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Some Operator Inequalities for Operator Means and Positive Linear Maps

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Abstract. In this note, some operator inequalities for operator means and positive linear maps are investigated. The conclusion based on operator means is presented as follows: Let $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ be a strictly positive unital linear map and $h_1^{-1}I_H \le A \le h_1 I_H$ and $h_2^{-1}I_H \le B \le h_2 I_H$ for positive real numbers $h_1, h_2 \ge 1$. Then for $p > 0$ and an arbitrary operator mean σ,

$$(\Phi(A)\sigma\Phi(B))^p \le \alpha_p \Phi^p(A\sigma^* B),$$

where $\alpha_p = \max\left\{\left(\frac{\alpha^2(h_1,h_2)}{4}\right)^p, \frac{1}{16}\alpha^{2p}(h_1,h_2)\right\}$, $\alpha(h_1,h_2) = (h_1 + h_1^{-1})\sigma(h_2 + h_2^{-1})$. Likewise, a p-th ($p \ge 2$) power of the Diaz-Metcalf type inequality is also established.

1. Introduction

Throughout, let $B(\mathcal{H})$ be the C^*-algebra of bounded linear operator on a complex Hilbert space $\mathcal{H}$ and the identity operator is denoted by I_H. For two self-adjoint operators $A, B \in B(\mathcal{H})$, $A \le (<) B$ means $B - A$ is a positive (invertible) operator. A linear map $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ is called positive (strictly positive, resp.) if it maps positive (invertible, resp.) operators into positive (invertible, resp.) operators and is said to be unital if it maps identity operator to identity operator.

The axiomatic theory for operator means for pairs of positive invertible operators have been developed by Kubo and Ando [12]. A binary operation σ defined on the set of positive invertible operators is called an operator mean provided that

1. $I_H \sigma I_H = I_H$;
2. $C^*(A\sigma B)C \le (C^*AC)\sigma(C^*BC)$;
3. $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n\sigma B_n \downarrow A\sigma B$, where $A_n \downarrow A$ means $A_1 \ge A_2 \ge \cdots$ and $A_n \to A$ in the strong operator topology;
4. $A \le B$ and $C \le D$ imply that $A\sigma C \le B\sigma D$.

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There exists an affine order isomorphism between the class of operator means and the class of positive monotone functions f defined on $(0,\infty)$ with $f(1)=1$ via $f(t)I_H = I_H\sigma(tI_H)$ $(t>0)$. Then f is called the representing function. In addition, $A\sigma B = A^{\frac{1}{2}}f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}$ for all positive invertible operators A and B, where f is the representing function of σ. A continuous real function f defined on an interval J is called an operator monotone function if $A \geq B$ implies $f(A) \geq f(B)$ for all self-adjoint operators A and B with spectra in J. For $A, B \geq 0$, the Lowner-Heinz inequality states that, if $A \leq B$, then $A^\alpha \leq B^\alpha$, where $0 \leq \alpha \leq 1$. Thus, t^α ($\alpha \in [0,1]$) is an operator monotone function. Other examples are the functions $((1-\alpha)+\alpha t^{-1})^{-1}$, $(1-\alpha)+\alpha t$ ($\alpha \in [0,1]$).

The operator means corresponding to operator monotone functions $((1-\alpha)+\alpha t^{-1})^{-1}$, $(1-\alpha)+\alpha t$ and t^α with $0 \leq \alpha \leq 1$ are called weighted harmonic, arithmetic and geometric means and denoted by $!_\alpha$, ∇_α and $\sharp_\alpha$, respectively. When $\alpha = \frac{1}{2}$, $!_{\frac{1}{2}}$, $\nabla_{\frac{1}{2}}$ and $\sharp_{\frac{1}{2}}$ are called harmonic, arithmetic and geometric means and simply written as $!$, ∇ and $\sharp$, respectively. It is well known that $A!_\alpha B \leq A\sharp_\alpha B \leq A\nabla_\alpha B$ for positive invertible operators A, B and $0 \leq \alpha \leq 1$.

Let σ be an operator mean with representing function f. Then for positive real numbers a and b,

$$\begin{aligned}(aI_H)\sigma(bI_H) &= (aI_H)^{\frac{1}{2}} f((aI_H)^{-\frac{1}{2}}(bI_H)(aI_H)^{-\frac{1}{2}})(aI_H)^{\frac{1}{2}} \\ &= (a^{\frac{1}{2}} f(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})a^{\frac{1}{2}})I_H \\ &=: (a\sigma b)I_H.\end{aligned}$$

Moreover, the operator mean with representing function $f(t^{-1})^{-1}$ is called the adjoint of σ and denoted by σ^*. It follows from the definition that

$$A\sigma^* B = (A^{-1}\sigma B^{-1})^{-1},$$

for $A, B > 0$.

Let $\Phi(\cdot)$ be a strictly positive unital linear map. Then for $A > 0$,

$$\Phi^{-1}(A) \leq \Phi(A^{-1}). \tag{1}$$

This is known as Choi's inequality [3, Theorem 2.3.6].

In 1990, a reverse of inequality (1) was established by Marshall and Olkin [15]: Let $\Phi(\cdot)$ be a strictly positive unital linear map. and $0 < mI_H \leq A \leq MI_H$ for positive real numbers $0 < m < M$. Then

$$\Phi(A^{-1}) \leq \frac{(M+m)^2}{4Mm}\Phi^{-1}(A). \tag{2}$$

In 2013, Lin [10] proved that inequality (2) is order preserving under squaring:

$$\Phi^2(A^{-1}) \leq \Big(\frac{(M+m)^2}{4Mm}\Big)^2 \Phi^{-2}(A), \tag{3}$$

where $A \in B(\mathcal{H})$ is a positive invertible operator with $0 < mI_H \leq A \leq MI_H$ for positive real numbers m and M.

Inequality (3) was further generalized by Fu and He [5] as follows: Let $\Phi(\cdot)$ be a strictly positive unital linear map and $0 < mI_H \leq A \leq MI_H$ for positive real numbers $0 < m < M$. Then

$$\Phi^p(A^{-1}) \leq \frac{(M+m)^{2p}}{16M^p m^p}\Phi^{-p}(A) \tag{4}$$

holds for $p \geq 2$.

Based on the similar above ideas, M. Khosravi, M. S. Moslehian and A. Sheikhhosseini got the following result [13, Theorem 2.5].

Theorem 1.1. *Let $0 < mI_H \leq A, B \leq MI_H$, σ be an arbitrary operator mean, Φ be a positive unital linear map and $p > 0$. Then*

$$\Phi^p(A\sigma B) \leq \alpha^p \Phi^p(A\sigma^* B), \tag{5}$$

where $\alpha = \max\{K, 4^{1-\frac{2}{p}}K\}$, $K = \frac{(M+m)^2}{4Mm}$.

Next, we present the p-th power of the Diaz-Metcalf type inequality, Obtained by C. Yang and C. Yang [17, Theorem 2.8].

Theorem 1.2. *Let $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ be a strictly positive unital linear map. If $m_1^2 \leq A \leq M_1^2$ and $m_2^2 \leq B \leq M_2^2$ for positive real numbers $m_1 \leq M_1$ and $m_2 \leq M_2$. Then for $p \geq 2$*

$$\Big(\frac{M_2 m_2}{M_1 m_1}\Phi(A) + \Phi(B)\Big)^p \leq \frac{1}{16}\Big\{\frac{(M_1 m_1(M_2^2 + m_2^2) + M_2 m_2(M_1^2 + m_1^2))^2}{2\sqrt{M_1 M_2 m_1 m_2} M_1^2 m_1^2 M_2 m_2}\Big\}^p \Phi^p(A\sharp B). \tag{6}$$

The Diaz-Metcalf inequality [16, Theorem 2.1] is

$$\frac{M_2 m_2}{M_1 m_1}\Phi(A) + \Phi(B) \leq \Big(\frac{M_2}{m_1} + \frac{m_2}{M_1}\Big)\Phi(A\sharp B),$$

where $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ is a positive unital linear map and $m_1^2 \leq A \leq M_1^2$ and $m_2^2 \leq B \leq M_2^2$ for positive real numbers $m_1 \leq M_1$ and $m_2 \leq M_2$.

It should be mentioned that many authors did the similar researches, see (e.g. [6], [8],[9], [11], [14], [17], [18]).

In this note, we are also concerned with the similar topics above. We present some operator inequalities for operator means and strictly positive unital linear maps. we give an operator inequality on operator means, which is a refinement of inequality (5) for $0 < p \leq 1$, and also present a further generalization of the Diaz-Metcalf type inequality.

2. Main results

We start this section with some lemmas. The Lemma 2.1 was obtained by Bhatia and Kittaneh [4, Theorem 1].

Lemma 2.1. *Let $A, B \in B(\mathcal{H})$ with $A, B \geq 0$. Then*

$$\|AB\|_\infty \leq \frac{1}{4}\big\|A + B\big\|_\infty^2,$$

where $\|\cdot\|_\infty$ is the operator norm.

The Lemma 2.2 was obtained by Ando and Zhan [2, Theorem 1].

Lemma 2.2. *For each $A, B > 0$ and $p > 1$,*

$$\|A^p + B^p\|_\infty \leq \|(A + B)^p\|_\infty.$$

The Lemma 2.3 can be found in [7].

Lemma 2.3. *Let $0 < mI_H \leq A \leq MI_H$ for positive real numbers $0 < m < M$. Then*

$$A + MmA^{-1} \leq (M + m)I_H.$$

In [1], Ando obtained the following inequality [1, Theorem 3]:

$$\Phi(A\sharp B) \leq \Phi(A)\sharp\Phi(B),$$

where Φ is a strictly positive linear map and $A, B > 0$. Actually, this inequality still holds for any operator mean.

Lemma 2.4. *Let Φ be a strictly positive linear map and $A, B > 0$. Then*

$$\Phi(A\sigma B) \leq \Phi(A)\sigma\Phi(B)$$

holds for any operator mean σ.

Proof. Consider the map Ψ defined by

$$\Psi(X) = \Phi^{-\frac{1}{2}}(A)\Phi(A^{\frac{1}{2}}XA^{\frac{1}{2}})\Phi^{-\frac{1}{2}}(A),$$

where $X \in B(\mathcal{H})$. Then Ψ is a strictly positive linear map as Φ and is unital.
Let f be the representing function of the operator mean σ. Then f is an operator monotone function. By [1,Theorem 4], we have

$$\Psi(f(X)) \leq f(\Psi(X)),$$

where $X \in B(\mathcal{H})$ is a positive invertible operator.
Hence,

$$\begin{aligned}
\Phi(A\sigma B) &= \Phi(A^{\frac{1}{2}}f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}) \\
&= \Phi^{\frac{1}{2}}(A)\Psi(f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}))\Phi^{\frac{1}{2}}(A) \\
&\leq \Phi^{\frac{1}{2}}(A)f(\Psi(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}))\Phi^{\frac{1}{2}}(A) \\
&= \Phi^{\frac{1}{2}}(A)f(\Phi^{-\frac{1}{2}}(A)\Phi(B)\Phi^{-\frac{1}{2}}(A))\Phi^{\frac{1}{2}}(A) \\
&= \Phi(A)\sigma\Phi(B).
\end{aligned}$$

This completes the proof. □

Lemma 2.5. *Let $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ be a strictly positive unital linear map and $h_1^{-1}I_H \leq A \leq h_1 I_H$ and $h_2^{-1}I_H \leq B \leq h_2 I_H$ for positive real numbers $h_1, h_2 \geq 1$. Then*

$$\Phi^{-1}(A\sigma^* B) + \Phi(A)\sigma\Phi(B) \leq \alpha(h_1, h_2)I_K$$

holds for an arbitrary operator mean σ, where $\alpha(h_1, h_2) = (h_1 + h_1^{-1})\sigma(h_2 + h_2^{-1})$.

Proof. By Lemma 2.3, we have

$$A^{-1} + A \leq (h_1 + h_1^{-1})I_H,$$

which implies

$$\Phi(A^{-1}) + \Phi(A) \leq (h_1 + h_1^{-1})I_K.$$

Similarly,

$$\Phi(B^{-1}) + \Phi(B) \leq (h_2 + h_2^{-1})I_K.$$

Then, by the subadditivity and the monotonicity of the operator mean σ, we have

$$\begin{aligned}\Phi(A)\sigma\Phi(B)+\Phi(A^{-1})\sigma\Phi(B^{-1}) &\leq (\Phi(A)+\Phi(A^{-1}))\sigma(\Phi(B)+\Phi(B^{-1}))\\ &\leq ((h_1+h_1^{-1})I_K)\sigma((h_2+h_2^{-1})I_K)\\ &= \alpha(h_1,h_2)I_K. \qquad (7)\end{aligned}$$

On the other hand, by Lemma 2.4 and Choi's inequality (1), we have

$$\Phi(A^{-1})\sigma\Phi(B^{-1}) \geq \Phi(A^{-1}\sigma B^{-1}) = \Phi((A\sigma^*B)^{-1}) \geq \Phi^{-1}(A\sigma^*B). \qquad (8)$$

Thus, the desired result follows from inequalities (7) and (8).
This completes the proof. □

Based on the same method as in [13], we obtain the following result.

Theorem 2.6. *Let $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ be a strictly positive unital linear map and $h_1^{-1}I_H \leq A \leq h_1 I_H$ and $h_2^{-1}I_H \leq B \leq h_2 I_H$ for positive real numbers h_1, $h_2 \geq 1$. Then for $p > 0$ and an arbitrary operator mean σ,*

$$(\Phi(A)\sigma\Phi(B))^p \leq \alpha_p \Phi^p(A\sigma^*B), \qquad (9)$$

where $\alpha_p = \max\left\{\left(\frac{\alpha^2(h_1,h_2)}{4}\right)^p, \frac{1}{16}\alpha^{2p}(h_1,h_2)\right\}$, $\alpha(h_1,h_2) = (h_1+h_1^{-1})\sigma(h_2+h_2^{-1})$.

Proof. If $0 < p \leq 2$, applying Lemmas 2.1 and 2.5, we get

$$\begin{aligned}\left\|\Phi^{-1}(A\sigma^*B)(\Phi(A)\sigma\Phi(B))\right\|_\infty &\leq \frac{1}{4}\left\|\Phi^{-1}(A\sigma^*B)+\Phi(A)\sigma\Phi(B)\right\|_\infty^2\\ &\leq \frac{1}{4}\left\|\alpha(h_1,h_2)I_K\right\|_\infty^2\\ &= \frac{1}{4}\alpha^2(h_1,h_2),\end{aligned}$$

which is equivalent to

$$(\Phi(A)\sigma\Phi(B))^2 \leq \frac{1}{16}\alpha^4(h_1,h_2)\Phi^2(A\sigma^*B).$$

Since $0 < \frac{p}{2} \leq 1$, by the Lowner-Heinz inequality, we have

$$(\Phi(A)\sigma\Phi(B))^p \leq \left(\frac{1}{4}\alpha^2(h_1,h_2)\right)^p \Phi^p(A\sigma^*B). \qquad (10)$$

If $p > 2$, by Lemmas 2.1 and 2.2 and 2.5, we obtain

$$\begin{aligned}\left\|\Phi^{-\frac{p}{2}}(A\sigma^*B)(\Phi(A)\sigma\Phi(B))^{\frac{p}{2}}\right\|_\infty &\leq \frac{1}{4}\left\|\Phi^{-\frac{p}{2}}(A\sigma^*B)+(\Phi(A)\sigma\Phi(B))^{\frac{p}{2}}\right\|_\infty^2\\ &\leq \frac{1}{4}\left\|\left(\Phi^{-1}(A\sigma^*B)+\Phi(A)\sigma\Phi(B)\right)^{\frac{p}{2}}\right\|_\infty^2\\ &\leq \frac{1}{4}\left\|\left(\alpha(h_1,h_2)I_K\right)^{\frac{p}{2}}\right\|_\infty^2\\ &= \frac{1}{4}\alpha^p(h_1,h_2),\end{aligned}$$

which gives

$$(\Phi(A)\sigma\Phi(B))^p \leq \frac{1}{16}\alpha^{2p}(h_1,h_2)\Phi^p(A\sigma^*B). \qquad (11)$$

Hence, inequality (9) follows from inequalities (10) and (11).
This completes the proof. □

Corollary 2.7. *Let* $0 < mI_H \leq A, B \leq MI_H$ *for positive real numbers* $0 < m \leq M$, σ *be an arbitrary operator mean,* $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ *be a strictly positive unital linear map and* $p > 0$. *Then*

$$(\Phi(A)\sigma\Phi(B))^p \leq \alpha^p \Phi^p(A\sigma^* B), \tag{12}$$

where $\alpha = \max\left\{K, 4^{1-\frac{2}{p}}K\right\}$, $K = \frac{(M+m)^2}{4Mm}$.

Proof. The condition $0 < mI_H \leq A, B \leq MI_H$ implies $0 < \sqrt{\frac{m}{M}} I_H \leq \frac{A}{\sqrt{Mm}}, \frac{B}{\sqrt{Mm}} \leq \sqrt{\frac{M}{m}} I_H$. Replacing A and B by $\frac{A}{\sqrt{Mm}}$ and $\frac{B}{\sqrt{Mm}}$, respectively and putting $h_1 = h_2 = \sqrt{\frac{M}{m}}$, then by Theorem 2.6, we can obtain the following inequality

$$\left(\Phi\left(\frac{A}{\sqrt{Mm}}\right)\sigma\Phi\left(\frac{B}{\sqrt{Mm}}\right)\right)^p \leq \alpha^p \Phi^p\left(\frac{A}{\sqrt{Mm}}\sigma^*\frac{B}{\sqrt{Mm}}\right),$$

where $\alpha_p = \max\left\{\left(\frac{\alpha^2(h_1,h_2)}{4}\right)^p, \frac{1}{16}\alpha^{2p}(h_1,h_2)\right\}$, $\alpha(h_1,h_2) = (h_1 + h_1^{-1})\sigma(h_2 + h_2^{-1})$.
On the other hand, we have

$$\Phi\left(\frac{A}{\sqrt{Mm}}\right)\sigma\Phi\left(\frac{B}{\sqrt{Mm}}\right) = \frac{1}{\sqrt{Mm}}\Phi(A)\sigma\Phi(B),$$

$$\Phi\left(\frac{A}{\sqrt{Mm}}\sigma^*\frac{B}{\sqrt{Mm}}\right) = \frac{1}{\sqrt{Mm}}\Phi(A\sigma^* B)$$

and

$$\alpha(h_1,h_2) = \frac{M+m}{\sqrt{Mm}}.$$

Hence, inequality (12) follows from the above relations.
This completes the proof. □

Remark 2.8. *Since* $\Phi(A)\sigma\Phi(B) \geq \Phi(A\sigma B)$ *for any operator mean* σ, *then by the Lowner-Heinz inequality, we have* $(\Phi(A)\sigma\Phi(B))^p \geq (\Phi(A\sigma B))^p$ *for* $0 < p \leq 1$. *Thus, inequality (12) is a refinement of inequality (5) for* $0 < p \leq 1$.

Remark 2.9. *Putting* $\sigma = \sharp_\alpha$ *for* $\alpha \in [0,1]$, *then* $\sigma^* = \sharp_\alpha$. *The conditions* $0 < m_1 I_H \leq B \leq M_1 I_H$ *and* $0 < m_2 I_H \leq B \leq M_2 I_H$ *implies* $0 < \sqrt{\frac{m_1}{M_1}} I_H \leq \frac{A}{\sqrt{M_1 m_1}} \leq \sqrt{\frac{M_1}{m_1}} I_H$ *and* $0 < \sqrt{\frac{m_2}{M_2}} I_H \leq \frac{B}{\sqrt{M_2 m_2}} \leq \sqrt{\frac{M_2}{m_2}} I_H$, *respectively. Replacing A and B by* $\frac{A}{\sqrt{M_1 m_1}}$ *and* $\frac{B}{\sqrt{M_2 m_2}}$, *respectively and putting* $h_1 = \sqrt{\frac{M_1}{m_1}}$ *and* $h_2 = \sqrt{\frac{M_2}{m_2}}$. *Noting that*

$$\alpha(h_1,h_2) = (h_1 + h_1^{-1})\sharp_\alpha(h_2 + h_2^{-1}) = \left(\frac{M_1+m_1}{\sqrt{M_1 m_1}}\right)^{1-\alpha}\left(\frac{M_2+m_2}{\sqrt{M_2 m_2}}\right)^\alpha,$$

$$\Phi\left(\frac{A}{\sqrt{M_1 m_1}}\right)\sharp_\alpha\Phi\left(\frac{B}{\sqrt{M_2 m_2}}\right) = \frac{1}{(\sqrt{M_1 m_1})^{1-\alpha}}\frac{1}{(\sqrt{M_2 m_2})^\alpha}\Phi(A)\sharp_\alpha\Phi(B),$$

and

$$\Phi\left(\frac{A}{\sqrt{M_1 m_1}}\sharp_\alpha\frac{B}{\sqrt{M_2 m_2}}\right) = \frac{1}{(\sqrt{M_1 m_1})^{1-\alpha}}\frac{1}{(\sqrt{M_2 m_2})^\alpha}\Phi(A\sharp_\alpha B),$$

then, inequality (9) gives

$$(\Phi(A)\sharp_\alpha\Phi(B))^p \leq \frac{1}{16}\left\{\frac{(M_1+m_1)^2((M_1+m_1)^{-1}(M_2+m_2))^{2\alpha}}{(m_2M_2)^\alpha(m_1M_1)^{1-\alpha}}\right\}^p \Phi^p(A\sharp_\alpha B)$$

for $p \geq 2$. *This is just the C. Yang and C. Yang's [17, Theorem 2.5] result.*

In the following, we give a further generalization related to the Diaz-Metcalf type inequality.

Theorem 2.10. *Let $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ be a strictly positive unital linear map and σ and τ be two operator means with $\sigma \leq \tau$. If $h_1^{-1}I_H \leq A \leq h_1 I_H$ and $h_2^{-1}I_H \leq B \leq h_2 I_H$ for positive real numbers h_1, $h_2 \geq 1$, then for $p \geq 2$, the following inequality holds*

$$(\Phi(A)\sigma^*\Phi(B))^p \leq \frac{1}{16}\beta^{2p}(h_1,h_2)\Phi^p(A\tau B), \tag{13}$$

where $\beta(h_1,h_2) = (h_1 + h_1^{-1})\sigma^*(h_2 + h_2^{-1})$.

Proof. By the proof of Lemma 2.5, we have

$$\Phi(A^{-1}) + \Phi(A) \leq (h_1 + h_1^{-1})I_K$$

and

$$\Phi(B^{-1}) + \Phi(B) \leq (h_2 + h_2^{-1})I_K.$$

According to Choi's inequality (1), Lemma 2.4 and the subadditivity and the monotonicity of the operator mean, we have

$$\begin{aligned}
\Phi^{-1}(A\tau B) + \Phi(A\sigma^*B) &\leq \Phi((A\tau B)^{-1}) + \Phi(A\sigma^*B)\\
&= \Phi(A^{-1}\tau^*B^{-1}) + \Phi(A\sigma^*B)\\
&\leq \Phi(A^{-1}\sigma^*B^{-1}) + \Phi(A\sigma^*B)\\
&\leq \Phi(A^{-1})\sigma^*\Phi(B^{-1}) + \Phi(A)\sigma^*\Phi(B)\\
&\leq (\Phi(A^{-1}) + \Phi(A))\sigma^*(\Phi(B^{-1}) + \Phi(B))\\
&\leq ((h_1 + h_1^{-1})I_K)\sigma^*((h_2 + h_2^{-1})I_K)\\
&= \beta(h_1,h_2)I_K.
\end{aligned}$$

Therefore, by Lemmas 2.1 and 2.2 and the above inequality, we obtain

$$\begin{aligned}
\left\|\Phi^{-\frac{p}{2}}(A\tau B)\Phi^{\frac{p}{2}}(A\sigma^*B)\right\|_\infty &\leq \frac{1}{4}\left\|\Phi^{-\frac{p}{2}}(A\tau B) + \Phi^{\frac{p}{2}}(A\sigma^*B)\right\|_\infty^2\\
&\leq \frac{1}{4}\left\|\left(\Phi^{-1}(A\tau B) + \Phi(A\sigma^*B)\right)^{\frac{p}{2}}\right\|_\infty^2\\
&\leq \frac{1}{4}\left\|\left(\beta(h_1,h_2)I_K\right)^{\frac{p}{2}}\right\|_\infty^2\\
&= \frac{1}{4}\beta^p(h_1,h_2),
\end{aligned}$$

which gives

$$(\Phi(A)\sigma^*\Phi(B))^p \leq \frac{1}{16}\beta^{2p}(h_1,h_2)\Phi^p(A\tau B). \tag{14}$$

This completes the proof. □

Based on Theorem 2.10, we can get the result of Theorem 1.2.

Corollary 2.11. *Let $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ be a strictly positive unital linear map. If $m_1^2 I_H \leq A \leq M_1^2 I_H$ and $m_2^2 I_H \leq B \leq M_2^2 I_H$ for positive real numbers $0 < m_1 \leq M_1$ and $0 < m_2 \leq M_2$. Then for $p \geq 2$*

$$\Big(\frac{M_2 m_2}{M_1 m_1}\Phi(A) + \Phi(B)\Big)^p \leq \frac{1}{16}\Big\{\frac{(M_1 m_1(M_2^2 + m_2^2) + M_2 m_2(M_1^2 + m_1^2))^2}{2\sqrt{M_1 M_2 m_1 m_2}M_1^2 m_1^2 M_2 m_2}\Big\}^p \Phi^p(A\sharp B). \tag{15}$$

Proof. The conditions $0 < m_1^2 I_H \leq A \leq M_1^2 I_H$ and $0 < m_2^2 I_H \leq B \leq M_2^2 I_H$ imply $0 < \frac{m_1}{M_1} I_H \leq \frac{A}{M_1 m_1} \leq \frac{M_1}{m_1} I_H$ and $0 < \frac{m_2}{M_2} I_H \leq \frac{B}{M_2 m_2} \leq \frac{M_2}{m_2} I_H$. Replacing A and B by $\frac{A}{M_1 m_1}$ and $\frac{B}{M_2 m_2}$, respectively and putting $h_1 = \frac{M_1}{m_1}$, $h_2 = \frac{M_2}{m_2}$ and $\tau = \sharp$, $\sigma =!$. Then,

$$\Phi\Big(\frac{A}{M_1 m_1}\Big)\sigma^*\Phi\Big(\frac{B}{M_2 m_2}\Big) = \frac{1}{2}\Big(\Phi\Big(\frac{A}{M_1 m_1}\Big) + \Phi\Big(\frac{B}{M_2 m_2}\Big)\Big) = \frac{1}{2M_2 m_2}\Big(\frac{M_2 m_2}{M_1 m_1}\Phi(A) + \Phi(B)\Big),$$

$$\Phi\Big(\frac{A}{M_1 m_1}\tau\frac{B}{M_2 m_2}\Big) = \Phi\Big(\frac{A}{M_1 m_1}\sharp\frac{B}{M_2 m_2}\Big) = \frac{1}{\sqrt{M_1 M_2 m_1 m_2}}\Phi(A\sharp B)$$

and

$$\beta(h_1, h_2) = \frac{h_1 + h_1^{-1} + h_2 + h_2^{-1}}{2} = \frac{M_2 m_2(M_1^2 + m_1^2) + M_1 m_1(M_2^2 + m_2^2)}{2M_1 M_2 m_1 m_2}.$$

Therefore, by Theorem 2.10, we can obtain this corollary.
This completes the proof. □

Remark 2.12. *Putting $p = 2$, then inequality (15) (or (6)) gives*

$$\Big(\frac{M_2 m_2}{M_1 m_1}\Phi(A) + \Phi(B)\Big)^2 \leq \Big\{\frac{(M_1 m_1(M_2^2 + m_2^2) + M_2 m_2(M_1^2 + m_1^2))^2}{8\sqrt{M_1 M_2 m_1 m_2} M_1^2 m_1^2 M_2 m_2}\Big\}^2 \Phi^2(A\sharp B).$$

This inequality is just the result of Theorem 2.14 obtained by Moslehian and Fu [14, Theorem 2.14].

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