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CR-Submanifolds of $(LCS)_n$-Manifolds with Respect to Quarter Symmetric Non-Metric Connection

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Abstract. The present paper deals with the study of CR-submanifolds of $(LCS)_n$-manifolds with respect to quarter symmetric non-metric connection. We investigate integrability of the distributions and the geometry of foliations. The totally umbilical CR-submanifolds of said ambient manifolds are also studied. An example is presented to illustrate the results.

1. Introduction

Quarter symmetric linear connection on smooth manifolds $\tilde{M}$ introduced in [12], is a linear connection $\bar{\nabla}$ such that its torsion tensor T is of the form

$$T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y \tag{1}$$

where η is an 1-form and ϕ is a $(1,1)$ type tensor. If $\phi X = X$, in particular then it reduces to semisymmetric connection introduced in [11]. Further, if $(\bar{\nabla}_X g)(Y,Z) \neq 0$ for all $X,\ Y,\ Z \in \chi(\tilde{M})$, then $\bar{\nabla}$ is said to be a quarter symmetric non-metric connection.

Lorentzian concircular structure manifolds (briefly, $(LCS)_n$-manifolds) introduced in [29] as a generalisation of LP-Sasakian manifold [25], has many applications in the general theory of relativity and cosmology ([32], [33]). In [23] it has shown that LCS- spacetimes coincide with generalised Robertson-Walker spacetimes. So, these manifolds are interesting for geometry as well as for physics. For detail study of this type of manifolds we may refer to ([7], [13], [30], [31], [35], [36], [37], [38]) and for study of submanifolds of $(LCS)_n$-manifolds we may refer ([4], [14]-[21], [39]).

CR-submanifolds was introduced by Bejancu in [5]. There are several research papers (see [3], [5], [8], [22], [28], [31]) on geometry of CR- submanifolds. Cohomology of CR-submanifolds is studied in [2], [9], [10]. In the present paper we have studied curvature properties and CR-submanifolds of $(LCS)_n$-manifolds $\tilde{M}$ with respect to quarter symmetric non-metric connection $\bar{\nabla}$. The totally umbilical CR-submanifolds of $\tilde{M}$ is also studied. Finally, we have presented an example of a submanifold of a $(LCS)_5$-manifold to illustrate the results.

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2. Preliminaries

A Lorentzian concircular structure manifold (briefly $(LCS)_n$-manifold) is a Lorentzian manifold $\tilde{M}$ of dimension n endowed with the unit timelike concircular vector field ξ, its associated 1-form η and an $(1,1)$ tensor field ϕ such that

$$\tilde{\nabla}_X \xi = \alpha \phi X, \tag{2}$$

α being a non-zero scalar function satisfying

$$\tilde{\nabla}_X \alpha = (X\alpha) = d\alpha(X) = \rho\eta(X) \tag{3}$$

where $\rho = -(\xi\alpha)$ is another scalar, and $\tilde{\nabla}$ is the Levi-Civita connection of the Lorentzian metric g. For $\alpha = 1$, a $(LCS)_n$-manifold reduces to the LP-Sasakian manifold ([25], [34]).
In a $(LCS)_n$-manifold $(n > 2)$ $\tilde{M}$, the following relations hold [29]:

$$\eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{4}$$

$$\phi^2 X = X + \eta(X)\xi, \tag{5}$$

$$(\tilde{\nabla}_X \eta)(Y) = \alpha\{g(X,Y) + \eta(X)\eta(Y)\}, \quad (\alpha \neq 0), \tag{6}$$

$$(\tilde{\nabla}_X \phi)Y = \alpha\{g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}, \tag{7}$$

$$(X\rho) = d\rho(X) = \beta\eta(X) \tag{8}$$

for all $X,\ Y,\ Z \in \Gamma(T\bar{M})$ and $\beta = -(\xi\rho)$ is a scalar function.

Let M be a submanifold of dimension m of a $(LCS)_n$-manifold $\tilde{M}$ $(m < n)$ with induced metric g and induced connections ∇ and $\nabla^\perp$ on TM and $T^\perp M$, respectively. Then for$X \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, Gauss and Weingarten formulae are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{9}$$

and

$$\tilde{\nabla}_X V = -A_V X + \nabla^\perp_X V \tag{10}$$

respectively, where h and A_V are second fundamental form and shape operator for the immersion of M satisfying the relation [40]

$$g(h(X,Y),V) = g(A_V X, Y). \tag{11}$$

M is totally umbilical if

$$h(X,Y) = g(X,Y)H \tag{12}$$

for each $X, Y \in \Gamma(TM)$, where H is the mean curvature vector on M and M becomes minimal if $H \equiv 0$, totally geodesic if $h \equiv 0$. Throughout the paper we have taken M is a submanifold of $\tilde{M}$.

Definition 2.1. *[6] A submanifold M of $\tilde{M}$ is called a CR-submanifold if ξ is tangent to M and there is a differential distribution D and its orthogonal complementary distribution $D^\perp$ such that*
(i)$\phi(D) \subseteq D$ and
(ii)$\phi(D^\perp) \subseteq T^\perp M$.

Here, D (resp. $D^\perp$) is called horizontal (resp. vertical) distribution. M is called ξ-horizontal (resp. ξ-vertical) if $\xi \in D$ (resp. $\xi \in D^\perp$). Now we have

$$TM = D \oplus D^\perp, \text{ and } T^\perp M = \phi(D^\perp) \oplus \mu, \tag{13}$$

where μ is a normal subbundle invariant to ϕ. For $X \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, we write

$$X = PX + QX, \tag{14}$$

and

$$\phi V = BV + CV \tag{15}$$

where $PX \in D$, $QX \in D^\perp$, $BV = \tan(\phi V)$ and $CV = \mathrm{nor}(\phi V)$.

3. $(LCS)_n$-manifolds with respect to quarter symmetric non-metric connection

We consider a linear connection $\bar{\tilde{\nabla}}$ on $\tilde{M}$ by

$$\bar{\tilde{\nabla}}_X Y = \tilde{\nabla}_X Y + \eta(Y)\phi X + a(X)\phi Y \tag{16}$$

where a is an 1-form associated to a vector field A on $\tilde{M}$ by

$$g(X, A) = a(X) \tag{17}$$

for every $X \in \chi(\tilde{M})$. If $\bar{\tilde{T}}$ be the torsion tensor of $\tilde{M}$ with respect to $\bar{\tilde{\nabla}}$, then from (16), we find

$$\bar{\tilde{T}}(X, Y) = \eta(Y)\phi X - \eta(X)\phi Y + a(X)\phi Y - a(Y)\phi X. \tag{18}$$

Furthermore

$$(\bar{\tilde{\nabla}}_X g)(Y, Z) = -\eta(Y)g(\phi X, Z) - \eta(Z)g(\phi X, Y) - 2a(X)g(\phi Y, Z). \tag{19}$$

Thus $\bar{\tilde{\nabla}}$, given in (16) which satisfies (18) and (19) is a quarter symmetric non-metric connection. The existence and uniqueness of such connection has shown in [27] for LP-Sasakian manifolds.
Let the curvature tensor of $\tilde{M}$ with respect to $\bar{\tilde{\nabla}}$ and $\tilde{\nabla}$ be $\bar{\tilde{R}}$ and $\tilde{R}$ respectively. Then we find

$$\begin{aligned}\bar{\tilde{R}}(X, Y)Z = {} & \tilde{R}(X, Y)Z + \alpha[g(\phi X, Z)\phi Y - g(\phi Y, Z)\phi X + \eta(Y)\eta(Z)X \\ & -\eta(X)\eta(Z)Y + a(Y)g(X, Z)\xi - a(X)g(Y, Z)\xi] \\ & +(2\alpha - 1)[a(Y)\eta(X)\eta(Z)\xi - a(X)\eta(Y)\eta(Z)\xi] \\ & +(\alpha - 1)[a(Y)\eta(Z)X - a(X)\eta(Z)Y] + da(X, Y)\phi Z.\end{aligned} \tag{20}$$

After contraction we obtain the Ricci tensor $\bar{\tilde{S}}$ as

$$\begin{aligned}\bar{\tilde{S}}(Y, Z) = {} & \tilde{S}(Y, Z) + \alpha\{1 - a(\xi)\}g(Y, Z) - \alpha\lambda g(\phi Y, Z) \\ + {} & \{n\alpha - (2\alpha - 1)a(\xi)\}\eta(Y)\eta(Z) + (n - 2)(\alpha - 1)a(Y)\eta(Z) + da(Y, \phi Z)\end{aligned} \tag{21}$$

and the scalar curvature $\bar{\tilde{r}}$ as

$$\bar{\tilde{r}} = \tilde{r} - (n - 1)a(\xi) - \lambda^2 + \mu \tag{22}$$

where $\lambda = trace\ \phi$ and $\mu = trace\ da$. Thus we have the following:

Theorem 3.1. *$\bar{\tilde{R}}$, $\bar{\tilde{S}}$ and $\bar{\tilde{r}}$ of $\tilde{M}$ with respect to $\bar{\tilde{\nabla}}$ are given in (20), (21) and (22) respectively.*

For $\tilde{M}$ with respect to $\bar{\tilde{\nabla}}$, we get

$$(\bar{\tilde{\nabla}}_X\phi)Y = \alpha g(X, Y)\xi + (\alpha - 1)\eta(Y)X + (2\alpha - 1)\eta(X)\eta(Y)\xi \tag{23}$$

and

$$\bar{\tilde{\nabla}}_X\xi = (\alpha - 1)\phi X. \tag{24}$$

4. CR-submanifolds M of $(LCS)_n$-manifold $\tilde{M}$ with respect to $\bar{\tilde{\nabla}}$

Let ∇ be the induced connection on M from the connection $\tilde{\nabla}$ and $\bar{\nabla}$ be the induced connection on M from the connection $\bar{\tilde{\nabla}}$. Let h and $\bar{h}$ be second fundamental form with respect to ∇ and $\bar{\nabla}$ respectively.
Then we have

$$\bar{\tilde{\nabla}}_X Y = \bar{\nabla}_X Y + \bar{h}(X, Y). \tag{25}$$

From (9), (16) and (25), we get

$$\bar{\nabla}_X Y + \bar{h}(X, Y) = \nabla_X Y + h(X, Y) + \eta(Y)\phi X + a(X)\phi Y. \tag{26}$$

Using (14) in (26), we get

$$\begin{aligned} P\bar{\nabla}_X Y + Q\bar{\nabla}_X Y + \bar{h}(X, Y) &= P\nabla_X Y + Q\nabla_X Y + h(X, Y) + \eta(Y)\phi PX \\ &\quad +\eta(Y)\phi QX + a(X)\phi PY + a(X)\phi QY. \end{aligned} \tag{27}$$

Comparing horizontal, vertical and normal part from both sides, we get

$$P\bar{\nabla}_X Y = P\nabla_X Y + \eta(Y)\phi PX + a(X)\phi PY, \tag{28}$$

$$Q\bar{\nabla}_X Y = Q\nabla_X Y, \tag{29}$$

$$\bar{h}(X, Y) = h(X, Y) + \eta(Y)\phi QX + a(X)\phi QY. \tag{30}$$

Now if $X,\ Y \in D$ then we obtain from (26) that

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X + a(X)\phi Y \tag{31}$$

and

$$\bar{h}(X, Y) = h(X, Y). \tag{32}$$

For $X,\ \xi \in D$, $\bar{h}(X, \xi) = h(X, \xi) = 0$. which means that $\bar{\nabla}$ is a quarter symmetric non-metric connection and the second fundamental forms are equal. This leads to the following:

Proposition 4.1. *If M is an invariant submanifold of $\tilde{M}$ admitting $\bar{\bar{\nabla}}$, then*
(i) The induced connection $\bar{\nabla}$ on M is also quarter symmetric non-metric.
(ii) The second fundamental forms h and $\bar{h}$ are equal.

Again if $Z,\ \ W \in D^{\perp}$, then we have

$$\bar{\nabla}_Z W = \nabla_Z W, \tag{33}$$

i.e., both the connections are identical and

$$\bar{h}(Z, W) = h(Z, W) + \eta(W)\phi Z + a(Z)\phi W. \tag{34}$$

If $X \in D$ and $Z \in D^{\perp}$ then

$$\bar{\nabla}_X Z = \nabla_X Z + \eta(Z)\phi X, \tag{35}$$

$$\bar{h}(X, Z) = h(X, Z) + a(X)\phi Z. \tag{36}$$

Again for $X \in TM$ and $V \in T^{\perp}M$ from Weingarten formula for quarter symmetric non-metric connection, we have

$$\bar{\bar{\nabla}}_X V = -\bar{A}_V X + \bar{\nabla}^{\perp}_X V. \tag{37}$$

Also from (10), (15) and (16), we get

$$\bar{\bar{\nabla}}_X V = -A_V X + \nabla^{\perp}_X V + a(X)BV + a(X)CV. \tag{38}$$

Thus from (37) and (38), we get

$$\bar{A}_V X = A_V X - a(X)BV \tag{39}$$

and

$$\bar{\nabla}^{\perp}_X V = \nabla^{\perp}_X V + a(X)CV. \tag{40}$$

Now, for $Z \in D^{\perp}$, $\phi Z \in T^{\perp}M$ and hence for any $X \in TM$, we get

$$\bar{\bar{\nabla}}_X \phi Z = -A_{\phi Z}X + a(X)\{\nabla^{\perp}_X \phi Z + Z + \eta(Z)\xi\}, \tag{41}$$

from which, we get

$$\bar{A}_{\phi Z}X = A_{\phi Z}X - a(X)\{Z + \eta(Z)\}\xi, \tag{42}$$

and

$$\bar{\nabla}^{\perp}_X \phi Z = \nabla^{\perp}_X \phi Z. \tag{43}$$

Lemma 4.2. *Let M be a CR-submanifold of $\tilde{M}$ with respect to quarter symmetric non-metric connection. Then*

$$P\bar{\nabla}_X \phi PY - P\bar{A}_{\phi QY}X = \phi P(\bar{\nabla}_X Y) + \alpha g(X,Y)P\xi + (\alpha-1)\eta(Y)PX + (2\alpha-1)\eta(X)\eta(Y)P\xi \tag{44}$$

$$Q\bar{\nabla}_X \phi PY - Q\bar{A}_{\phi QY}X = B\bar{h}(X,Y) + \alpha g(X,Y)Q\xi + (\alpha-1)\eta(Y)QX + (2\alpha-1)\eta(X)\eta(Y)Q\xi \tag{45}$$

$$\bar{h}(X,\phi PY) + \bar{\nabla}^{\perp}_X \phi QY = \phi(Q\bar{\nabla}_X Y) + C\bar{h}(X,Y) \tag{46}$$

for all $X,\ Y \in TM$.

Proof. From (23), we get

$$\bar{\bar{\nabla}}_X \phi Y - \phi(\bar{\bar{\nabla}}_X Y) = \alpha g(X,Y)\xi + (\alpha-1)\eta(Y)X + (2\alpha-1)\eta(X)\eta(Y)\xi.$$

Using (14), (15), (25) and (37) in above equation, we get

$$\begin{aligned} &P\bar{\nabla}_X \phi PY + Q\bar{\nabla}_X \phi PY + \bar{h}(X,\phi PY) - P\bar{A}_{\phi QY}X - Q\bar{A}_{\phi QY}X \\ &+\bar{\nabla}^{\perp}_X \phi QY - \phi(P\bar{\nabla}_X Y) - \phi(Q\bar{\nabla}_X Y) - B\bar{h}(X,Y) - C\bar{h}(X,Y) = \\ &\alpha g(X,Y)P\xi + g(X,Y)Q\xi + (\alpha-1)\eta(Y)PX + (\alpha-1)\eta(Y)QX \\ &\qquad +(2\alpha-1)\eta(X)\eta(Y)P\xi + (2\alpha-1)\eta(X)\eta(Y)Q\xi. \end{aligned} \tag{47}$$

Equating horizontal, vertical and normal components of (47), the result follows. □

5. Integrability of the distributions

Lemma 5.1. *Let M be a CR-submanifold of $\tilde{M}$ with respect to $\bar{\bar{\nabla}}$. Then*

$$\begin{aligned} \phi P[W,Z] = {}& A_{\phi W}Z - A_{\phi Z}W + [a(W)Z - a(Z)W] + [a(W)\eta(Z) - a(Z)\eta(W)]\xi \\ & +(\alpha-1)[\eta(W)Z - \eta(Z)W] \end{aligned} \tag{48}$$

for all $W,\ Z \in D^{\perp}$.

Proof. For any $W,\ Z \in D^{\perp}$ we have

$$\bar{\bar{\nabla}}_Z \phi W = (\bar{\bar{\nabla}}_Z \phi)W + \phi(\bar{\bar{\nabla}}_Z W).$$

Using (14), (15), (23), (25) and (37) in above equation, we get

$$\begin{aligned} \bar{\bar{\nabla}}^{\perp}_Z \phi W = {}& \bar{A}_{\phi W}Z + \phi P(\bar{\nabla}_Z W) + \phi(Q\bar{\nabla}_Z W) + B\bar{h}(W,Z) + C\bar{h}(W,Z) \\ & +\alpha g(W,Z)\xi + (\alpha-1)\eta(W)Z + (2\alpha-1)\eta(Z)\eta(W)\xi. \end{aligned} \tag{49}$$

Also from (46), we get

$$\bar{\bar{\nabla}}^{\perp}_Z \phi W = \phi(Q\bar{\nabla}_Z W) + C\bar{h}(Z,W). \tag{50}$$

From (49) and (50), we get

$$\phi(P\bar{\nabla}_Z W) = -\bar{A}_{\phi W}Z - B\bar{h}(W,Z) - \alpha g(W,Z)\xi - (\alpha-1)\eta(Y)Z - (2\alpha-1)\eta(Z)\eta(W)\xi \tag{51}$$

which implies that

$$\phi P[W,Z] \quad = \quad \bar{A}_{\phi W}Z - \bar{A}_{\phi Z}W + (\alpha - 1)\{\eta(W)Z - \eta(Z)W\}. \tag{52}$$

In view of (42), (52) yields

$$\phi P[W,Z] \quad = \quad A_{\phi W}Z - A_{\phi Z}W - a(Z)[W + \eta(Y)\xi] + a(W)[Z + \eta(Z)\xi] + (\alpha - 1)[\eta(W)Z - \eta(Z)W], \tag{53}$$

from which (48) follows. □

Theorem 5.2. *Let M be a CR-submanifold of $\tilde{M}$ with respect to $\bar{\tilde{\nabla}}$. Then the distribution $D^{\perp}$ is integrable if and only if*

$$A_{\phi W}Z - A_{\phi Z}W \quad = \quad a(Z)W - a(W)Z + (a(Z)\eta(W) - a(W)\eta(Z))\xi + (\alpha - 1)(\eta(Z)W - \eta(W)Z) \tag{54}$$

for all $W,\ Z \in D^{\perp}$.

Proof. From Lemma 5.1, it is obvious. □

Corollary 5.3. *Let M be a ξ-horizontal CR-submanifold of $\tilde{M}$ with respect to $\bar{\tilde{\nabla}}$. Then the distribution $D^{\perp}$ is integrable if and only if*

$$A_{\phi W}Z - A_{\phi Z}W = a(Z)W - a(W)Z$$

for all $W,\ Z \in D^{\perp}$.

Remark 1. Let M be a CR-submanifold of $\tilde{M}$ with respect to $\tilde{\nabla}$. Then the distribution $D^{\perp}$ is integrable if and only if

$$A_{\phi W}Z - A_{\phi Z}W = \alpha\,[\eta(Z)W - \eta(W)Z]$$

for all $W,\ Z \in D^{\perp}$.

Remark 2. Let M be a ξ-horizontal CR-submanifold of $\tilde{M}$ with respect to $\tilde{\nabla}$. Then the distribution $D^{\perp}$ is integrable if and only if

$$A_{\phi W}Z = A_{\phi Z}W$$

for all $W,\ Z \in D^{\perp}$.

Theorem 5.4. *Let M be a CR-submanifold of $\tilde{M}$ with respect to $\bar{\tilde{\nabla}}$. Then the distribution D is integrable if and only if*

$$h(X,\phi Y) = h(Y,\phi X), \quad \textit{for all } X,\ Y \in D. \tag{55}$$

Proof. For $X,\ Y \in D$, we have from (32) and (46) that

$$\phi(Q\bar{\nabla}_X Y) = h(X,\phi Y) - Ch(X,Y), \tag{56}$$

from which we get

$$\phi Q[X,Y] = h(X,\phi Y) - h(Y,\phi X). \tag{57}$$

Therefore D is integrable if and only if the relation (55) holds. □

Remark 3. Let M be a CR-submanifold of $\tilde{M}$ with respect to $\tilde{\nabla}$. Then the distribution D is integrable if and only if $h(X,\phi Y) = h(Y,\phi X)$ for all $X,\ Y \in D$.

Theorem 5.5. *Let M be a CR-submanifold of $\tilde{M}$ with respect to $\bar{\bar{\nabla}}$. If the distribution D is integrable and the leaf of D is totally geodesic in M then*

$$g(h(X,Y),\phi Z)+(\alpha-1)\eta(Z)g(X,Y)+(2\alpha-1)\eta(X)\eta(Y)\eta(Z) \;=\; 0 \tag{58}$$

for all $X,\ Y\in D$ and $Z\in D^{\perp}$.

Proof. If D is integrable and leaf of D is totally geodesic in M then $\bar{\nabla}_X\phi Y\in D$ for $X,\ Y\in D$. Now for $X\in D$ and $Z\in D^{\perp}$ we have from (47) that

$$\phi P(\bar{\nabla}_X Z) \;=\; -\bar{A}_{\phi Z}X+\bar{\nabla}^{\perp}_X\phi Z-\phi(Q\bar{\nabla}_X Z)-\phi\bar{h}(X,Z)-(\alpha-1)\eta(Z)X-(2\alpha-1)\eta(X)\eta(Z)\xi. \tag{59}$$

From (14), (15) and (59), we find

$$\begin{aligned} 0 &= g(\bar{\nabla}_X\phi Y,Z)=-g(\phi Y,\bar{\nabla}_X Z)=-g(\phi Y,P\bar{\nabla}_X Z)=-g(Y,\phi P\bar{\nabla}_X Z)\\ &= g(\bar{A}_{\phi Z}X+B\bar{h}(X,Z),Y)+(\alpha-1)\eta(Z)g(X,Y)+(2\alpha-1)\eta(X)\eta(Y)\eta(Z)\end{aligned}$$

for all $X,\ Y\in D$ and $Z\in D^{\perp}$.
Now using (11) and (32) in the above relation, we get (58). □

Corollary 5.6. *Let M be a ξ-horizontal CR-submanifold of $\tilde{M}$ with respect to $\bar{\bar{\nabla}}$. Then the distribution D is integrable and the leaf of D is totally geodesic in M if and only if*

$$g(h(X,Y),\phi Z)=0,\quad \text{for all } X,\ Y\in D \text{ and } Z\in D^{\perp}. \tag{60}$$

Proof. The direct part follows from Theorem 5.5. For converse part, let the relation (60) holds. Then using (7) in (60), we get

$$0=g(h(X,Y),\phi Z)=g(\bar{\bar{\nabla}}_X\phi Y,\phi Z)=g(\bar{\nabla}_X Y,Z),$$

which implies that $\bar{\nabla}_X Y\in D$ for any $X,\ Y\in D$ and the leaf of D is totally geodesic in M with respect to quarter symmetric non-metric connection. This completes the proof. □

Theorem 5.7. *Let M be a CR-submanifold of $\tilde{M}$ with respect to $\bar{\bar{\nabla}}$. Then the distribution $D^{\perp}$ is integrable and the leaf of $D^{\perp}$ is totally geodesic in M if and only if*

$$g(h(X,Z),\phi W)+a(X)g(Z,W)+a(X)\eta(Z)\eta(W)+\alpha g(Z,W)\eta(X)+(2\alpha-1)\eta(X)\eta(Z)\eta(W) \;=\; 0 \tag{61}$$

for all $X\in D$ and $Z,\ W\in D^{\perp}$.

Proof. For all $Z,\ W\in D^{\perp}$, we have from (47) that

$$\begin{aligned}\phi P\bar{\nabla}_Z W \;=\; & -\bar{A}_{\phi W}Z+\bar{\nabla}^{\perp}_Z\phi W-\phi(Q\bar{\nabla}_Z W)-\phi\bar{h}(Z,W)\\ & -\alpha g(Z,W)\xi-(2\alpha-1)\eta(Z)\eta(W)\xi-(\alpha-1)\eta(W)Z.\end{aligned} \tag{62}$$

Now, taking inner product of (62) with $X\in D$ we get

$$g(\phi P\bar{\nabla}_Z W,X)=-g(\bar{A}_{\phi W}Z,X)-\alpha g(Z,W)\eta(X)-(2\alpha-1)\eta(X)\eta(Z)\eta(W).$$

Using (11) and (36) in the above equation, we get

$$\begin{aligned} g(\phi P\bar{\nabla}_Z W,X) \;=\; & g(h(X,Z),\phi W)+a(X)g(Z,W)+a(X)\eta(Z)\eta(W)\\ & +\alpha g(Z,W)\eta(X)+(2\alpha-1)\eta(X)\eta(Z)\eta(W),\end{aligned} \tag{63}$$

from which (61) follows. The converse part is trivial. □

Corollary 5.8. *Let M be a ξ-horizontal CR-submanifold of $\tilde{M}$ with respect to $\bar{\bar{\nabla}}$. Then the distribution $D^{\perp}$ is integrable and the leaf of $D^{\perp}$ is totally geodesic in M if and only if*

$$g(h(X,Z),\phi W)+a(X)g(Z,W)+\alpha g(Z,W)\eta(X) \;=\; 0 \tag{64}$$

for all $X\in D$ and $Z,\ W\in D^{\perp}$.

Corollary 5.9. *Let M be a ξ-vertical CR-submanifold of $\tilde{M}$ with respect to $\bar{\bar{\nabla}}$. Then the distribution $D^\perp$ is integrable and the leaf of $D^\perp$ is totally geodesic in M if and only if*

$$g(h(X,Z),\phi W) + a(X)g(Z,W) + a(X)\eta(Z)\eta(W) = 0 \tag{65}$$

for all $X \in D$ and $Z,\ W \in D^\perp$.

Definition 5.10 ([1], [24]). *A CR-submanifold M of a $(LCS)_n$-manifold $\tilde{M}$ with respect to $\bar{\bar{\nabla}}$ is called Lorentzian contact CR-product if M is locally a Riemannain product of M_T and $M_\perp$, where M_T and $M_\perp$ denotes the leaves of the distribution D and $D^\perp$ respectively.*

Theorem 5.11. *Let M be a ξ-horizontal CR-submanifold of $\tilde{M}$ with respect to $\bar{\bar{\nabla}}$. Then M is a Lorentzian contact CR-product if and only if*

$$A_{\phi W}X + \alpha\eta(X)W + a(X)W = 0 \tag{66}$$

for all $X \in D$ and $W \in D^\perp$.

Proof. As the leaves of $D^\perp$ are totally geodesic, we have from (64) that

$$g(A_{\phi W}X + \alpha\eta(X)W + a(X)W, Z) = 0$$

for all $X \in D$ and $Z,\ W \in D^\perp$, which implies that

$$A_{\phi W}X + \alpha\eta(X)W + a(X)W \in D. \tag{67}$$

Now for $X,\ Y \in D$ and $W \in D^\perp$, we have

$$\begin{aligned} g(A_{\phi W}X + \alpha\eta(X)W + a(X)W, Y) &= g(A_{\phi W}X, Y) = g(\phi(\bar{\bar{\nabla}}_X Y - \bar{\nabla}_X Y), W) \\ &= g(\bar{\bar{\nabla}}_X \phi Y, W) = g(\bar{\nabla}_X \phi Y, W) = 0, \end{aligned}$$

which means that

$$A_{\phi W}X + \alpha\eta(X)W + a(X)W \in D^\perp. \tag{68}$$

From (67) and (68), we get (66). Conversely, let (66) holds. Then, for $Z \in D^\perp$, we get

$$g(h(X,Z),\phi W) + a(X)g(Z,W) + \alpha\eta(X)g(Z,W) = 0,$$

which implies that the leaves of $D^\perp$ are totally geodesic. Next for all $X, Y \in D$ and $W \in D^\perp$, we have

$$\begin{aligned} g(\bar{\nabla}_X Y, W) &= g(\bar{\bar{\nabla}}_X Y, W) = g(\phi\bar{\bar{\nabla}}_X, \phi W) \\ &= g(\bar{\bar{\nabla}}_X \phi Y, \phi W) = g(h(X,\phi Y), \phi W) \\ &= g(A_{\phi W}X, \phi Y) \\ &= g(-\alpha\eta(Y)W - a(X)W, \phi Y) \\ &= 0. \end{aligned}$$

Therefore, the leaves of D are totally geodesic in M. So, M is a Lorentzian contact CR-product. □

6. Totally umbilical CR-submanifolds

In this section, we study totally umbilical CR-submanifolds of $(LCS)_n$-manifolds. Let M be a totally umbilical CR-submanifolds of $\tilde{M}$ with respect to $\tilde{\nabla}$.
Then for $Z,\ W \in D^{\perp}$ we have from (7) that

$$\tilde{\nabla}_Z \phi W - \phi(\tilde{\nabla}_Z W) = \alpha[g(Z,W)\xi + 2\eta(Z)\eta(W)\xi + \eta(W)Z]. \tag{69}$$

Using (9), (10) and (14) in (69), we get

$$-A_{\phi W}Z + \nabla^{\perp}_Z \phi W = \phi(P\nabla_Z W) + \phi(Q\nabla_Z W) + \phi h(Z,W) + \alpha\{g(Z,W)\xi + 2\eta(Z)\eta(W)\xi + \eta(W)Z\}. \tag{70}$$

Taking inner product of (70) with $Z \in D^{\perp}$ and using (11), we get

$$-g(h(Z,Z),\phi W) = g(\phi h(Z,W),Z) + \alpha\{g(Z,W)\eta(Z) + 2\eta^2(Z) + \eta(W)g(Z,Z)\}. \tag{71}$$

In view of (12), (71) yields

$$g(H,\phi W) = -\frac{1}{||Z||^2}[g(Z,W)g(\phi H,Z) + \alpha\{g(Z,W)\eta(Z) + 2\eta^2(Z) + \eta(W)||Z||^2\}]. \tag{72}$$

Interchanging Z and W in (72), we obtain

$$g(H,\phi Z) = -\frac{1}{||W||^2}\Big[g(Z,W)g(\phi H,W) + \alpha\{g(Z,W)\eta(W) + 2\eta^2(W) + \eta(Z)||W||^2\}\Big]. \tag{73}$$

Substituting (72) in (73), we get after simplification

$$\left[1 - \frac{g(Z,W)^2}{||Z||^2||W||^2}\right]g(H,\phi Z) - \alpha\left[\frac{\eta(W)g(Z,W)}{||W||^2} - \eta(Z)\right] - 2\alpha\frac{\eta(Z)\eta(W)}{||W||^2}\left[\frac{\eta(Z)g(Z,W)}{||Z||^2} - \eta(W)\right] - \alpha\frac{g(Z,W)}{||W||^2}\left[\frac{\eta(Z)g(Z,W)}{||Z||^2} - \eta(W)\right] = 0. \tag{74}$$

Hence we get the following theorems:

Theorem 6.1. *Let M be a ξ-horizontal totally umbilical CR-submanifold of $\tilde{M}$ with respect to $\tilde{\nabla}$. Then one of the following holds:*
(i) M is minimal in $\tilde{M}$,
(ii) $\dim D^{\perp} = 1$,
(iii) $H \in \Gamma(\mu)$.

Theorem 6.2. *Let M be a ξ-vertical totally umbilical CR-submanifold of $\tilde{M}$ with respect to $\tilde{\nabla}$. Then $\dim D^{\perp} = 1$.*

Remark 4. The Theorem 6.1 and Theorem 6.2 also holds good in case of considering $\tilde{M}$ with respect to $\bar{\tilde{\nabla}}$.

7. Cohomology

In this section we have studied cohomology of CR-submanifold of $\tilde{M}$ with respect to $\bar{\tilde{\nabla}}$ and obtain the following:

Lemma 7.1. *Let M be a ξ-vertical CR-submanifold of $\tilde{M}$ with respect to $\bar{\tilde{\nabla}}$. Then the invariant distribution D is minimal if*

$$g(A_{\phi Z}X, \phi X) = -\alpha\eta(Z)g(X,\phi X) \tag{75}$$

for every $X \in D$ and $Z \in D^{\perp}$.

Proof. For $X \in D$ and $Z \in D^{\perp}$, we have from (16) that

$$g(Z, \bar{\nabla}_X X) = g(Z, \bar{\tilde{\nabla}}_X X) = g(Z, \tilde{\nabla}_X X) \tag{76}$$

By virtue of (2), (4) and (7), (76) yields

$$g(Z, \bar{\nabla}_X X) = -g(\tilde{\nabla}_X \phi Z, \phi X) + \alpha\eta(Z)g(X, \phi X). \tag{77}$$

Using (10) in (77), we find

$$g(Z, \bar{\nabla}_X X) = g(A_{\phi Z} X, \phi X) + \alpha\eta(Z)g(X, \phi X). \tag{78}$$

Replacing X by ϕX in (78), we obtain

$$g(Z, \bar{\nabla}_{\phi X} \phi X) = g(A_{\phi Z} X, \phi X) + \alpha\eta(Z)g(X, \phi X). \tag{79}$$

From (78) and (79), we get

$$g(Z, \bar{\nabla}_X X) + g(Z, \bar{\nabla}_{\phi X} \phi X) = 2g(A_{\phi Z} X, \phi X) + 2\alpha\eta(Z)g(X, \phi X). \tag{80}$$

Thus the result follows from (80). □

Let $\{e_1, \cdots, e_q, e_{q+1} = \phi e_1, \cdots, e_{2q} = \phi e_q, e_{2q+1}, \cdots, e_{m-1} = e_{2q+p-1}, e_m = e_{2q+p} = \xi\}$ is a local pseudo orthonormal basis of $\chi(M)$ such that $\{e_1, \cdots, e_{2q}\}$ is a local basis of D and $\{e_{2q+1}, \cdots, e_{2q+p}\}$ is a local basis of $D^{\perp}$. We take $\{\omega^1, \cdots, \omega^{2q}\}$ as dual basis of $\{e_1, \cdots, e_{2q}\}$ and $\{\theta^{2q+1}, \cdots, \theta^{2q+p-1}, \eta\}$ as the dual basis of $\{e_{2q+1}, \cdots, e_{2q+p-1}, \xi\}$. Let $\nu = \omega^1 \wedge \omega^2 \cdots \wedge \omega^{2q}$ is the transversal volume form of a foliation $\mathcal{F}^{\perp}$ defined by $D^{\perp}$ on M. Then

$$d\nu = (-1)^j \omega^1 \wedge \omega^2 \cdots \wedge d\omega^j \wedge \cdots \wedge \omega^{2q}.$$

Thus $d\nu = 0$ if

$$d\nu(W_1, W_2, X_1, \cdots, X_{2q-1}) = 0 \tag{81}$$

and

$$d\nu(W_1, X_1, \cdots, X_{2q}) = 0 \tag{82}$$

for any $X_1,\ X_2, \cdots, X_{2q} \in D$ and $W_1, W_2 \in D^{\perp}$.
By straightforward we can say that (81) holds if $D^{\perp}$ is integrable and (82) holds if D is minimal. Consequently ν is closed if (54) and (75) holds simultaneously.
Again we take the p-form $\nu^{\perp} = \theta^{2q+1} \wedge \cdots \wedge \theta^{2q+p-1} \wedge \eta$ so that
$\theta^i(e_j) = \delta^i_j$, $\theta^i_{/D} = 0$, $i,\ j = \overline{2q+1, 2q+p-1}$. Then by similar argument ν is closed if $D^{\perp}$ is minimal and D is integrable i.e. $D^{\perp}$ is minimal and $h(X, \phi Y) = h(Y, \phi X)$ for $X,\ Y \in D$. Thus we get the following theorem:

Theorem 7.2. *Let M be a compact CR-submanifold of $\tilde{M}$ with respect to $\bar{\tilde{\nabla}}$. Then the transversal volume form ν defines a cohomology class $c(\nu) := [\nu] \in H^{2q}(M; \mathbb{R}), 2q = dimD$ if (54) and (75) holds simultaneously.*
Furthermore if $D^{\perp}$ is minimal and $h(X, \phi Y) = h(Y, \phi X)$ for $X,\ Y \in D$ holds then $H^{2i}(M, \mathbb{R}) \neq 0$ for any $i \in \{1, \cdots, q\}$.

8. Example

In this section we construct an example of a $(LCS)_5$-manifold as similar in [20], then we verify Proposition 4.1 and the relation (20).

Example 8.1. *Let us consider the manifold* $\tilde{M} = \{(x, y, z, u, v) \in \mathbb{R}^5 : (x, y, z, u, v) \neq (0, 0, 0, 0, 0)\}$. *We take the linearly independent vector fields at each point of* $\tilde{M}$ *as*
$e_1 = e^{-kz}(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})$, $e_2 = e^{-kz}\frac{\partial}{\partial y}$, $e_3 = e^{-2kz}\frac{\partial}{\partial z}$, $e_4 = e^{-kz}(u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v})$, $e_5 = e^{-kz}\frac{\partial}{\partial v}$ *for some scalar k.*

Let $\tilde{g}$ *be the metric defined by*

$$\tilde{g}(e_i, e_j) = \begin{cases} 1, \; for\ i = j \neq 3, \\ 0, \; for\ i \neq j, \\ -1, \; for\ i = j = 3. \end{cases}$$

Here $i,\ j \in \{1, 2, \cdots, 5\}$.
Let η *be the 1-form defined by* $\eta(Z) = \tilde{g}(Z, e_3)$, *for any vector field* $Z \in \chi(\tilde{M})$. *Let* ϕ *be the (1,1) tensor field defined by* $\phi e_1 = e_1$, $\phi e_2 = e_2$, $\phi e_3 = 0$, $\phi e_4 = e_4$, $\phi e_5 = e_5$. *Then using the linearity property of* ϕ *and* $\tilde{g}$ *we have* $\eta(e_3) = -1$, $\phi^2 U = U + \eta(U)\xi$ *and* $\tilde{g}(\phi U, \phi V) = \tilde{g}(U, V) + \eta(U)\eta(V)$, *for every* $U,\ V \in \chi(\tilde{M})$.*Thus for* $e_3 = \xi$, $(\phi, \xi, \eta, \tilde{g})$ *defines a Lorentzian paracontact structure on* $\tilde{M}$. *Let* $\tilde{\nabla}$ *be the Levi-Civita connection on* $\tilde{M}$ *with respect to the metric* $\tilde{g}$. *Then we have* $[e_1, e_2] = -e^{-kz}e_2$, $[e_1, e_3] = ke^{-2kz}e_1$, $[e_1, e_4] = 0$, $[e_1, e_5] = 0$, $[e_2, e_3] = ke^{-2kz}e_2$, $[e_2, e_4] = 0$, $[e_2, e_5] = 0$, $[e_4, e_3] = ke^{-2kz}e_4$, $[e_5, e_3] = ke^{-2kz}e_5$, $[e_4, e_5] = 0$.
Now, using Koszul's formula for $\tilde{g}$, *it can be calculated that* $\tilde{\nabla}_{e_1}e_1 = ke^{-2kz}e_3$, $\tilde{\nabla}_{e_1}e_3 = ke^{-2kz}e_1$, $\tilde{\nabla}_{e_2}e_1 = e^{-kz}e_2$, $\tilde{\nabla}_{e_2}e_2 = -e^{-kz}e_1 + ke^{-2kz}e_3$, $\tilde{\nabla}_{e_2}e_3 = ke^{-2kz}e_2$, $\tilde{\nabla}_{e_4}e_3 = ke^{-2kz}e_4$, $\tilde{\nabla}_{e_4}e_4 = ke^{-2kz}e_3$, $\tilde{\nabla}_{e_5}e_3 = ke^{-2kz}e_5$, $\tilde{\nabla}_{e_5}e_4 = e^{-kz}e_5$, *and* $\tilde{\nabla}_{e_5}e_5 = -e^{-kz}e_4 + ke^{-2kz}e_3$.
and rest of the terms are zero.
Since $\{e_1, e_2, e_3, e_4, e_5\}$ *is a frame field, then any vector field* $X, Y \in T\tilde{M}$ *can be written as*

$$\begin{aligned} X &= x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + x_5e_5, \\ Y &= y_1e_1 + y_2e_2 + y_3e_3 + y_4e_4 + y_5e_5, \end{aligned}$$

where $x_i, y_i \in \mathbb{R}$, $i = 1, 2, 3, 4, 5$ *such that*

$$x_1y_1 + x_2y_2 - x_3y_3 + x_4y_4 + x_5y_5 \neq 0$$

and hence

$$\tilde{g}(X, Y) = (x_1y_1 + x_2y_2 - x_3y_3 + x_4y_4 + x_5y_5). \tag{83}$$

Therefore,

$$\begin{aligned} \tilde{\nabla}_X Y = \; & ke^{-2kz}[x_1y_3e_1 + x_2y_3e_2 + (x_1y_1 + x_2y_2 + x_4y_4 + x_5y_5)e_3 \\ & + x_4y_3e_4 + x_5y_3e_5] + e^{-kz}[-x_2y_1e_1 + x_2y_1e_2 - x_5y_5e_4 + x_5y_4e_5]. \end{aligned} \tag{84}$$

From the above it can be easily seen that $(\phi, \xi, \eta, \tilde{g})$ *is a* $(LCS)_5$ *structure on* $\tilde{M}$ *with* $\alpha = ke^{-2kz} \neq 0$ *such that* $X(\alpha) = \rho\eta(X)$, *where* $\rho = 2k^2e^{-4kz}$.
We set $A = e_1$. *Then* $a(X) = g(X, A) = x_1$. *Hence from (16), we get*

$$\begin{aligned} \bar{\tilde{\nabla}}_X Y = \; & ke^{-2kz}[x_1y_3e_1 + x_2y_3e_2 + (x_1y_1 + x_2y_2 + x_4y_4 + x_5y_5)e_3 \\ & + x_4y_3e_4 + x_5y_3e_5] + e^{-kz}(-x_2y_2e_1 + x_2y_1e_2 - x_5y_5e_4 + x_5y_4e_5) \\ & - y_3(x_1e_1 + x_2e_2 + x_4e_4 + x_5e_5) + x_1(y_1e_1 + y_2e_2 + y_4e_4 + y_5e_5). \end{aligned} \tag{85}$$

Also, for $Z = z_1e_1 + z_2e_2 + z_3e_3 + z_4e_4 + z_5e_5, z_i \in \mathbb{R}, i = 1$ *to* 5, *we have*

$$\begin{aligned} (\bar{\tilde{\nabla}}_X \tilde{g})(Y, Z) &= z_3(x_1y_1 + x_2y_2 + x_4y_4 + x_5y_5) - 2x_1(y_1z_1 + y_2z_2 + y_4z_4 + y_5z_5) \\ &\neq 0. \end{aligned}$$

Thus in an $(LCS)_5$*-manifold the quarter symmetric non-metric connection is given by (85). Let* f *be an isometric immersion from* M *to* $\tilde{M}$ *defined by* $f(x, y, z) = (x, y, z, 0, 0)$. *Let* $M = \{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \neq (0, 0, 0)\}$, *where* (x, y, z) *are the standard coordinates in* $\mathbb{R}^3$. *The vector fields*

$e_1 = e^{-kz}(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})$, $e_2 = e^{-kz}\frac{\partial}{\partial y}$, $e_3 = e^{-2kz}\frac{\partial}{\partial z}$ *are linearly independent at each point of M.*
Let g be the induced metric defined by

$$g(e_i, e_j) = \begin{cases} 1, \; for\, i = j \neq 3, \\ 0, \; for\; i \neq j, \\ -1, \; for\, i = j = 3. \end{cases}$$

Here i and j runs over 1 to 3.
Let ∇ *be the Levi-Civita connection on M with respect to the metric g. Then we have* $[e_1, e_2] = -e^{-kz}e_2$, $[e_1, e_3] = ke^{-2kz}e_1$, $[e_2, e_3] = ke^{-2kz}e_2$. *Clearly* $\{e_4, e_5\}$ *is the frame field for the normal bundle* $T^{\perp}M$. *If we take* $Z \in TM$ *then* $\phi Z \in TM$ *and therefore M is an invariant submanifold of* $\tilde{M}$. *If we take* $X,\ Y \in TM$ *then we can express them as*

$$\begin{aligned} X &= x_1e_1 + x_2e_2 + x_3e_3, \\ Y &= y_1e_1 + y_2e_2 + y_3e_3. \end{aligned}$$

Therefore

$$\nabla_X Y = ke^{-2kz}[x_1y_3e_1 + x_2y_3e_2 + (x_1y_1 + x_2y_2 + x_4y_4 + x_5y_5)e_3] + e^{-kz}[-x_2y_2e_1 + x_2y_1e_2].$$

Now from (85), the tangential part of $\bar{\tilde{\nabla}}_X Y$ *is given by*

$$\begin{aligned} \bar{\nabla}_X Y &= ke^{-2kz}[x_1y_3e_1 + x_2y_3e_2 + (x_1y_1 + x_2y_2)e_3] + e^{-kz}(-x_2y_2e_1 + x_2y_1e_2) \\ &\quad -y_3(x_1e_1 + x_2e_2) + x_1(y_1e_1 + y_2e_2) \\ &= \nabla_X Y + \eta(Y)\phi X + a(X)\phi Y. \end{aligned}$$

And

$$\begin{aligned} (\bar{\nabla}_X g)(Y, Z) &= z_3(x_1y_1 + x_2y_2) - 2x_1(y_1z_1 + y_2z_2), \\ &\neq 0. \end{aligned}$$

which means M admits quarter symmetric non-metric connection. Also, it is easy to see that

$$\bar{h}(X, Y) = h(X, Y) = ke^{-2kz}(x_4y_3e_4 + x_5y_3e_5) + e^{-kz}(-x_5y_5e_4 + x_5y_4e_5).$$

Thus the Proposition 4.1 is verified.
Now, if R and $\bar{R}$ *be the curvature tensors of M with respect to* ∇ *and* $\bar{\nabla}$ *respectively then we can easily calculate*

$$\begin{aligned} R(e_1, e_2)e_2 &= k^2e^{-4kz}e_1 - e^{-2kz}e_1 \\ R(e_1, e_3)e_3 &= k^2e^{-4kz}e_1 \\ R(e_2, e_1)e_1 &= k^2e^{-4kz}e_2 - e^{-2kz}e_2 \\ R(e_2, e_3)e_3 &= k^2e^{-4kz}e_2 \\ R(e_3, e_1)e_1 &= -k^2e^{-4kz}e_3 \\ R(e_3, e_2)e_2 &= -k^2e^{-4kz}e_3 \\ R(e_1, e_2)e_3 &= 0. \end{aligned} \tag{86}$$

Again from (16), we have
$\bar{\nabla}_{e_1}e_1 = ke^{-2kz}e_3 + e_1$, $\bar{\nabla}_{e_1}e_2 = e_2$, $\bar{\nabla}_{e_1}e_3 = (ke^{-2kz} - 1)e_1$, $\bar{\nabla}_{e_2}e_1 = e^{-kz}e_2$, $\bar{\nabla}_{e_2}e_2 = -e^{-kz}e_1 + ke^{-2kz}e_3$, $\bar{\nabla}_{e_2}e_3 = (ke^{-2kz} - 1)e_2$ *and rest of the terms are zero. Therefore*

$$\begin{aligned} \bar{R}(e_1, e_2)e_2 &= k^2e^{-4kz}e_1 - e^{-2kz}e_1 - ke^{-2kz}e_1 - ke^{-2kz}e_3 \\ \bar{R}(e_1, e_3)e_3 &= k^2e^{-4kz}e_1 + ke^{-2kz}e_1 \\ \bar{R}(e_2, e_1)e_1 &= k^2e^{-4kz}e_2 - e^{-2kz}e_2 - ke^{-2kz}e_2 \\ \bar{R}(e_2, e_3)e_3 &= k^2e^{-4kz}e_2 + ke^{-2kz}e_2 \\ \bar{R}(e_3, e_1)e_1 &= -k^2e^{-4kz}e_3 - ke^{-2kz}e_3 \\ \bar{R}(e_3, e_2)e_2 &= -k^2e^{-4kz}e_3 \\ \bar{R}(e_1, e_2)e_3 &= (ke^{-2kz} - 1)e_2. \end{aligned} \tag{87}$$

Now from (86), (87) and using the relation $da(X, Y) = \frac{1}{2}\{Xa(Y) - Ya(X)\} - a[X, Y]$, *we can easily verify (20).*

References

[1] Ahmad, M., *CR-submanifolds of a Lorentzian Para-Sasakian manifold endowed with quarter symmetric metric connection*, Bull. Korean Math. Soc., **49** (2012), 25–32.
[2] Al-Ghefari, R., Shahid, M. H. and Al-Solamy, F. R, *Submersion of CR-submanifolds of locally conformal Kaehler manifold*, Beiträge Zur Algebra und Geom. **47** (2006), 147–159.
[3] Atceken, M. and Dirik, S., *On Contact CR-submanifolds of Kenmotsu manifolds*, Acta Univ. Sapientiac, Math., **4** (2012), 182–198.
[4] Ateceken, M. and Hui, S. K., *Slant and pseudo-slant submanifolds of LCS-manifolds*, Czechoslovak Math. J., **63** (2013), 177–190.
[5] Bejancu, A., *CR-submanifolds of a Kaehler manifold*, Proc. of Amr. Math. Soc., **69** (1978), 135–142.
[6] Bejancu, A., *Geometry of CR-submanifolds*, D. Reidel Publ. Co., Dordrecht, Holland, **1986**.
[7] Chandra, S., Hui, S. K. and Shaikh, A. A., *Second order parallel tensors and Ricci solitons on* $(LCS)_n$*-manifolds*, Commun. Korean Math. Soc., **30** (2015), 123–130.
[8] Chen, B. Y., *Geometry of warped product CR-submanifolds in Kaehler manifold*, Monatsh. Math., **133** (2001), 177–195.
[9] Chen, B. Y., *Cohomology of CR-submanifolds*, Ann. Fac. Sci. Toulouse, **3** (1981), 167–172.
[10] Chen, B. Y. and Piccini, P., *The canonical foliations of a locally conformal Kaehler manifold*,Ann. Mat. Pura Appl., IV, Ser., **141** (1985), 289–305.
[11] Friedmann, A. and Schouten, J. A., *Über die geometric derhalbsymmetrischen Übertragung*, Math. Zeitscr., **21** (1924), 211–223.
[12] Golab, S., *On semi-symmetric and quarter symmetric linear connections*, Tensor, N. S., **29** (1975), 249-254.
[13] Hui, S. K., *On* ϕ*-pseudo symmetries of* $(LCS)_n$*-manifolds*, Kyungpook Math. J., **53** (2013), 285–294.
[14] Hui, S. K. and Atceken, M., *Contact warped product semi-slant submanifolds of* $(LCS)_n$*-manifolds*, Acta Univ. Sapientiae Math., **3** (2011), 212–224.
[15] Hui, S. K., Atceken, M. and Nandy, S., *Contact CR-warped product submanifolds of* $(LCS)_n$*-manifolds*, Acta Math. Univ. Comenianae, **86** (2017), 101–109.
[16] Hui, S. K., Atceken, M. and Pal, T., *Warped product pseudo slant submanifolds of* $(LCS)_n$*-manifolds*, New Trends in Math. Sci., **5** (2017), 204–212.
[17] Hui, S. K., Atceken, M., Pal, T. and Mishra, L. N., *On contact CR-submanifolds of* $(LCS)_n$*-manifolds* to appear in Thai J. of Math.
[18] Hui, S. K., Mishra, V. N., Pal, T. and Vandana, *Some classes of invariant submanifolds of* $(LCS)_n$*-manifolds*, Italian J. of Pure Appl. Math., **39** (2018), 359–372.
[19] Hui, S. K. and Pal, T., *Totally real submanifolds of* $(LCS)_n$*-manifolds*, Facta Universitatis (NIS), Ser. Math. Inform, **33(2)** (2018), 141–152.
[20] Hui, S. K., Piscoran, L. I. and Pal, T., *Invariant submanifolds of* $(LCS)n$*-manifolds with respect to quarter symmetric metric connection*, Acta Math. Univ. Comenianae, **87** (2018), 205–221.
[21] Hui, S. K., Prasad, R. and Pal, T., *Ricci solitons on submanifolds of* $(LCS)_n$*-manifolds*, Ganita, Bharat Ganit Parishad, **68(1)** (2018), 53–63.
[22] Laha, B., Das, B. and Bhattacharyya, A., *Contact CR-submanifolds of an indefinite Lorentzian para-Sasakian manifold*, Acta Univ. Sapientae, Mathematica, **5** (2013), 157–168.
[23] Mantica, C. A. and Molinari, L. G., *A note on concircular structure space-times*, arXiv: 1804.02272 [math. DG].
[24] Matsumoto, K., *On contact CR-submanifolds of Sasakian manifolds*, Internat. J. Math. and Math. Sci., **6**(2) (1983), 313–326.
[25] Matsumoto, K., *On Lorentzian almost paracontact manifolds*, Bull. of Yamagata Univ. Nat. Sci., **12** (1989), 151–156.
[26] O'Neill, B., *Semi Riemannian geometry with applications to relativity*, Academic Press, New York, **1983**.
[27] Prakash, A. and Narain, D., *On a quarter symmetric non-metric connection in an Lorentzian para Sasakian manifoldes*, Int. Electronic J. of Geom., **4** (2011), 129–137.
[28] Shahid, M. H., *CR-submanifolds of a trans-Sasakian manifold*, Indian J. Pure Appl. Math., **22** (1991), 1007–1012.
[29] Shaikh, A. A., *On Lorentzian almost paracontact manifolds with a structure of the concircular type*, Kyungpook Math. J., **43** (2003), 305–314.
[30] Shaikh, A. A., *Some results on* $(LCS)_n$*-manifolds*, J. Korean Math. Soc., **46** (2009), 449–461.
[31] Shaikh, A. A. and Ahmad, H., *Some transformations on* $(LCS)_n$*-manifolds*, Tsukuba J. Math., **38** (2014), 1–24.
[32] Shaikh, A. A. and Baishya, K. K., *On concircular structure spacetimes*, J. Math. Stat., **1** (2005), 129–132.
[33] Shaikh, A. A. and Baishya, K. K., *On concircular structure spacetimes II*, American J. Appl. Sci., **3(4)** (2006), 1790–1794.
[34] Shaikh, A. A. and Baishya, K. K., *Some results of LP-Sasakian manifolds*, Bull. Math. Soc. Sci. Math. Roumanie, **49** (2006), 193–205.
[35] Shaikh, A. A., Basu, T. and Eyasmin, S., *On locally* ϕ*-symmetric* $(LCS)_n$*-manifolds*, Int. J. of Pure and Appl. Math., **41(8)** (2007), 1161–1170.
[36] Shaikh, A. A., Basu, T. and Eyasmin, S., *On the existence of* ϕ*-recurrent* $(LCS)_n$*-manifolds*, Extracta Mathematicae, **23(1)** (2008), 71–83.
[37] Shaikh, A. A. and Binh, T. Q., *On weakly symmetric* $(LCS)_n$*-manifolds*, J. Adv. Math. Studies, **2** (2009), 75–90.
[38] Shaikh, A. A. and Hui, S. K., *On generalized* ϕ*-recurrent* $(LCS)_n$*-manifolds*, AIP Conf. Proc., **1309** (2010), 419–429.
[39] Shaikh, A. A., Matsuyama, Y. and Hui, S. K., *On invariant submanifold of* $(LCS)_n$*-manifolds*, J. of Egyptian Math. Soc., **24** (2016), 263–269.
[40] Yano, K. and Kon, M., *Structures on manifolds*, World Scientific publishing, 1984.