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Theta Function Identities of Level 6 and their Application to Partitions

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Abstract. M. Somos has conjectured many theta function identities belonging to different levels. He has done so with the use of a computer but has not chosen to validate these identities. We find that the mentioned identities are analogous to those discovered by Srinivasa Ramanujan. The intent to prove some of the identities prepared by Somos concerning theta function identities of a level six and to also establish certain partition-theoretic interpretations of these identities which we have been successfully proved here.

1. Introduction

All through this paper, we have assumed $|q| < 1$ and used the standard notation

$$(a;q)_\infty := \prod_{m=0}^{\infty}(1-aq^m).$$

The general theta function $f(a,b)$ as given by S. Ramanujan is stated below

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} \qquad |ab| < 1.$$

By Jacobi's triple product identity [5, p. 35], we have

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

Ramanujan has defined the following special cases of $f(a,b)$ [5, p. 36]

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty.$$

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In addition to Ramanujan, others too have defined

$$\chi(q) := (-q;q^2)_\infty.$$

Recently H. M. Srivastava et. al. [11] proved two q-identities which provide relationship between $f(-q)$, $\varphi(q)$ and $\psi(q)$, by using well known Jacobi's triple product identity. Note that, if $q = e^{2\pi i\tau}$ then $f(-q) = e^{-\pi i\tau/12}\eta(\tau)$, where $\eta(\tau)$ denotes the classical Dedekind η-function for any complex number τ and $\mathrm{Im}(\tau) > 0$. The theta-function identity, which relates $f(-q)$ to $f(-q^n)$ is called theta function identity of level n. Ramanujan recorded several identities which involve $f(-q), f(-q^2), f(-q^n)$ and $f(-q^{2n})$ in his second notebook [7] as well as his 'Lost' notebook [8]. For example [6, p. 204, Entry 51],

$$f_1^8 f_2^8 f_3^4 f_6^4 + 9q f_1^4 f_2^4 f_3^8 f_6^8 = q f_1^{12} f_6^{12} + f_2^{12} f_3^{12},$$

where $f_n = f(-q^n)$. After the publication of [6], several authors including N. D. Baruah [3, 4], and K. R. Vasuki [14, 15] have found many modular equations of the above type. Recently C. Adiga et. al. [1] have established several modular relations for the Rogers−Ramanujan type functions of order eleven which analogous to Ramanujan's forty identities for Rogers−Ramanujan functions and also they established certain interesting partition-theoretic interpretation of some of the modular relations. H. M. Srivastava and M. P. Chaudhary [10] established a set of four new results which depict the interrelationships between q-product identities, continued fraction identities and combinatorial partition identities. C. Adiga et. al. [2], have established some relation between Ramanujan's continued fraction $F(a;b;q)$ and obtained three equivalent integral representations of $F(-1;1;q)$ and some modular equations for the same. Also they found continued fraction representations for the Ramanujan-Weber class invariants. Further, they deduced some algebraic numbers and transcendental numbers involving $F(-1;1;q) + 1$, the Ramanujan-Göllnitz-Gordon continued fraction $H(q)$ and the Dedekind η-function. Of late the public domain as made famous the discovery by M. Somos [9] of theta-function identities of different levels numbering around 6200. It is notable that these identities are very similar to the work of Ramanujan. Somos uses PARI/GP scripts to verify these identities but does not attempt to prove them. B. Yuttanan [17] has used Ramanujan's modular equations to prove identities of level 4, 6 and 8 for those of Somos theta-functions. K. R. Vasuki and R. G. Veeresha [16] too have proved the identities discovered by Somos of theta-function of a level 14. B. R. Srivatsa Kumar and R. G. Veeresha [12] have also succeeded in obtaining partition identities for the same. B. R. Srivatsa Kumar and D. Anu Radha [13] proved the identities of level 10 and obtained certain interesting partitions discovered by Somos. We see that Somos [9] work focusing on a 100 new elegant theta-function identities of level 6 have been achieved by using PARI/GP scripts without offering any proof. These identities have the arguments in $f(-q), f(-q^2), f(-q^3)$ and $f(-q^6)$, namely $-q, -q^2, -q^3$ and $-q^6$ all have exponents dividing the number 6, which is thus made equal to the 'level' of the identity six. We intend to prove in this paper new theta function identities of level 6 focused by Somos with the intention of establishing partition-theoretic interpretations for a few of them. In this paper, we prove 15 identities and the remaining Somos identities of level 6 can also be proved in the same method. With the intention of stating and proving Somos identities we wish to initially base our work on particular modular equations and theta-function identities which we will require in sequel.

2. Preliminaries

Let us now take up a modular equation as given in the literature. A modular equation of degree n is an equation relating α and β that is induced by

$$n\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\beta\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)},$$

where

$$ {}_2F_1(a,b;c;z) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n \qquad |z| < 1, $$

denotes an ordinary hypergeometric function with

$$ (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}. $$

Then, we say that β is of degree n over α and call the ratio

$$ m := \frac{z_1}{z_n}, $$

the multiplier, where $z_1 = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)$ and $z_n = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)$. On page 230 of his second notebook [7] and [5, pp. 230−238, Entry 5], Ramanujan recorded many modular equations of degree 3 and two among them are as follows:
If β has degree three over α and m is the multiplier of degree three, then

$$ m = \frac{1 - 2\left(\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right)^{1/8}}{1-2(\alpha\beta)^{1/4}}, \tag{1} $$

$$ \frac{3}{m} = \frac{2\left(\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right)^{1/8} - 1}{1-2(\alpha\beta)^{1/4}} \tag{2} $$

and if $P := \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}$ and $Q := \left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{1/4}$, then

$$ Q + \frac{1}{Q} + 2\sqrt{2}\left(P - \frac{1}{P}\right) = 0. \tag{3} $$

From (1) and (2), we have

$$ \frac{m^2}{3} = \frac{1 - 2\left(\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right)^{1/8}}{2\left(\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right)^{1/8} - 1} \tag{4} $$

On transforming (3) and (4) in terms theta functions by employing [5, pp. 122−124, Entry 10(i) and Entry 12(v)], we deduce

$$ \frac{q\chi^6(q)}{\chi^6(q^3)} + \frac{\chi^6(q^3)}{\chi^6(q)} = \chi^3(q)\chi^3(q^3) - \frac{8q}{\chi^3(q)\chi^3(q^3)} \tag{5} $$

and

$$ \frac{\varphi^4(q)}{3\varphi^4(q^3)} = \frac{1 - 4q\frac{\chi^3(q)}{\chi^9(q^3)}}{\frac{4\chi^3(q^3)}{\chi^9(q)} - 1}. \tag{6} $$

By using q−identities, one can easily deduce the following:

$$ \varphi(q) = \frac{f_2^5}{f_1^2 f_4^2}, \qquad \chi(q) = \frac{f_2^2}{f_1 f_4} \qquad \text{and} \qquad \chi(-q) = \frac{f_1}{f_2}. \tag{7} $$

From (7), we observe that

$$\frac{\varphi(q)}{\varphi(q^3)} = \frac{\chi^2(q)}{\chi^2(q^3)}\frac{f_2}{f_6}. \tag{8}$$

Before concluding this section, for convenience we write

$$x := x(q) = q^{-1/24}\chi(q) \qquad \text{and} \qquad y := y(q) = q^{-1/8}\chi(q^3).$$

3. Main Results: Somos identities of level 6

Theorem 3.1. *We have*

$$f_1^9 f_3 f_6^4 + 2f_2^9 f_3^4 f_6 + 6q f_1^4 f_2 f_6^9 - 3f_1 f_2^4 f_3^9 = 0.$$

Proof. Multiplying (5) throughout by $2y^{-4}$, we obtain

$$2y^8 + \frac{16x^3}{y} + \frac{2x^{12}}{y^4} - 2x^9y^5 = 0, \tag{9}$$

which is equivalent to

$$y^8\left(\frac{x^9}{y^3}+2\right)\left(1-\frac{4x^3}{y^9}\right) = x^9\left(\frac{2x^3}{y^4}+y^5\right)\left(\frac{4y^3}{x^9}-1\right).$$

By employing (6) in the above, we see that

$$y^8\left(\frac{x^9}{y^3}+2\right) = 3x^9\left(\frac{2x^3}{y^4}+y^5\right)\frac{\varphi^4(q^3)}{\varphi^4(q)}.$$

By using (8) in the above, we obtain

$$\frac{x^9}{y^3}+2 = 3q^{2/3}\left(\frac{2x^4}{y^4}+xy^5\right)\frac{f_6^4}{f_2^4}.$$

Further we have deduced the desired result by substituting q to $-q$ in the above, representing $x(-q)$ and $y(-q)$ by means of f_n through (7) and then multiplying throughout by $f_2^9 f_3^4 f_6$. □

Theorem 3.2. *We have*

$$f_1^5 f_2^5 f_3 f_6 + 8q f_2^6 f_6^6 - f_1^6 f_3^6 - 9q f_1 f_2 f_3^5 f_6^5 = 0.$$

Proof. Multiplying (5) throughout by $(1-16x^{-6}y^{-6})$, we obtain

$$x^{12} - x^9y^9 - 16\frac{x^6}{y^6} - \frac{128}{x^3y^3} - 16\frac{y^6}{x^6} + 24x^3y^3 + y^{12} = 0,$$

which is equivalent to

$$3y^{12}\left(1-\frac{4x^3}{y^9}\right)^2 + 3x^{12}\left(\frac{4y^3}{x^9}-1\right)^2 = x^9y^9\left(1+\frac{8}{x^6y^6}\right)\left(1-\frac{4x^3}{y^9}\right)\left(\frac{4y^3}{x^9}-1\right).$$

By employing (6) in the above, we find that

$$y^{12}\frac{\varphi^4(q)}{\varphi^4(q^3)} + 9x^{12}\frac{\varphi^4(q^3)}{\varphi^4(q)} = x^3y^3(x^6y^6+8).$$

By using (8) in the above, we see that

$$\frac{q^{-2/3}}{xy^5}\frac{f_2^4}{f_6^4}+\frac{9q^{2/3}}{x^5y}\frac{f_6^4}{f_2^4}=1+\frac{8}{x^6y^6}.$$

Further, we have deduced the desired result by substituting q to $-q$ in the above, representing $x(-q)$ and $y(-q)$ by means of f_n through (7) and then multiplying throughout by $f_1^6f_3^6$. □

Theorem 3.3. *We have*

$$f_1^9f_3f_6^4+12qf_1^4f_2f_6^9+3f_1f_2^4f_3^9-4f_2^9f_3^4f_6=0.$$

Proof. The identity (5) is equivalent to

$$\left(\frac{y}{x}\right)^8\left(\left(\frac{x}{y}\right)^8-\frac{4}{xy^5}\right)\left(1-\frac{4x^3}{y^9}\right)=\left(\frac{4x^3}{y^9}-1\right)\left(\frac{4y^3}{x^9}-1\right).$$

By employing (6) in the above, we have

$$\left(\frac{y}{x}\right)^8\left(\left(\frac{x}{y}\right)^8-\frac{4}{xy^5}\right)\frac{\varphi^4(q)}{\varphi^4(q^3)}=3\left(\frac{4x^3}{y^9}-1\right).$$

By using (8) in the above, we find that

$$q^{-2/3}\left(\left(\frac{x}{y}\right)^8-\frac{4}{xy^5}\right)\frac{f_2^4}{f_6^4}=3\left(\frac{4x^3}{y^9}-1\right).$$

Further we have deduced the desired result by substituting q to $-q$ in the above, representing $x(-q)$ and $y(-q)$ by means of f_n through (7) and then multiplying throughout by $f_1f_2^4f_3^9$. □

Theorem 3.4. *We have*

$$16f_2^9f_3^4f_6-7f_1^9f_3f_6^4-72qf_1^4f_2f_6^9-9f_1f_2^4f_3^9=0.$$

Proof. Multiplying (9) throughout by $2x^{-9}y^3$, we obtain

$$\frac{32y^2}{x^6}+\frac{4y^{11}}{x^9}+\frac{4x^3}{y}-4y^8=0,$$

which is equivalent to

$$3\left(\frac{x}{y}\right)^8\left(\frac{4y^3}{x^9}-1\right)\left(\frac{8}{x^5y}-\frac{y^8}{x^8}\right)=\left(7-\frac{16y^3}{x^9}\right)\left(1-\frac{4x^3}{y^9}\right).$$

By employing (6) in the above, we obtain

$$\frac{9x^8}{y^8}\left(\frac{8}{x^5y}-\frac{y^8}{x^8}\right)\frac{\varphi^4(q^3)}{\varphi^4(q)}=7-\frac{16y^3}{x^9}.$$

By using (8) in the above, we find that

$$9q^{2/3}\left(\frac{8}{x^5y}-\frac{y^8}{x^8}\right)\frac{f_6^4}{f_2^4}=7-\frac{16y^3}{x^9}.$$

Further we have deduced the desired result by substituting q to $-q$ in the above, representing $x(-q)$ and $y(-q)$ by means of f_n through (7) and then multiplying throughout by $f_1^9f_3f_6^4$. □

Theorem 3.5. *We have*

$$f_1^9f_3f_6^4+16qf_1^4f_2f_6^9+7f_1f_2^4f_3^9-8f_2^9f_3^4f_6=0.$$

Proof. Multiplying (9) throughout by $2/xy^5$, we deduce that

$$3y^8\left(\frac{x^8}{y^8}-\frac{8}{xy^5}\right)\left(1-\frac{4x^3}{y^9}\right)=x^8\left(\frac{16x^3}{y^9}-7\right)\left(\frac{4y^3}{x^9}-1\right).$$

By employing (6) in the above, we see that

$$y^8\left(\frac{x^8}{y^8}-\frac{8}{xy^5}\right)\frac{\varphi^4(q)}{\varphi^4(q^3)}=x^8\left(\frac{16x^3}{y^9}-7\right).$$

By using (8) in the above, we find that

$$q^{-2/3}\left(\frac{x^8}{y^8}-\frac{8}{xy^5}\right)\frac{f_2^4}{f_6^4}=\frac{16x^3}{y^9}-7.$$

Further we have deduced the desired result by substituting q to $-q$ in the above, representing $x(-q)$ and $y(-q)$ by means of f_n through (7) and then multiplying throughout by $f_1f_2^4f_3^9$. □

Theorem 3.6. *We have*

$$f_1^9f_3f_6^4+f_2^9f_3^4f_6+7qf_1^4f_2f_6^9-2f_1f_2^4f_3^9=0.$$

Proof. Multiplying (9) throughout by $5/2ab^5$, we deduce that

$$3y^8\left(\frac{x^8}{y^8}+\frac{1}{xy^5}\right)\left(1-\frac{4x^3}{y^9}\right)=x^8\left(2+\frac{7x^3}{y^9}\right)\left(\frac{4y^3}{x^9}-1\right).$$

By employing (6) in the above, we see that

$$y^8\left(\frac{x^8}{y^8}+\frac{1}{xy^5}\right)\frac{\varphi^4(q)}{\varphi^4(q^3)}=x^8\left(2+\frac{7x^3}{y^9}\right).$$

By using (8) in the above, we find that

$$q^{-2/3}\left(\frac{x^8}{y^8}+\frac{1}{xy^5}\right)\frac{f_2^4}{f_6^4}=2+\frac{7x^3}{y^9}.$$

Further we have deduced the desired result by substituting q to $-q$ in the above, representing $x(-q)$ and $y(-q)$ by means of f_n through (7) and then multiplying throughout by $f_1f_2^4f_3^9$. □

Theorem 3.7. *We have*

$$2f_1^9f_3f_6^4+7f_2^9f_3^4f_6+9qf_1^4f_2f_6^9-9f_1f_2^4f_3^9=0.$$

Proof. The identity (6) is equivalent to

$$y^8\left(\frac{2x^9}{y^3}+7\right)\left(1-\frac{4x^3}{y^9}\right)=3x^8\left(\frac{x^4}{y^4}+xy^5\right)\left(\frac{4y^3}{x^9}-1\right).$$

By employing (6) in the above, we see that

$$y^8\left(\frac{2x^9}{y^3}+7\right)=9x^8\left(\frac{x^4}{y^4}+xy^5\right)\frac{\varphi^4(q^3)}{\varphi^4(q)}.$$

By using (8) in the above, we find that

$$\frac{2x^9}{y^3}+7=9q^{2/3}\left(\frac{x^4}{y^4}+xy^5\right)\frac{f_6^4}{f_2^4}.$$

Further we have deduced the desired result by substituting q to $-q$ in the above, representing $x(-q)$ and $y(-q)$ by means of f_n through (7) and then multiplying throughout by $f_2^9f_3^4f_6$. □

Theorem 3.8. *We have*

$$f_1^{14}f_6^3+81qf_1f_2^5f_3^7f_6^4-f_1^5f_2^9f_3^3-72qf_2^{10}f_3^2f_6^5=0.$$

Proof. Multiplying (5) throughout by $x^{-6}y^{-9}(16x^9-48y^3-x^6y^9)$, we obtain

$$x^9y^9-17x^{12}+\frac{16x^{15}}{y^9}-\frac{48y^6}{x^6}+56x^3y^3-y^{12}+\frac{80x^6}{y^6}-\frac{384}{x^3y^3}=0,$$

which is equivalent to

$$y^{12}\left(\frac{x^9}{y^3}-1\right)\left(1-\frac{4x^3}{y^9}\right)^2-9x^{12}\left(\frac{4y^3}{x^9}-1\right)^2+24x^3y^3\left(\frac{4y^3}{x^9}-1\right)\left(1-\frac{4x^3}{y^9}\right)=0.$$

By employing (6) in the above, we obtain

$$y^{12}\left(\frac{x^9}{y^3}-1\right)-81x^{12}\frac{\varphi^8(q^3)}{\varphi^8(q)}+72x^3y^3\frac{\varphi^4(q^3)}{\varphi^4(q)}=0.$$

By using (8) in the above, we find that

$$\frac{x^9}{y^3}-1-\frac{81q^{4/3}y^4}{x^4}\frac{f_6^8}{f_2^8}+\frac{72q^{2/3}}{x^5y}\frac{f_6^4}{f_2^4}=0.$$

Further we have deduced the desired result by substituting q to $-q$ in the above, representing $x(-q)$ and $y(-q)$ by means of f_n through (7) and then multiplying throughout by $f_1^5f_2^9f_3^3$. □

Theorem 3.9. *We have*

$$f_1^9f_2^5f_6^3+8f_2^{14}f_3^3-81qf_1^5f_2f_3^4f_6^7-9f_1^{10}f_3^5f_6^2=0.$$

Proof. Multiplying (5) throughout by $y^{-6}(3x^9y^6-16x^3-8y^9)$, we obtain

$$3x^{21}+11x^9y^{12}+32x^{12}y^3-\frac{16x^{15}}{y^6}-80x^3y^6-\frac{128x^6}{y^3}-3x^{18}y^9-8y^{15}=0,$$

which is equivalent to

$$y^{15}\left(\frac{x^9}{y^3}+8\right)\left(1-\frac{4x^3}{y^9}\right)^2=3x^{18}y^9\left(1-\frac{4x^3}{y^9}\right)\left(\frac{4y^3}{x^9}-1\right)-9x^{21}\left(\frac{4y^3}{b^9}-1\right)^2.$$

By employing (6) in the above, we see that

$$y^{15}\left(\frac{x^9}{y^3}+8\right)=9x^{18}y^9\frac{\varphi^4(q^3)}{\varphi^4(q)}-81x^{21}\frac{\varphi^8(q^3)}{\varphi^8(q)}.$$

By using (8) in the above, we find that

$$\frac{x^9}{y^3}+8=9q^{2/3}x^{10}y^2\frac{f_6^4}{f_2^4}-81q^{4/3}x^5y\frac{f_6^8}{f_2^8}.$$

Further we have deduced the desired result by substituting q to $-q$ in the above, representing $x(-q)$ and $y(-q)$ by means of f_n through (7) and then multiplying throughout by $f_2^{14}f_3^3$. □

Theorem 3.10. *We have*

$$f_1^5f_2^2f_3^{10}+qf_2^3f_3^9f_6^5-f_1^4f_2^7f_3^5f_6-8q^2f_1^3f_6^{14}=0.$$

Proof. Multiplying (5) throughout by $x^{-6}(8x^9+16y^3-3x^6y^9)$, we obtain

$$8x^{15}-32x^3y^{12}+3x^9y^{18}-11x^{12}y^9+\frac{16y^{15}}{x^6}+\frac{128y^6}{x^3}+80x^6y^3-3y^{21}=0,$$

which is equivalent to

$$3x^9y^{18}\left(\frac{4y^3}{x^9}-1\right)\left(1-\frac{4x^3}{y^9}\right)-x^{12}\left(\frac{4y^3}{x^9}-1\right)^2(y^9+8x^3)-9y^{21}\left(1-\frac{4x^3}{y^9}\right)^2=0.$$

By employing (6) and then (8) above, we find that

$$q^{2/3}x^5y^{10}\frac{f_6^4}{f_2^4}-x^4y^5-q^{4/3}(y^9+8x^3)\frac{f_6^8}{f_2^8}=0.$$

Further we have deduced the desired result by substituting q to $-q$ in the above, representing $x(-q)$ and $y(-q)$ by means of f_n through (7) and then multiplying throughout by $f_1^4f_2^7f_3^5f_6$. □

Theorem 3.11. *We have*

$$f_2^3f_3^{14}+qf_1^3f_3^5f_6^9-f_1^7f_2^4f_3f_6^5-8qf_1^2f_2^5f_6^{10}=0.$$

Proof. Multiplying (5) throughout by $x^{-9}y^{-2}(48x^3-16y^9+x^9y^6)$, we obtain

$$\frac{16y^{19}}{x^9}+\frac{80y^{10}}{x^6}-17y^{16}+x^9y^{13}+56x^3y^7-x^{12}y^4-\frac{48x^6}{y^2}-\frac{384y}{x^3}=0,$$

which is equivalent to

$$x^9y^4(y^9-x^3)\left(\frac{4y^3}{x^9}-1\right)^2-9y^{16}\left(1-\frac{4x^3}{y^9}\right)^2+24x^3y^7\left(\frac{4y^3}{x^9}-1\right)\left(1-\frac{4x^3}{y^9}\right)=0.$$

By employing (6) in the above, we find that

$$x^9(y^{13}-x^3y^4)\frac{\varphi^8(q^3)}{\varphi^8(q)}-y^{16}+8x^3y^7\frac{\varphi^4(q^3)}{\varphi^4(q)}=0.$$

By using (8) in the above, we obtain

$$q^{4/3}\left(\frac{y^{13}}{x^7}-\frac{y^4}{x^4}\right)\left(\frac{f_6}{f_2}\right)^8-1+\frac{8q^{2/3}}{x^5y}\left(\frac{f_6}{f_2}\right)^4=0.$$

Further we have deduced the desired result by substituting q to $-q$ in the above, representing $x(-q)$ and $y(-q)$ by means of f_n through (7) and then multiplying throughout by $f_1^7f_2^4f_3f_6^5$. □

Theorem 3.12. *We have*

$$f_1^{13}f_3^2f_6^3+9qf_2^5f_3^9f_6^4-f_1^4f_2^9f_3^5-72q^2f_1^3f_2^2f_6^{13}=0.$$

Proof. Multiplying (5) throughout by $x^{-6}y^{-5}(8x^9 - 16y^3 - x^6y^9)$, we obtain

$$x^9y^{13} - 9x^{12}y^4 + \frac{8x^{15}}{y^5} + \frac{48x^6}{y^2} - \frac{16y^{10}}{x^6} - \frac{128b}{x^3} + 16x^3y^7 - y^{16} = 0,$$

which is equivalent to

$$y^{16}\left(\frac{x^9}{y^3} - 1\right)\left(1 - \frac{4x^3}{y^9}\right)^2 = x^{16}\left(\frac{y^4}{x^4} + \frac{8}{xy^5}\right)\left(\frac{4y^3}{x^9} - 1\right)^2.$$

By employing (6) in the above, we deduce

$$y^{16}\left(\frac{x^9}{y^3} - 1\right) - 9x^{16}\left(\frac{y^4}{x^4} + \frac{8}{xy^5}\right)\frac{\varphi^8(q^3)}{\varphi^8(q)} = 0.$$

By using (8) in the above, we obtain

$$\left(\frac{x^9}{y^3} - 1\right) - 9q^{4/3}\left(\frac{y^4}{x^4} + \frac{8}{xy^5}\right)\frac{f_6^8}{f_2^8} = 0.$$

Further we have deduced the desired result by substituting q to $-q$ in the above, representing $x(-q)$ and $y(-q)$ by means of f_n through (7) and then multiplying throughout by $f_1^4f_2^9f_3^5$. □

Theorem 3.13. *We have*

$$f_1^{14}f_6^4 + 72qf_1f_2^5f_3^7f_6^5 - f_1^6f_2^4f_3^8 - 64qf_2^{10}f_3^2f_6^6 = 0.$$

Proof. Multiplying (5) throughout by $4x^{-15}y^{-9}(12x^9 - 32y^3 - x^6y^9)$, we obtain

$$52x^3 - \frac{48x^6}{y^9} + \frac{128y^6}{x^{15}} - \frac{144y^3}{x^6} + \frac{1024}{x^{12}y^3} - \frac{256}{x^3y^6} + \frac{4y^{12}}{x^9} - 4y^9 = 0,$$

which is equivalent to

$$3y^9\left(1 - \frac{4x^3}{y^9}\right)^2 - 24x^3\left(\frac{4y^3}{x^9} - 1\right)^2 + y^9\left(\frac{64}{x^6y^6} - 1\right)\left(\frac{4y^3}{x^9} - 1\right)\left(1 - \frac{4x^3}{y^9}\right) = 0.$$

By employing (6) in the above, we find that

$$\frac{\varphi^4(q)}{\varphi^4(q^3)} - \frac{72x^3}{y^9}\frac{\varphi^4(q^3)}{\varphi^4(q)} + \frac{64}{x^6y^6} - 1 = 0.$$

By using (8) in the above, we see that

$$q^{-2/3}\left(\frac{x}{y}\right)^8\frac{f_2^4}{f_6^4} - \frac{72q^{2/3}}{x^5y}\frac{f_6^4}{f_2^4} + \frac{64}{x^6y^6} - 1 = 0.$$

Further we have deduced the desired result by substituting q to $-q$ in the above, representing $x(-q)$ and $y(-q)$ by means of f_n through (7) and then multiplying throughout by $f_1^6f_2^4f_3^8$. □

Theorem 3.14. *We have*

$$f_1^{10}f_3^6f_6^2 + qf_1^4f_2^6f_6^8 + 9qf_1^5f_2f_3^5f_6^7 - f_2^{14}f_3^4 = 0.$$

Proof. Multiplying (5) throughout by $y^{-6}(x^9y^6 - 4x^3 - 3y^9)$, we obtain

$$x^{21} + 4x^9y^{12} + 9x^{12}y^3 - \frac{32x^6}{y^3} - \frac{4x^{15}}{y^6} - x^{18}y^9 - 28x^3y^6 - 3y^{15} = 0,$$

which is equivalent to

$$x^{12}y^3(x^6y^6 - 1)\left(\frac{4y^3}{x^9} - 1\right)\left(1 - \frac{4x^3}{y^9}\right) - 3x^{21}\left(\frac{4y^3}{x^9} - 1\right)^2 - 3y^{15}\left(1 - \frac{4x^3}{y^9}\right)^2 = 0.$$

By employing (6) in the above, we find that

$$x^{12}y^3(x^6y^6 - 1)\frac{\varphi^4(q^3)}{\varphi^4(q)} - 9x^{21}\frac{\varphi^8(q)}{\varphi^8(q)} - y^{15} = 0.$$

By using (8) in the above, we obtain

$$q^{2/3}x^4(x^6y^6 - 1)\frac{f_6^4}{f_2^4} - 9q^{4/3}x^5y^5\frac{f_6^8}{f_2^8} - y^4 = 0.$$

Further we have deduced the desired result by substituting q to $-q$ in the above, representing $x(-q)$ and $y(-q)$ by means of f_n through (7) and then multiplying throughout by $f_2^{14}f_3^4$. □

Theorem 3.15. *We have*

$$f_1^6f_2^2f_3^{10} + qf_2^8f_3^4f_6^6 - f_1^5f_2^7f_3^5f_6 - 9q^2f_1^4f_6^{14} = 0.$$

Proof. Multiplying (5) throughout by $x^{-6}(3x^9 + 4y^3 - x^6y^9)$, we obtain

$$y^{21} + 9x^3y^{12} - x^9y^{18} - \frac{4y^{15}}{x^6} - \frac{32y^6}{x^3} + 4x^{12}y^9 - 28x^6y^3 - 3x^{15} = 0,$$

which is equivalent to

$$x^3y^{12}(x^6y^6 - 1)\left(\frac{4y^3}{x^9} - 1\right)\left(1 - \frac{4x^3}{y^9}\right) - 3y^{21}\left(1 - \frac{4x^3}{y^9}\right)^2 - 3x^{15}\left(\frac{4y^3}{x^9} - 1\right)^2 = 0.$$

By employing (6) in the above, we find that

$$x^6y^{12}(x^6y^6 - 1) - y^{21}\frac{\varphi^4(q)}{\varphi^4(q^3)} - 9x^{15}\frac{\varphi^4(q^3)}{\varphi^4(q)} = 0.$$

By using (8) in the above, we deduce

$$(x^6y^6 - 1)y^4 - q^{-2/3}x^5y^5\frac{f_2^4}{f_6^4} - 9q^{2/3}x^4\frac{f_6^4}{f_2^4} = 0.$$

Further we have deduced the desired result by substituting q to $-q$ in the above, representing $x(-q)$ and $y(-q)$ by means of f_n through (7) and then multiplying throughout by $f_1^6f_2^2f_3^{10}$. □

Remark:

Since the proof of Somos Dedekind η-function identities of level 6 are similar, we omit the proof of the following identities.

$$f_1^9 f_2^4 f_6^5 + 8 f_2^{13} f_3^3 f_6^2 - 9q f_1^5 f_3^4 f_6^9 - 9 f_1^2 f_2^3 f_3^{13} = 0.$$
$$f_1^8 f_3^6 f_6^4 + 64q f_1^2 f_2^6 f_6^{10} + 8 f_1^7 f_2^5 f_3 f_6^5 - 9 f_2^4 f_3^{14} = 0.$$
$$f_1^{14} f_3^2 f_6^4 + 8q f_2^{10} f_3^4 f_6^6 - f_1^6 f_2^4 f_3^{10} - 72q^2 f_1^4 f_2^2 f_6^{14} = 0.$$
$$f_1^{10} f_3^6 f_6^4 + 8 f_2^{14} f_3^4 f_6^2 - 8q f_1^4 f_2^6 f_6^{10} - 9 f_1^2 f_2^4 f_3^{14} = 0.$$
$$f_1^{18} f_6^6 + 81q f_1^5 f_2^5 f_3^7 f_6^7 - f_2^{18} f_3^6 - 63q f_1^4 f_2^{10} f_3^2 f_6^8 = 0.$$
$$f_1^{18} f_6^6 + 63 f_1^{10} f_2^4 f_3^8 f_6^2 + 648q f_1^5 f_2^5 f_3^7 f_6^7 - 64 f_2^{18} f_3^6 = 0.$$
$$8 f_2^{17} f_3^7 + q f_1^{13} f_6^{11} - 8 f_1^{10} f_2^3 f_3^9 f_6^2 - 81q f_1^5 f_2^4 f_3^8 f_6^7 = 0.$$
$$8 f_1^9 f_2^8 f_3^4 f_6^3 + 81q f_1^5 f_2^4 f_3^8 f_6^7 - 8 f_2^{17} f_3^7 - 9q f_1^{13} f_6^{11} = 0.$$
$$f_2^{16} f_3^8 + 9 f_1^{10} f_2^2 f_3^{10} f_6^2 - 10 f_1^9 f_2^7 f_3^5 f_6^3 - 81q^2 f_1^8 f_6^{16} = 0.$$
$$f_1^{10} f_2^2 f_3^{10} f_6^2 + q^2 f_1^8 f_6^{16} + 10q f_1^5 f_2^3 f_3^9 f_6^7 - f_2^{16} f_3^8 = 0.$$

and many more.

4. Application of Somos identities to Colored Partitions

The identities discovered by Somos which have been proved here in Section 3 have applications to the theory of partitions. To illustrate this, we present partition-theoretic interpretations for Theorem 3.1 and 3.2. For the sake of simplicity, we adopt this standard notation

$$(a_1, a_2, \cdots, a_n; q)_\infty := (a_1; q)_\infty (a_2; q)_\infty \dots (a_n; q)_\infty,$$

and define

$$(q_k^{r\pm}; q^s)_\infty := (q^r, q^{s-r}; q^s)_\infty^k, \qquad (r < s); r, s \in \mathbb{N}. \tag{10}$$

For example, $(q_4^{2\pm}; q^8)_\infty$ means $(q^2, q^6; q^8)_\infty^4$ which is $(q^2; q^8)_\infty^4 \, (q^6; q^8)_\infty^4$.

Now we define colored partition as given in the literature. "A positive integer n has l colors if there are l copies of n available colors and all of them are viewed as distinct objects. Partitions of a positive integer into parts with colors are called colored partitions."

For example, if 1 is allowed to have 2 colors, then all the colored partitions of 2 are 2, $1_r + 1_r$, $1_g + 1_g$ and $1_r + 1_g$, where we use the indices r (red) and g (green) to distinguish the two colors of 1. Also

$$(q^a; q^b)_\infty^{-m},$$

is known as the generating function for the number of partitions of n where all the parts are congruent to a modulo b and have m colors.

Theorem 4.1. *If $\alpha(n)$ represent the number of partitions of n being divided into parts congruent to ± 1 or ± 3 modulo 6 with nine and six colors respectively. If $\beta(n)$ indicates the number of partitions of n being split into parts congruent to $\pm 1, \pm 2$ or $+3$ modulo 6 with five, four and six colors respectively. If $\gamma(n)$ is taken to represent the number of partitions of n into several parts congruent to ± 1 or ± 2 modulo 6 with eight and four colors respectively. Then the following identity holds true.*

$$2\alpha(n) + 6\beta(n-1) - 3\gamma(n) = 0, \qquad n \geq 1.$$

Proof. On dividing Theorem 3.1 by $f_1^9 f_2^9 f_3^9 f_6^9$, simplifying subject to common base q^6 and then using (10), we deduce

$$1 + \frac{2}{(q_9^{1\pm}, q_6^{3+}; q^6)_\infty} + \frac{6q}{(q_5^{1\pm}, q_4^{2\pm}, q_6^{3+}; q^6)_\infty} - \frac{3}{(q_8^{1\pm}, q_4^{2\pm}; q^6)_\infty} = 0.$$

We observe that the above identity generates $\alpha(n)$, $\beta(n)$ and $\gamma(n)$ as the generating functions and hence we have

$$1 + 2\sum_{n=0}^{\infty} \alpha(n)q^n + 6q\sum_{n=0}^{\infty} \beta(n)q^n - 3\sum_{n=0}^{\infty} \gamma(n)q^n = 0,$$

where we set $\alpha(0) = \beta(0) = \gamma(0) = 1$. Further on extracting the coefficients of q^n in the above, we obtain the desired result. □

Now we verify the result when $n = 2$ for Theorem 4.1.

$\alpha(2) = 45$:	$1_r + 1_r, 1_g + 1_g, 1_y + 1_y, 1_w + 1_w, 1_{bl} + 1_{bl}, 1_o + 1_o, 1_b + 1_b,$ $1_i + 1_i, 1_p + 1_p, 1_r + 1_g, 1_r + 1_y, 1_r + 1_w, 1_r + 1_{bl}, 1_r + 1_o,$ $1_r + 1_b, 1_r + 1_i, 1_r + 1_p, 1_g + 1_y, 1_g + 1_w, 1_g + 1_{bl}, 1_g + 1_o,$ $1_g + 1_b, 1_g + 1_i, 1_g + 1_p, 1_y + 1_w, 1_y + 1_{bl}, 1_y + 1_o, 1_y + 1_b,$ $1_y + 1_i, 1_y + 1_p, 1_w + 1_{bl}, 1_w + 1_o, 1_w + 1_b, 1_w + 1_i, 1_w + 1_p,$ $1_{bl} + 1_o, 1_{bl} + 1_b, 1_{bl} + 1_i, 1_{bl} + 1_p, 1_o + 1_b, 1_o + 1_i, 1_o + 1_p,$ $1_b + 1_i, 1_b + 1_p, 1_i + 1_p.$
$\beta(1) = 5$:	$1_r, 1_g, 1_y, 1_w, 1_p.$
$\gamma(2) = 40$:	$2_r, 2_o, 2_w, 2_g, 1_r + 1_r, 1_g + 1_g, 1_y + 1_y, 1_w + 1_w, 1_{bl} + 1_{bl}, 1_o + 1_o,$ $1_b + 1_b, 1_i + 1_i, 1_r + 1_g, 1_r + 1_y, 1_r + 1_w, 1_r + 1_{bl}, 1_r + 1_o, 1_r + 1_b,$ $1_r + 1_i, 1_g + 1_y, 1_g + 1_w, 1_g + 1_{bl}, 1_g + 1_o, 1_g + 1_b, 1_g + 1_i, 1_y + 1_w,$ $1_y + 1_{bl}, 1_y + 1_o, 1_y + 1_b, 1_y + 1_i, 1_w + 1_{bl}, 1_w + 1_o, 1_w + 1_b, 1_w + 1_i,$ $1_{bl} + 1_o, 1_{bl} + 1_b, 1_{bl} + 1_i, 1_o + 1_b, 1_o + 1_i, 1_b + 1_i.$

Theorem 4.2. *If $\alpha(n)$ is taken to represent the number of partitions of n being divided into parts that are congruent to ±1 or $+3$ modulo 6 with one and six colors respectively. If $\beta(n)$ is taken to be the number of partitions of n being split into parts which are congruent to $\pm1, \pm2$ or $+3$ modulo 6 with six, four and twelve colors respectively. If $\gamma(n)$ stands for the number of partitions of n into different parts that are congruent to ±2 modulo 6 with four colors. If $\delta(n)$ is chosen to represent the number of partitions of n into many parts that are congruent to $\pm1, \pm2$ or $+3$ modulo 6 with five, eight and six colors respectively. Then the following relation holds true.*

$$\alpha(n) + 8\beta(n-1) - \gamma(n) - 9\delta(n-1) = 0, \qquad n \geq 1.$$

Proof. Dividing Theorem 3.2 by $f_1^6 f_2^6 f_3^6 f_6^6$, simplifying subject to the common base q^6 and then using (10), we deduce

$$\frac{1}{(q_1^{1\pm}, q_6^{3+}; q^6)_\infty} + \frac{8q}{(q_6^{1\pm}, q_4^{2\pm}, q_{12}^{3+}; q^6)_\infty} - \frac{1}{(q_4^{2\pm}; q^6)_\infty} - \frac{9q}{(q_5^{1\pm}, q_8^{2\pm}, q_6^{3+}; q^6)_\infty} = 0.$$

We observe that the above identity generates $\alpha(n)$, $\beta(n)$, $\gamma(n)$ and $\delta(n)$ as the generating functions and hence we have

$$\sum_{n=0}^{\infty} \alpha(n)q^n + 8q\sum_{n=0}^{\infty} \beta(n)q^n - \sum_{n=0}^{\infty} \gamma(n)q^n - 9q\sum_{n=0}^{\infty} \delta(n)q^n = 0,$$

where we set $\alpha(0) = \beta(0) = \gamma(0) = \delta(0) = 1$. On equating the coefficients of q^n in the above, yields the desired result. □

Now we verify the case when $n = 3$ for Theorem 4.2.

$\alpha(3) = 7$:	$1 + 1 + 1, 3_r, 3_g, 3_y, 3_o, 3_b, 3_w$.
$\beta(2) = 25$:	$1_r + 1_r, 1_g + 1_g, 1_y + 1_y, 1_o + 1_o, 1_b + 1_b, 1_w + 1_w, 1_r + 1_y, 1_r + 1_g, 1_r + 1_b, 1_r + 1_o, 1_r + 1_w, 1_g + 1_o, 1_g + 1_y, 1_g + 1_b, 1_g + 1_w, 1_y + 1_o, 1_y + 1_w, 1_y + 1_b, 1_o + 1_w, 1_o + 1_b, 1_b + 1_w, 2_r, 2_g, 2_y, 2_b$.
$\gamma(3) = 0$:	
$\delta(2) = 23$:	$1_r + 1_r, 1_g + 1_g, 1_y + 1_y, 1_o + 1_o, 1_b + 1_b, 1_r + 1_y, 1_r + 1_g, 1_r + 1_o, 1_r + 1_b, 1_g + 1_o, 1_g + 1_y, 1_g + 1_b, 1_y + 1_o, 1_y + 1_b, 1_o + 1_b, 2_r, 2_g, 2_y, 2_b, 2_o, 2_w, 2_{bl}, 2_v$.

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