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On the Convexity of Functions

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Abstract. Let A, B, and X be bounded linear operators on a separable Hilbert space such that A, B are positive, $X \geq \gamma I$, for some positive real number γ, and $\alpha \in [0,1]$. Among other results, it is shown that if $f(t)$ is an increasing function on $[0,\infty)$ with $f(0) = 0$ such that $f\left(\sqrt{t}\right)$ is convex, then

$$\gamma \left|\left|\left| f(\alpha A + (1-\alpha)B) + f(\beta |A-B|) \right|\right|\right| \leq \left|\left|\left| \alpha f(A) X + (1-\alpha) X f(B) \right|\right|\right|$$

for every unitarily invariant norm, where $\beta = \min(\alpha, 1-\alpha)$. Applications of our results are given.

1. Introduction

Let $\mathbb{B}(\mathbb{H})$ be the algebra of all bounded linear operators on a separable Hilbert space $\mathbb{H}$. For a compact operator $A \in \mathbb{B}(\mathbb{H})$, let $s_1(A), s_2(A), ...$ denote the singular values of A, i.e , the eigenvalues of the operator $|A| = (A^*A)^{1/2}$.

In addition to the usual operator norm, which is defined on all of $\mathbb{B}(\mathbb{H})$, a unitarily invariant (or symmetric) norm $|||\cdot|||$ is a norm defined on a norm ideal contained in the ideal of compact operators and satisfies the invariance property $|||UAV||| = |||A|||$ for all operators A and for all unitary operators U and V in $\mathbb{B}(\mathbb{H})$. For the sake of brevity, we will make no explicit mention of this ideal. Thus, when we consider $|||A|||$ we are assuming that the operator A belongs to the norm ideal associated with $|||\cdot|||$. Moreover, each unitarily invariant norm $|||\cdot|||$ is a symmetric gauge function of the singular values. For the general theory of unitarily invariant norms, we refer to [2] or [5].

For $1 \leq p < \infty$, the Schatten p-norm of a compact operator $A \in \mathbb{B}(\mathbb{H})$ is defined by

$$\|A\|_p = \left(\sum_{j=1}^{n} s_j^p(A) \right)^{1/p}.$$

In particular, when $p = 2$, this norm is called the Hilbert-Schmidt norm or the Frobenious norm. It can be seen that Schatten p-norms are typical examples of unitarily invariant norms. Moreover, one of the useful

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properties of the Hilbert-Schmidt norm is that if $A \in \mathbb{B}(\mathbb{H})$, then

$$\|A\|_2 = \left(\sum_{i,j=1}^{n} \left| \left\langle Ae_j, f_i \right\rangle \right|^2 \right)^{1/2},$$

where $\{e_j\}$ and $\{f_j\}$ are two orthonormal bases of the Hilbert space $\mathbb{H}$.

A non-negative function f on $[0, \infty)$ is said to be convex if

$$f(\alpha a + (1-\alpha) b) \leq \alpha f(a) + (1-\alpha) f(b) \tag{1}$$

for $a, b \in [0, \infty)$ and $\alpha \in [0, 1]$.

A generalization of the inequality (1) for operators in $\mathbb{B}(\mathbb{H})$ (see, [1]) asserts that if $A, B \in \mathbb{B}(\mathbb{H})$ are positive, $\alpha \in [0, 1]$, and $f(t)$ is a non-negative convex function on $[0, \infty)$, then

$$\left|\left|\left| f(\alpha A + (1-\alpha) B) \right|\right|\right| \leq \left|\left|\left| \alpha f(A) + (1-\alpha) f(B) \right|\right|\right| \tag{2}$$

for every unitarily invariant norm.

Applying the inequality (2) to the convex function $f(t) = t^r, t \in [0, \infty)$ for $r \geq 1$ we have

$$\left|\left|\left| (\alpha A + (1-\alpha) B)^r \right|\right|\right| \leq |||\alpha A^r + (1-\alpha) B^r||| \tag{3}$$

for every unitarily invariant norm. Specializing the inequality (3) to the Hilbert-Schmidt norm and letting $r = 2$, we have

$$\left\| (\alpha A + (1-\alpha) B)^2 \right\|_2 \leq \left\| \alpha A^2 + (1-\alpha) B^2 \right\|_2. \tag{4}$$

In this paper, we introduce inequalities for convex functions. In Section 2, we introduce a refinement of the inequality (2) and we give applications of this refinement. Section 3, is devoted to give refinements of the inequality (4).

2. A refinement of the inequality (2)

In this section we introduce a refinement of the inequality (2). In order to do that we need the following two lemmas. The first lemma a consequence of the Spectral Theorem of operators (see, e.g., [2, p. 5]) and the second lemma is given in [7].

Lemma 2.1. *Let $A \in \mathbb{B}(\mathbb{H})$ be positive and let f be an increasing convex function on $[0, \infty)$. Then $f\left(s_j(A)\right) = s_j(f(A))$ for $j = 1, 2, ...$*

Lemma 2.2. *Let $A, B \in \mathbb{B}(\mathbb{H})$ be positive and let $f(t)$ be a non-negative convex function on $[0, \infty)$ with $f(0) = 0$. Then $\left|\left|\left| f(A) + f(B) \right|\right|\right| \leq \left|\left|\left| f(A+B) \right|\right|\right|$ for every unitarily invariant norm.*

Under certain conditions, an improvement of the inequality (2) can be seen in the following result.

Theorem 2.3. *Let $A, B \in \mathbb{B}(\mathbb{H})$ be positive, $\alpha \in [0, 1]$, and let f be an increasing function on $[0, \infty)$ with $f(0) = 0$ such that $f\left(\sqrt{t}\right)$ is convex. Then*

$$\left|\left|\left| f(\alpha A + (1-\alpha) B) + f(\beta |A - B|) \right|\right|\right| \leq \left|\left|\left| \alpha f(A) + (1-\alpha) f(B) \right|\right|\right|$$

for every unitarily invariant norm, where $\beta = \min(\alpha, 1-\alpha)$.

Proof.

Case 1: Suppose that $\alpha \in \left[\frac{1}{2}, 1\right]$. By direct computations it can be seen that

$$\begin{aligned} &\alpha A^2 + (1-\alpha) B^2 - (\alpha A + (1-\alpha) B)^2 - (1-\alpha)^2 (A-B)^2 \\ &= (1-\alpha)(2\alpha-1) A^2 - (1-\alpha)(2\alpha-1)(AB+BA) + (1-\alpha)(2\alpha-1) B^2 \\ &= (1-\alpha)(2\alpha-1)\left(A^2 - AB - BA + B^2\right) \\ &= (1-\alpha)(2\alpha-1)(A-B)^2 \\ &\geq 0 \ \ (\text{since } \alpha \in \left[\frac{1}{2}, 1\right] \text{ and } (A-B)^2 \geq 0) \end{aligned}$$

Consequently,

$$(\alpha A + (1-\alpha) B)^2 + (1-\alpha)^2 (A-B)^2 \leq \alpha A^2 + (1-\alpha) B^2.$$

So,

$$s_j\left((\alpha A + (1-\alpha) B)^2 + (1-\alpha)^2 (A-B)^2\right) \leq s_j\left(\alpha A^2 + (1-\alpha) B^2\right) \tag{5}$$

for $j = 1, 2, \ldots$. Let $g(t) = f\left(\sqrt{t}\right)$, $t \in [0, \infty)$. Then g is an increasing convex function on $[0, \infty)$. It follows that

$$\begin{aligned} &s_j\left(g\left((\alpha A + (1-\alpha) B)^2 + (1-\alpha)^2 (A-B)^2\right)\right) \\ &= g\left(s_j\left((\alpha A + (1-\alpha) B)^2 + (1-\alpha)^2 (A-B)^2\right)\right) \text{ (by Lemma 2.1)} \\ &\leq g\left(s_j\left(\alpha A^2 + (1-\alpha) B^2\right)\right) \text{ (by the inequality (5))} \\ &= s_j\left(g\left(\alpha A^2 + (1-\alpha) B^2\right)\right) \text{ (by Lemma 2.1)} \end{aligned} \tag{6}$$

for $j = 1, 2, \ldots$, so

$$\begin{aligned} &\left|\left|\left| g\left((\alpha A + (1-\alpha) B)^2 + (1-\alpha)^2 (A-B)^2\right) \right|\right|\right| \\ &\leq \left|\left|\left| g\left(\alpha A^2 + (1-\alpha) B^2\right) \right|\right|\right| \text{ (by the inequality (6))} \\ &\leq \left|\left|\left| \alpha g\left(A^2\right) + (1-\alpha) g\left(B^2\right) \right|\right|\right| \text{ (by the inequality (2))} \\ &= \left|\left|\left| \alpha f(A) + (1-\alpha) f(B) \right|\right|\right|. \end{aligned} \tag{7}$$

Also,

$$\begin{aligned} &\left|\left|\left| g\left((\alpha A + (1-\alpha) B)^2 + (1-\alpha)^2 (A-B)^2\right) \right|\right|\right| \\ &\geq \left|\left|\left| \left(g\left((\alpha A + (1-\alpha) B)^2\right) + g\left((1-\alpha)^2 (A-B)^2\right)\right) \right|\right|\right| \text{ (by Lemma 2.2)} \\ &= \left|\left|\left| f((\alpha A + (1-\alpha) B) + f((1-\alpha)|A-B|)) \right|\right|\right|. \end{aligned} \tag{8}$$

Thus,

$$\left|\left|\left| f(\alpha A + (1-\alpha) B) + f(\beta |A-B|) \right|\right|\right|$$

$$
\begin{aligned}
&= \left|\left|\left| f(\alpha A + (1-\alpha)B) + f((1-\alpha)|A-B|) \right|\right|\right| \quad \left(\text{since } \alpha \in \left[\frac{1}{2}, 1\right]\right) \\
&\leq \left|\left|\left| \alpha f(A) + (1-\alpha) f(B) \right|\right|\right| \quad \text{(by the inequalities (7) and (8))}.
\end{aligned}
$$

Case 2: Suppose that $\alpha \in \left[0, \frac{1}{2}\right]$. Let $\tilde{\alpha} = 1 - \alpha$ and $\tilde{\beta} = \min(\tilde{\alpha}, 1 - \tilde{\alpha})$. Then $\tilde{\alpha} \in \left[\frac{1}{2}, 1\right]$ and $\tilde{\beta} = \beta$. It follows from Case 1, by interchanging the operators A, B and replacing α by $\tilde{\alpha}$, that

$$
\begin{aligned}
&\left|\left|\left| f(\alpha A + (1-\alpha)B) + f(\beta|A-B|) \right|\right|\right| \\
&= \left|\left|\left| f(\tilde{\alpha} B + (1-\tilde{\alpha})A) + f\left(\tilde{\beta}|B-A|\right) \right|\right|\right| \\
&\leq \left|\left|\left| \tilde{\alpha} f(B) + (1-\tilde{\alpha}) f(A) \right|\right|\right| \quad \text{(by Case 1)} \\
&= \left|\left|\left| \alpha f(A) + (1-\alpha) f(B) \right|\right|\right|,
\end{aligned}
$$

as required. □

Remark 2.4. *It can be seen that the convexity of the function $f\left(\sqrt{t}\right)$ given in Theorem 2.3 is essential and can not be replaced by taking $f(t)$ to be convex. Indeed, let $f(t) = t, \alpha \in (0, 1)$ and $A, B \in \mathbb{B}(\mathbb{H})$ be positive. Then*

$$
\begin{aligned}
\left|\left|\left| f(\alpha A + (1-\alpha)B) + f(\beta|A-B|) \right|\right|\right| &= \left|\left|\left| \alpha A + (1-\alpha)B + \beta|A-B| \right|\right|\right| \\
&\geq |||\alpha A + (1-\alpha)B||| \\
&= \left|\left|\left| \alpha f(A) + (1-\alpha) f(B) \right|\right|\right|.
\end{aligned}
$$

An application of Theorem 2.3 can be seen as follows. In this result we introduce equality conditions of the inequality (2).

Corollary 2.5. *Let $A, B \in \mathbb{B}(\mathbb{H})$ be positive, $\alpha \in [0, 1]$, and let f be a nonzero increasing function on $[0, \infty)$ with $f(0) = 0$ such that $f\left(\sqrt{t}\right)$ is convex. Then*

$$
\left|\left|\left| f(\alpha A + (1-\alpha)B) \right|\right|\right| = \left|\left|\left| \alpha f(A) + (1-\alpha) f(B) \right|\right|\right|
$$

for every unitarily invariant norm if and only if $A = B$, $\alpha = 0$, or $\alpha = 1$.

Proof. Suppose that $\left|\left|\left| f(\alpha A + (1-\alpha)B) \right|\right|\right| = \left|\left|\left| \alpha f(A) + (1-\alpha) f(B) \right|\right|\right|$. Then Theorem 2.3 implies that $f(\beta|A-B|) = 0$ and since f is a nonzero function we have $\beta|A-B| = 0$. Consequently, $A = B$ or $\beta = 0$ and so $A = B$, $\alpha = 0$, or $\alpha = 1$. The converse is trivial and the proof is complete. □

The following lemma can be found in [3].

Lemma 2.6. *Let $A, B, X \in \mathbb{B}(\mathbb{H})$ such that A, B are self-adjoint and $X \geq \gamma I$, for some positive real number γ. Then*

$$
\gamma\, |||A + B||| \leq |||AX + XB|||
$$

for every unitarily invariant norm.

Based on Lemma 2.6, an application of Theorem 2.3 can be seen in the following result.

Corollary 2.7. *Let $A, B, X \in \mathbb{B}(\mathbb{H})$ such that A, B are positive, $X \geq \gamma I$, for some positive real number γ, $\alpha \in [0, 1]$, and let f be an increasing function on $[0, \infty)$ with $f(0) = 0$ such that $f\left(\sqrt{t}\right)$ is convex. Then*

$$
\gamma \left|\left|\left| f(\alpha A + (1-\alpha)B) + f(\beta|A-B|) \right|\right|\right| \leq \left|\left|\left| \alpha f(A) X + (1-\alpha) X f(B) \right|\right|\right|
$$

for every unitarily invariant norm, where $\beta = \min(\alpha, 1-\alpha)$ and I is the identity operator in $\mathbb{B}(\mathbb{H})$.

Proof. In Lemma 2.6, replacing A and B by $\alpha f(A)$ and $(1-\alpha) f(B)$, respectively, we have

$$\gamma \left|\left|\left|\alpha f(A) + (1-\alpha) f(B)\right|\right|\right| \leq \left|\left|\left|\alpha f(A) X + (1-\alpha) X f(B)\right|\right|\right|. \tag{9}$$

Now, the result follows from Theorem 2.3 and the inequality (9). □

It is clear that Corollary 2.7 presents a generalization of the Theorem 2.3 which can be retained by taking $X = I$ and $\gamma = 1$.

Specializing Corollary 2.7 to some particular functions can be seen in the following result.

Corollary 2.8. *Let $A, B, X \in \mathbb{B}(\mathbb{H})$ such that A, B are positive, $X \geq \gamma I$, for some positive real number γ, $\alpha \in [0,1]$, and let $r \geq 2$. Then*

$$\gamma \left|\left|\left|e^{(\alpha A+(1-\alpha)B)^2} + e^{\beta^2|A-B|^2} - 2I\right|\right|\right| \leq \left|\left|\left|\alpha e^{A^2} X + (1-\alpha) X e^{B^2} - X\right|\right|\right| \tag{10}$$

and

$$\gamma \left|\left|\left|(\alpha A + (1-\alpha) B)^r + \beta^r |A-B|^r\right|\right|\right| \leq |||\alpha A^r X + (1-\alpha) X B^r||| \tag{11}$$

for every unitarily invariant norm, where $\beta = \min(\alpha, 1-\alpha)$.

Proof. The inequalities (10) and (11) follow by applying Corollary 2.7 to the functions $f(t) = e^{t^2} - 1$ and $f(t) = t^r$, respectively. □

In order to have another application of Theorem 2.3, we need the following two lemmas the first lemma is given in [3], while for the second lemma see, e.g., [6, p. 124].

Lemma 2.9. *Let $A, B, X \in \mathbb{B}(\mathbb{H})$ such that A, B are self-adjoint and $X \geq \pm(A+B)$. Then*

$$\left|\left|\left|(A+B)^2\right|\right|\right| \leq |||AX + XB\,|||$$

for every unitarily invariant norm.

Lemma 2.10. *Let $A, B \in \mathbb{B}(\mathbb{H})$ be positive and let g be an increasing convex function on $[0,\infty)$ with $g(0) = 0$. Then*

$$|||A||| \leq |||B|||$$

for every unitarily invariant norm implies that

$$\left|\left|\left|g(A)\right|\right|\right| \leq \left|\left|\left|g(B)\right|\right|\right|$$

for every unitarily invariant norm.

Corollary 2.11. *Let $A, B, X \in \mathbb{B}(\mathbb{H})$ such that A, B are positive, $\alpha \in [0,1]$, and let f be an increasing function on $[0,\infty)$ with $f(0) = 0$ such that $f\left(\sqrt{t}\right)$ is convex. If $X \geq \pm(\alpha f(A) + (1-\alpha) f(B))$, then*

$$\left|\left|\left|f^2(\alpha A + (1-\alpha) B) + f^2(\beta |A-B|)\right|\right|\right| \leq \left|\left|\left|\alpha f(A) X + (1-\alpha) X f(B)\right|\right|\right|$$

for every unitarily invariant norm, where $\beta = \min(\alpha, 1-\alpha)$.

Proof. In Lemma 2.9, replacing A and B by $\alpha f(A)$ and $(1-\alpha) f(B)$, respectively, we have

$$\left|\left|\left|(\alpha f(A) + (1-\alpha) f(B))^2\right|\right|\right| \leq \left|\left|\left|\alpha f(A) X + (1-\alpha) X f(B)\,\right|\right|\right|. \tag{12}$$

The convexity of the function $g(t) = t^2$ together with Theorem 2.3 and Lemma 2.10 implies that

$$\left|\left|\left|(f(\alpha A + (1-\alpha) B) + f(\beta |A-B|))^2\right|\right|\right| \leq \left|\left|\left|(\alpha f(A) + (1-\alpha) f(B))^2\right|\right|\right|. \tag{13}$$

Also, the convexity of the function $g(t) = t^2$ together with Lemma 2.2 implies that

$$\begin{aligned} &\left|\left|\left|(f(\alpha A + (1-\alpha)B) + f(\beta|A-B|))^2\right|\right|\right| \\ &\geq \left|\left|\left|f^2(\alpha A + (1-\alpha)B) + f^2(\beta|A-B|)\right|\right|\right| \end{aligned} \tag{14}$$

Now, the result follows from the inequalities (12), (13), and (14). □

In the rest of this section, we give inequalities that involve direct sum of operators. In order to do that we need the following two lemmas. The first lemma follows from the basic properties of unitarily invariant norms and for the second lemma see, e.g., [2, p. 97].

Lemma 2.12. *Let $A, B \in \mathbb{B}(\mathbb{H})$. Then $|||A||| \leq |||B|||$ for every unitarily invariant norm if and only if $|||A \oplus 0||| \leq |||B \oplus 0|||$ for every unitarily invariant norm.*

Lemma 2.13. *Let $A, B \in \mathbb{B}(\mathbb{H})$ be positive. Then $|||A \oplus B||| \leq |||(A+B) \oplus 0|||$ for every unitarily invariant norm.*

Corollary 2.14. *Let $A, B, X \in \mathbb{B}(\mathbb{H})$ such that A, B are positive, $X \geq \gamma I$, for some positive real number γ, $\alpha \in [0,1]$, and let f be an increasing function on $[0,\infty)$ with $f(0) = 0$ such that $f\left(\sqrt{t}\right)$ is convex. Then*

$$\gamma \left|\left|\left|f(\alpha A + (1-\alpha)B) \oplus f(\beta|A-B|)\right|\right|\right| \leq \left|\left|\left|(\alpha f(A)X + (1-\alpha)Xf(B)) \oplus 0\right|\right|\right|$$

for every unitarily invariant norm, where $\beta = \min(\alpha, 1-\alpha)$.

Proof. Corollary 2.7 together with Lemma 2.12 implies that

$$\begin{aligned} &\gamma\left|\left|\left|(f(\alpha A + (1-\alpha)B) + f(\beta|A-B|)) \oplus 0\right|\right|\right| \\ &\leq \left|\left|\left|(\alpha f(A)X + (1-\alpha)Xf(B)) \oplus 0\right|\right|\right| \end{aligned} \tag{15}$$

Also, Lemma 2.13 implies that

$$\begin{aligned} &\left|\left|\left|(f(\alpha A + (1-\alpha)B) + f(\beta|A-B|)) \oplus 0\right|\right|\right| \\ &\geq \left|\left|\left|f(\alpha A + (1-\alpha)B) \oplus f(\beta|A-B|)\right|\right|\right|. \end{aligned} \tag{16}$$

Now, the result follows from the inequalities (15) and (16). □

Corollary 2.15. *Let $A, B, X \in \mathbb{B}(\mathbb{H})$ such that A, B are positive, $\alpha \in [0,1]$, and let f be an increasing function on $[0,\infty)$ with $f(0) = 0$ such that $f\left(\sqrt{t}\right)$ is convex. If $X \geq \pm(\alpha f(A) + (1-\alpha)f(B))$, then*

$$\begin{aligned} &\left|\left|\left|f^2(\alpha A + (1-\alpha)B) \oplus f^2(\beta|A-B|)\right|\right|\right| \\ &\qquad\qquad \leq \left|\left|\left|(\alpha f(A)X + (1-\alpha)Xf(B)) \oplus 0\right|\right|\right| \end{aligned}$$

for every unitarily invariant norm, where $\beta = \min(\alpha, 1-\alpha)$.

We close this section by specializing Corollary 2.14 to the Schatten p-norms.

Corollary 2.16. *Let $A, B, X \in \mathbb{B}(\mathbb{H})$ be positive, $X \geq \gamma I$, for some positive real number γ, $\alpha \in [0,1]$, and let f be an increasing function on $[0,\infty)$ with $f(0) = 0$ such that $f\left(\sqrt{t}\right)$ is convex. Then*

$$\gamma^{p/2}\left(\left\|f(\alpha A + (1-\alpha)B)\right\|_p^p + \left\|f(\beta|A-B|)\right\|_p^p\right) \leq \left\|\alpha f(A)X + (1-\alpha)Xf(B)\right\|_p^p$$

for $p \geq 2$, where $\beta = \min(\alpha, 1-\alpha)$. In particular, letting $X = I$ and $\gamma = 1$, we have

$$\left\|f(\alpha A + (1-\alpha)B)\right\|_p^p + \left\|f(\beta|A-B|)\right\|_p^p \leq \left\|\alpha f(A) + (1-\alpha)f(B)\right\|_p^p$$

for $p \geq 2$.

Proof. Let $g(t) = t^{p/2}, t \geq 0$. Then g is convex. It follows from Corollary 2.14 and Lemma 2.10 that

$$\gamma^{p/2} \left|\left|\left| f^{p/2}(\alpha A + (1-\alpha) B) \oplus f^{p/2}((1-\alpha)|A-B|) \right|\right|\right| \leq \left|\left|\left| \left|\alpha f(A) X + (1-\alpha) X f(B)\right|^{p/2} \right|\right|\right|. \tag{17}$$

Applying the inequality (17) to the Frobenious norm $\|\cdot\|_2$ we have

$$\begin{aligned} &\gamma^{p/2}\left(\left\|f(\alpha A + (1-\alpha) B)\right\|_p^p + \left\|f((1-\alpha)|A-B|)\right\|_p^p\right) \\ &= \gamma^{p/2}\left(\left\|f^{p/2}(\alpha A + (1-\alpha) B)\right\|_2^2 + \left\|f^{p/2}((1-\alpha)|A-B|)\right\|_2^2\right) \\ &= \gamma^{p/2}\left\|f^{p/2}(\alpha A + (1-\alpha) B) \oplus f^{p/2}((1-\alpha)|A-B|)\right\|_2^2 \\ &\leq \left\|\left|\alpha f(A) X + (1-\alpha) X f(B)\right|^{p/2}\right\|_2^2 \text{ (by the inequality (17))} \\ &= \left\|\alpha f(A) X + (1-\alpha) X f(B)\right\|_p^p, \end{aligned}$$

as required. □

3. A refinement of the inequality (4)

It can be seen that Corollary 2.16 constitute of a refinement of the inequality (4). By taking $f(t) = t^2$, this refinement asserts that if $A, B \in \mathbb{B}(\mathbb{H})$ are positive and $\alpha \in [0,1]$, then

$$\left\|(\alpha A + (1-\alpha) B)^2\right\|_2^2 + \left\|\beta^2 |A-B|^2\right\|_2^2 \leq \left\|\alpha A^2 + (1-\alpha) B^2\right\|_2^2,$$

where $\beta = \min(\alpha, 1-\alpha)$.

In this section, we are interested in introducing another refinement of the inequality (4). In order to do that we need to start with the Binomial Theorem for scalars. The Binomial Theorem for scalars asserts that if $a, b \in \mathbb{R}$, then $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$, where $n \in \mathbb{N}$. Now, we need the following lemma (see, e.g., [8, p. 76]).

Lemma 3.1. *Let $a, b \in (0, \infty)$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then $f(t) = (\alpha a^t + (1-\alpha) b^t)^{1/t}$ is an increasing function on $(0, \infty)$.*

Based on Lemma 3.1 and the Binomial Theorem for scalars, we get the following result.

Theorem 3.2. *Let $A, B, X \in \mathbb{B}(\mathbb{H})$ such that A and B are positive, $\alpha \in (0,1)$, and $m, l \in \mathbb{N}$ with $l \leq m$. Then*

$$\left\|\sum_{k=0}^{m} \binom{m}{k} \alpha^{m-k} (1-\alpha)^k A^{l(m-k)} X B^{lk}\right\|_2 \leq \left\|\sum_{k=0}^{l} \binom{l}{k} \alpha^{l-k} (1-\alpha)^k A^{m(l-k)} X B^{mk}\right\|_2.$$

Proof. Since A and B are positive, there exist two orthonormal bases $\{e_j\}$ and $\{f_j\}$ of $\mathbb{C}^n$ and two sequences $\{s_j(A)\}$ and $\{s_j(B)\}$ of positive real number such that $Af_j = s_j(A) f_j, Be_j = s_j(B) e_j$ for $j = 1, 2, \ldots$. For simplicity, let $\lambda_j = s_j(A)$ and $\mu_j = s_j(B)$ for $j = 1, 2, \ldots$. So,

$$\left\|\sum_{k=0}^{m} \binom{m}{k} \alpha^{m-k} (1-\alpha)^k A^{\frac{m-k}{m}} X B^{\frac{k}{m}}\right\|_2^2$$

$$= \sum_{i,j=1}^{\infty}\left|\sum_{k=0}^{m}\binom{m}{k}\alpha^{m-k}(1-\alpha)^{k}\left\langle A^{\frac{m-k}{m}}XB^{\frac{k}{m}}e_j,f_i\right\rangle\right|^2$$
$$= \sum_{i,j=1}^{\infty}\left|\sum_{k=0}^{m}\binom{m}{k}\alpha^{m-k}(1-\alpha)^{k}\lambda_i^{\frac{m-k}{m}}\mu_j^{\frac{k}{m}}\right|^2\left|\left\langle Xe_j,f_i\right\rangle\right|^2$$
$$= \sum_{i,j=1}^{\infty}\left|\sum_{k=0}^{m}\binom{m}{k}\left(\alpha\lambda_i^{\frac{1}{m}}\right)^{m-k}\left((1-\alpha)\mu_j^{\frac{1}{m}}\right)^{k}\right|^2\left|\left\langle Xe_j,f_i\right\rangle\right|^2$$
$$= \sum_{i,j=1}^{\infty}\left(\alpha\lambda_i^{\frac{1}{m}}+(1-\alpha)\mu_j^{\frac{1}{m}}\right)^{2m}\left|\left\langle Xe_j,f_i\right\rangle\right|^2$$
$$= \sum_{i,j=1}^{\infty}f^2\left(\frac{1}{m}\right)\left|\left\langle Xe_j,f_i\right\rangle\right|^2, \tag{18}$$

where $f(t)=\left(\alpha\lambda_i^t+(1-\alpha)\mu_j^t\right)^{1/t}$. Since $\frac{1}{m}\leq\frac{1}{l}$, Lemma 3.1 implies that $f\left(\frac{1}{m}\right)\leq f\left(\frac{1}{l}\right)$, and so the inequality (18) implies that

$$\left\|\sum_{k=0}^{m}\binom{m}{k}\alpha^{m-k}(1-\alpha)^{k}A^{\frac{m-k}{m}}XB^{\frac{k}{m}}\right\|_2^2$$
$$\leq \sum_{i,j=1}^{\infty}f^2\left(\frac{1}{l}\right)\left|\left\langle Xe_j,f_i\right\rangle\right|^2$$
$$= \sum_{i,j=1}^{\infty}\left(\alpha\lambda_i^{\frac{1}{l}}+(1-\alpha)\mu_j^{\frac{1}{l}}\right)^{2l}\left|\left\langle Xe_j,f_i\right\rangle\right|^2$$
$$= \sum_{i,j=1}^{\infty}\left|\sum_{k=0}^{l}\binom{l}{k}\left(\alpha\lambda_i^{\frac{1}{l}}\right)^{l-k}\left((1-\alpha)\mu_j^{\frac{1}{l}}\right)^{k}\right|^2\left|\left\langle Xe_j,f_i\right\rangle\right|^2$$
$$= \sum_{i,j=1}^{\infty}\left|\sum_{k=0}^{l}\binom{l}{k}\alpha^{l-k}(1-\alpha)^{k}\left\langle A^{\frac{l-k}{l}}XB^{\frac{k}{l}}e_j,f_i\right\rangle\right|^2$$
$$= \left\|\sum_{k=0}^{l}\binom{l}{k}\alpha^{l-k}(1-\alpha)^{k}A^{\frac{l-k}{l}}XB^{\frac{k}{l}}\right\|_2^2. \tag{19}$$

Now, the result follows from the inequality (19) by replacing A and B by A^{lm} and B^{lm}, respectively. □

An application of Theorem 3.2 can be seen as follows

Corollary 3.3. *Let $A,B\in\mathbb{B}(\mathbb{H})$ be positive and $\alpha\in[0,1]$. If $m\in\mathbb{N}$, then*

$$\left\|\sum_{k=0}^{m}\binom{m}{k}\alpha^{m-k}(1-\alpha)^{k}A^{(m-k)}XB^{k}\right\|_2\leq\left\|\alpha A^mX+(1-\alpha)XB^m\right\|_2. \tag{20}$$

In particular, letting $m=2$ and $\alpha=\frac{1}{2}$, we have

$$\left\|A^2X+2AXB+XB^2\right\|_2\leq 2\left\|A^2X+XB^2\right\|_2.$$

Proof. The inequality (20) follows from Theorem 3.2 by letting $l = 1$. □

Also, we need the following lemma (see, e.g., [4]).

Lemma 3.4. *Let $T \in \mathbb{B}(\mathbb{H})$. Then*

$$\|T\|_2^2 = \|\mathrm{Re}\, T\|_2^2 + \|\mathrm{Im}\, T\|_2^2 .$$

Our second refinement of the inequality (4) can be stated as follows.

Corollary 3.5. *Let $A, B \in \mathbb{B}(\mathbb{H})$ be positive semidefinite and $\alpha \in [0, 1]$. Then*

$$\left\|(\alpha A + (1-\alpha) B)^2\right\|_2^2 + \alpha^2 (1-\alpha)^2 \|AB - BA\|_2^2 \le \left\|\alpha A^2 + (1-\alpha) B^2\right\|_2^2 .$$

Proof. In Theorem 3.2, letting $X = I, l = 1$, and $m = 2$, we have

$$\left\|\sum_{k=0}^{2} \binom{2}{k} \alpha^{2-k} (1-\alpha)^k A^{(2-k)} X B^k\right\|_2 \le \left\|\sum_{k=0}^{1} \binom{1}{k} \alpha^{1-k} (1-\alpha)^k A^{2(1-k)} X B^{2k}\right\|_2$$

and so

$$\left\|\alpha^2 A^2 + 2\alpha (1-\alpha) AB + (1-\alpha)^2 B^2\right\|_2 \le \left\|\alpha A^2 + (1-\alpha) B^2\right\|_2 . \tag{21}$$

By direct computations we have

$$\mathrm{Re}\left(\alpha^2 A^2 + 2\alpha (1-\alpha) AB + (1-\alpha)^2 B^2\right) = (\alpha A + (1-\alpha) B)^2$$

and

$$\mathrm{Im}\left(\alpha^2 A^2 + 2\alpha (1-\alpha) AB + (1-\alpha)^2 B^2\right) = -i\alpha (1-\alpha) (AB - BA) .$$

So, Lemma 3.4 implies that

$$\begin{aligned} &\left\|\alpha^2 A^2 + 2\alpha (1-\alpha) AB + (1-\alpha)^2 B^2\right\|_2^2 \\ =\ & \left\|(\alpha A + (1-\alpha) B)^2\right\|_2^2 + \|-i\alpha (1-\alpha)(AB - BA)\|_2^2 \\ =\ & \left\|(\alpha A + (1-\alpha) B)^2\right\|_2^2 + \alpha^2 (1-\alpha)^2 \|AB - BA\|_2^2 . \end{aligned} \tag{22}$$

Now, the result follows from the inequality (21) and the identity (22). □

References

[1] J. Aujla and F. Silva, Weak majorization inequalities and convex functions, Linear Algebra Appl. 369 (2003) 217–233.
[2] R. Bhatia, Matrix Analysis, Springer-Verlag, New York, 1997.
[3] R. Bhatia, C. Davis, and F. Kittaneh, Some inequalities for commutators and an application to spectral variation, J. Aequationes Math. 41 (1991) 70–78.
[4] R. Bhatia and X. Zhan X, Norm inequalities for operators with positive real parts, J. Operator Theory 50 (2003) 67-76.
[5] I.C. Gohberg and M.G. Krein, Introduction to the Theory of Linear Nonselfadjoint Operators, Transl. Math. Monographs, Vol. 18, Amer. Math. Soc., Providence, RI, 1969.
[6] F. Hiai, Log-majorization and norm inequalities for exponential operators, Linear Operators, Banach Center Publications, Polish Academy of Sciences, Warszawa, 38 (1997) 119–181.
[7] T. Kosem, Inequalities between $\|f(A+B)\|$ and $\|f(A)+f(B)\|$, Linear Algebra Appl. 418 (2006) 153-160.
[8] D. S. Mitrinović, Analytic inequalities, Springer-Verlag, 1970.