2}(b)\beta^{-1}(a_l)] a_\psi \otimes \gamma(c_{l(1)})^\psi \\ &= \sum_l \langle \gamma^{*-1}(c^*), c_{l(2)} \rangle \beta^{-1}(b) [\beta^{-1}(a_l)\beta^{-1}(a_\psi)] \otimes \gamma(c_{l(1)})^\psi \\ &= b \cdot \left[\sum_l \langle \gamma^{*-1}(c^*), c_{l(2)} \rangle \beta^{-1}(a_l)\beta^{-1}(a_\psi) \otimes c_{l(1)}^\psi \right] \end{aligned}$$

$$\begin{aligned}
 &= b \cdot \left[\sum_I a_i \beta^{-1}(a)_{\psi} \otimes \gamma(c_{I(1)})^{\psi} \langle \gamma^{*-2}(c^*), c_{I(2)} \rangle \right] \\
 &= b \cdot \phi(c^* \otimes a).
 \end{aligned}$$

Next we need to show that ϕ is right (C, γ) -colinear. In fact,

$$\begin{aligned}
 &\phi((c^* \otimes a)_{[0]}) \otimes (c^* \otimes a)_{[1]} \\
 &= \phi(\gamma^{*-1}(d_i^*) * \gamma^{*-2}(c^*) \otimes \beta^{-1}(a)_{\psi}) \otimes \gamma(d_i^{\psi}) \\
 &= \sum_{i,l} a_i \beta^{-1}(\beta^{-1}(a)_{\psi})_{\psi} \otimes \gamma(c_{I(1)})^{\psi} \langle \gamma^{*-2}(\gamma^{*-1}(d_i^*) * \gamma^{*-2}(c^*)), c_{I(2)} \rangle \otimes \gamma(d_i^{\psi}) \\
 &= \sum_{i,l} a_i \beta^{-2}(a)_{\psi\psi} \otimes \gamma(c_{I(1)})^{\psi} \langle \gamma^{*-3}(d_i^*), c_{I(2)(1)} \rangle \langle \gamma^{*-4}(c^*), c_{I(2)(2)} \rangle \otimes \gamma(d_i^{\psi}) \\
 &= \sum_{i,l} a_i \beta^{-2}(a)_{\psi\psi} \otimes \gamma(c_{I(1)})^{\psi} \langle \gamma^{*-4}(c^*), c_{I(2)(2)} \rangle \otimes \gamma^3(c_{I(2)(1)})^{\psi} \\
 &= \sum_{i,l} \beta(a_i) \beta^{-1}(a)_{\psi\psi} \otimes \gamma^2(c_{I(1)(1)})^{\psi} \langle \gamma^{*-3}(c^*), c_{I(2)} \rangle \otimes \gamma^2(c_{I(1)(2)})^{\psi} \\
 &= \sum_{i,l} a_i \beta^{-1}(a)_{\psi\psi} \otimes \gamma(c_{I(1)(1)})^{\psi} \langle \gamma^{*-2}(c^*), c_{I(2)} \rangle \otimes \gamma(c_{I(1)(2)})^{\psi} \\
 &= \rho^r(\phi((c^* \otimes a))).
 \end{aligned}$$

Conversely, let $\phi \in W_2$ and put $z = \phi(\varepsilon \otimes 1_A) = \sum_I a_i \otimes c_i$, we obtain $a(\varepsilon \otimes 1_A) = (\varepsilon \otimes 1_A)a$, for all $a \in A$. Hence $az = a\phi(\varepsilon \otimes 1_A) = \phi(a(\varepsilon \otimes 1_A)) = \phi((\varepsilon \otimes 1_A)a) = \phi(\varepsilon \otimes 1_A)a = za$, and $z \in W_1$. Take $z = \sum_I a_i \otimes c_i \in W_1$. Then

$$\begin{aligned}
 \beta_1^{-1}(\beta_1(z)) &= \beta_1^{-1}(\phi) = \phi(\varepsilon \otimes 1_A) = \sum_I a_i \beta^{-1}(1_A)_{\psi} \otimes \gamma^2(c_{I(1)})^{\psi} \langle \varepsilon, c_{I(2)} \rangle \\
 &= \sum_I \beta(a_i) \otimes \gamma^2(c_{I(1)}) \langle \varepsilon, c_{I(2)} \rangle \\
 &= \sum_I \beta(a_i) \otimes \gamma(c_i) = z.
 \end{aligned}$$

Finally, take $\phi \in W_2$, and write $\beta_1^{-1}(\phi) = \phi(\varepsilon \otimes 1_A) = \sum_I a_i \otimes c_i$. Then $C^* \otimes A$ and $A \otimes C$ are right (C, γ) -Hom-comodules and left (C^*, γ^*) -Hom-modules. Since ϕ is right (A, β) -linear, right (C, γ) -colinear and left (C^*, γ^*) -linear, we see

$$\begin{aligned}
 \phi(c^* \otimes a) &= \phi(\gamma^{*-1}(c^*) \otimes 1_A)a = [\gamma^{*-1}(c^*)\phi(\varepsilon \otimes 1_A)]a \\
 &= [\gamma^{*-1}(c^*) \cdot \sum_I a_i \otimes c_i]a \\
 &= \sum_I a_i \beta^{-1}(a)_{\psi} \otimes \gamma^2(c_{I(1)})^{\psi} \langle \gamma^{*-3}(c^*), c_{I(2)} \rangle \\
 &= \beta_1(z)(c^* \otimes a),
 \end{aligned}$$

and it follows that $\phi = \beta_1(z) = \beta_1(\beta_1^{-1}(\phi))$, as required. □

Theorem 4.9. Let (A, C, ψ) be a right-right Hom-entwining structure and assume that (C, γ) is finitely generated projective as a k -module. Let $F : \widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C \rightarrow \widetilde{\mathcal{H}}(\mathcal{M}_k)_A$ be the forgetful functor, and $G : \widetilde{\mathcal{H}}(\mathcal{M}_k)_A \rightarrow \widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C$ its adjoint. Then the following statements are equivalent:

- 1) (F, G) is a Frobenius pair;
- 2) There exist $z = \sum a_i \otimes c_i \in W_1$ and $\theta \in V_1$ such that the maps

$$\phi : C^* \otimes A \rightarrow A \otimes C \quad \text{and} \quad \bar{\phi} : A \otimes C \rightarrow C^* \otimes A,$$

given by

$$\phi(c^* \otimes a) = \sum_I a_i \beta^{-1}(a)_{\Psi} \otimes \gamma(c_{I(1)})^{\Psi} \langle \gamma^{*-2}(c^*), c_{I(2)} \rangle, \tag{4.14}$$

$$\bar{\phi}(a \otimes c) = \gamma^*(d_i^*) \otimes \beta^{-1}(a)_{\Psi} \theta(\gamma^{-1}(d_i^{\Psi}) \otimes \gamma^{-2}(c)), \tag{4.15}$$

are inverses of each other;

3) $C^* \otimes A$ and $A \otimes C^*$ are isomorphic as objects in ${}_A \widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C$.

Proof. 1) \Rightarrow 2). Let $z \in W_1$ and $\theta \in V_1$ be as in Theorem (4.4). Then $\phi = \beta_1(z)$ and $\bar{\phi} = \alpha_1(\theta)$ are morphisms in ${}_A \widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C$, and

$$\begin{aligned} \bar{\phi}(\phi(\varepsilon \otimes 1_A)) &= \bar{\phi}(z) = \sum_I \gamma^*(d_i^*) \otimes \beta^{-1}(a_i)_{\Psi} \theta(\gamma^{-1}(d_i^{\Psi}) \otimes \gamma^{-2}(c_i)) \\ &= \sum_I \gamma^*(d_i^*) \otimes a_{I\Psi} \theta(\gamma^{-1}(d_i^{\Psi}) \otimes \gamma^{-1}(c_i)) \\ &= \sum_I \gamma^*(d_i^*) \otimes \varepsilon(d_i) 1_A = \varepsilon \otimes 1_A. \end{aligned}$$

The fact that ϕ and $\bar{\phi}$ are right (A, β) -linear and left (C^*, γ^*) -linear implies that $\bar{\phi} \circ \phi = I_{C^* \otimes A}$. Similarly, for all $c \in C$,

$$\begin{aligned} \phi(\bar{\phi}(1_A \otimes c)) &= \phi(\gamma^*(d_i^*) \otimes \theta(d_i \otimes c)) \\ &= \sum_I a_i \beta^{-1}(\theta(d_i \otimes c))_{\Psi} \otimes \gamma(c_{I(1)})^{\Psi} \langle d_i^*, c_{I(2)} \rangle \\ &= \sum_I a_i \beta^{-1}(\theta(c_{I(2)} \otimes c))_{\Psi} \otimes \gamma(c_{I(1)})^{\Psi} \\ &= \sum_I a_i \theta(\gamma^{-1}(c_i) \otimes \gamma^{-2}(d)) \otimes \gamma(c_{(2)}) \\ &= \varepsilon(c_{(1)}) 1_A \otimes \gamma(c_{(2)}) \\ &= 1_A \otimes c. \end{aligned}$$

2) \Rightarrow 3). Obviously, since ϕ and $\bar{\phi}$ are in ${}_A \widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C$.

3) \Rightarrow 1). Let $\phi : C^* \otimes A \rightarrow A \otimes C$ be an isomorphism, and put $z = \phi(\varepsilon \otimes 1_A) = \sum_I a_i \otimes c_i \in W_1, \theta = \alpha_1^{-1}(\phi^{-1}) \in V_1$, we have

$$\varepsilon \otimes 1_A = \phi^{-1}(\phi(\varepsilon \otimes 1_A)) = \gamma^*(d_i^*) \otimes \beta^{-1}(a_i)_{\Psi} \theta(\gamma^{-1}(d_i^{\Psi}) \otimes \gamma^{-2}(c_i)).$$

Evaluating this equality at $c \in C$, one obtains (4.15). For all $c \in C$,

$$1_A \otimes c = \phi(\phi^{-1}(1_A \otimes c)) = \sum_I a_i \theta(\gamma^{-1}(c_i) \otimes \gamma^{-2}(d)) \otimes \gamma(c_{(2)}).$$

Applying ε to the second factor gives (4.14). Thus (F, G) is a Frobenius pair. □

Recall from [17] that let (A, β_A) and (B, β_B) be two Hom-associative algebras and a linear map $R : A \otimes B \rightarrow B \otimes A$ with $R(b \otimes a) = a_R \otimes b_R = a_r \otimes b_r$ satisfying the conditions:

$$\beta_A(a_R) \otimes \beta_B(b_R) = \beta_A(a)_R \otimes \beta_B(b)_R, \tag{4.16}$$

$$(aa')_R \otimes \beta_B(b)_R = a_R a'_r \otimes \beta_B((b_R)_r), \tag{4.17}$$

$$\beta_A(a)_R \otimes (bb')_R = \beta_A((a_R)_r) \otimes b_r b'_r, \tag{4.18}$$

$$1_{B_R} \otimes a = 1_B \otimes a, \tag{4.19}$$

$$b_R \otimes 1_{A_R} = b \otimes 1_A, \tag{4.20}$$

for all $a, a' \in A$ and $b, b' \in B$. Define a new multiplication on $A \otimes B$ by $(a \otimes b)(a' \otimes b') = aa'_R \otimes b_R b'$. Then $A \otimes B$ with this multiplication is a Hom-associative algebra with structure map $\beta_A \otimes \beta_B$, we denote it by $B\#_R A$.

Next we want to examine when $B\#_R A/A$ is Frobenius. In fact, this is a direct application of Theorem 4.9 under the hypotheses of that (B, β_B) is finitely generated and projective as a k -module. Let (A, C, ψ) be a Hom-entwining structure, with (C, γ) finitely generated and projective, and put $B = (C^*)^{op}$. Let $\{c_i, c_i^*, i = 1, \dots, n\}$ be a dual basis for (C, γ) . There is a bijective between Hom-entwining structures (A, C, ψ) and Hom smash product structures $((C^*)^{op}, A, R)$, where R and ψ can be recovered from each other using the formulae

$$R(a \otimes c^*) = \langle \gamma^{*-1}(c^*), c_i^\psi \rangle \gamma^*(c_i^*) \otimes a_\psi, \quad \psi(c \otimes a) = \langle (c_i^*)_R, g^{-1}(c) \rangle \gamma(c_i) \otimes a_R.$$

Moreover, there are isomorphisms of categories

$$\widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C \cong \widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_{B\#A} \quad \text{and} \quad {}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C \cong {}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_{B\#A}.$$

In particular, $B\#_R A$ can be made into an object of ${}_A\widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C$, and this explains the structure on $C^* \otimes A$ using in Section 4. Combining Theorem 4.9, we find that the forgetful functor $F : \widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C \rightarrow \widetilde{\mathcal{H}}(\mathcal{M}_k)_A$ and its adjoint form a Frobenius pair if and only if $A \otimes C$ and $C^* \otimes A$ are isomorphic as $(A; C^{*op}\#A)$ -bimodules if and only if the extension $(C^*)^{op} \otimes A/A$ is Frobenius.

5. Maschke type theorems

The classical Maschke’s Theorem states that a group ring of a finite group is semisimple if and only if the characteristic of the field does not divide the order of the group. Several generalisations of Maschke’s theorem for Hom-Hopf algebras and Hom-comodule algebras can be found in [6, 9, 10]. In [10], a Maschke-type theorem was formulated for Doi Hom-Hopf modules. Following [10] we define

Definition 5.1. Let (A, C, ψ) be a Hom-entwining structure. A k -module map $\theta : (C, \gamma) \otimes (C, \gamma) \rightarrow (A, \beta)$ satisfying $\theta \circ (\gamma \otimes \gamma) = \beta \circ \theta$ is called a normalized (A, β) -integral, if θ satisfies the following conditions:

(1) For all $c, d \in C$,

$$\beta(\beta^{-1}(a)_{\varphi_\psi})\theta(d^\psi \otimes c^\varphi) = \theta(d \otimes c)a, \tag{5.1}$$

(2) For all $a \in A, c, d \in C$,

$$\theta(\gamma^{-1}(d) \otimes c_{(1)}) \otimes \gamma(c_{(2)}) = \theta(d_{(2)} \otimes \gamma^{-1}(c))_\psi \otimes d_{(1)}^\psi, \tag{5.2}$$

(3) For all $c \in C$,

$$\theta(c_{(1)} \otimes c_{(2)}) = 1_A \varepsilon(c). \tag{5.3}$$

Theorem 5.2. Let (A, C, ψ) be a Hom-entwining structure and $(M, \mu), (N, \nu) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C$. Suppose that there exists a normalized (A, β) -integral $\theta : (C, \gamma) \otimes (C, \gamma) \rightarrow (A, \beta)$. Then a monomorphism (resp. epimorphism) $f : (M, \mu) \rightarrow (N, \nu)$ splits in $\widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C$, if the monomorphism (resp. epimorphism) f splits as an (A, β) -linear morphism.

Proof. Let $M, N \in \widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C$ and assume that $f : M \rightarrow N \in \widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C$ has a section $g : N \rightarrow M$ as an (A, β) -module morphism. Define $\tilde{g} : N \rightarrow M$ by $\tilde{g}(n) = \sum \mu(g(n_{[0]})_{[0]}) \cdot \phi(n_{[1]} \otimes \gamma^{-1}(g(n_{[0]})_{[1]}))$. By the partial normalized integral θ , one can easily check that \tilde{g} is right (A, β) -Hom-module morphism and right (C, γ) -Hom-comodule morphism. Now we show that \tilde{g} is a section of f in $\widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi)_A^C$. In fact,

$$\begin{aligned} f\tilde{g}(n) &= \sum f(\mu(g(n_{[0]})_{[0]}) \cdot \theta(n_{[1]} \otimes \gamma^{-1}(g(n_{[0]})_{[1]}))) \\ &= \sum (fg(\mu(n_{[0]}))_{[0]} \cdot \theta(n_{[1]} \otimes (fg\gamma^{-1}(n_{[0]}))_{[1]})) \\ &= \sum (\mu(n_{[0][0]}) \cdot \theta(n_{[1]} \otimes \gamma^{-1}(n_{[0][1]}))) \\ &= \sum n_{[0]} \cdot \theta(n_{[1(1)} \otimes n_{[1(2)}))1_A \\ &\stackrel{(5.3)}{=} n. \end{aligned}$$

Similar computation shows that if g is a retraction of f , then so is \tilde{g} .

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