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Certain Results of Hybrid Families of Special Polynomials Associated with Appell Sequences

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Abstract. In this article, the Legendre-Gould-Hopper polynomials are combined with Appell sequences to introduce certain mixed type special polynomials by using operational method. The generating functions, determinant definitions and certain other properties of Legendre-Gould-Hopper based Appell polynomials are derived. Operational rules providing connections between these formulae and known special polynomials are established. The 2-variable Hermite Kampé de Fériet based Bernoulli polynomials are considered as an member of Legendre-Gould-Hopper based Appell family and certain results for this member are also obtained.

1. Introduction and preliminaries

One of the important classes of polynomial sequences is the class of Appell polynomial sequences [1]. These polynomial sequences have been well studied from different aspect due to their remarkable applications in various fields (see for example [18, 19]). The Appell polynomial sequences arise in theoretical physics, chemistry and numerous problems of pure and applied mathematics such as the study of polynomial expansions of analytic functions, number theory and numerical analysis. The recent applications of Appell polynomials in probability theory and statistics are considered in [2, 16]. The generalized Appell polynomials as tools for approximating 3D-mappings were introduced for the first time in [14] in combination with Clifford analysis methods. The representation theoretic results like those of [3, 13] provide new examples of applications of Appell polynomials and gave evidence to the central role of Appell polynomials as orthogonal polynomials.

The Appell sets [1] may be defined by either of the following equivalent conditions [17, p.398]: $\{A_n(x)\}(n = 0, 1, 2, ...)$ is an Appell set ($A_n(x)$ being of degree exactly n) if either

(i) $\frac{d}{dx}A_n(x) = n\, A_{n-1}(x), \quad n = 0, 1, 2, ...,$ or

(ii) there exists a formal power series

$$A(t) = \sum_{n=0}^{\infty} A_n \frac{t^n}{n!}, \qquad A_0 \neq 0, \tag{1.1}$$

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such that (again formally)

$$A(t)\, exp(xt) = \sum_{n=0}^{\infty} A_n(x)\, \frac{t^n}{n!}. \tag{1.2}$$

The Appell polynomials have shown to be quasi-monomials [9] and characterized by the fact that the relevant derivative operator is just the ordinary derivative.

Recently, the Legendre-Gould-Hopper polynomials (LeGHP) ${}_SH_n^{(s)}(x,y,z)$ and $\frac{{}_RH_n^{(s)}(x,y,z)}{n!}$ are introduced in [21] which are defined by means of the generating functions

$$exp(yt + zt^s)\, C_0(-xt^2) = \sum_{n=0}^{\infty} {}_SH_n^{(s)}(x,y,z) \frac{t^n}{n!} \tag{1.3}$$

and

$$exp(zt^s)\, C_0(xt)\, C_0(-yt) = \sum_{n=0}^{\infty} \frac{{}_RH_n^{(s)}(x,y,z)}{n!} \frac{t^n}{n!}, \tag{1.4}$$

respectively, where $C_0(x)$ denotes the Tricomi function of order zero [7] which is given by the following operational definition:

$$C_0(\alpha x) = exp(-\alpha D_x^{-1})\{1\}, \tag{1.5}$$

where D_x^{-1} denotes the inverse of the derivative operator $D_x := \frac{\partial}{\partial x}$ and $D_x^{-n}\{1\} = \frac{x^n}{n!}$.

The LeGHP ${}_SH_n^{(s)}(x,y,z)$ and $\frac{{}_RH_n^{(s)}(x,y,z)}{n!}$ are shown to be quasi-monomial [7, 20] under the action of the following multiplicative and derivative operators [21]:

$$\hat{M}_{SH} := y + 2D_x^{-1}\frac{\partial}{\partial y} + sz\frac{\partial^{s-1}}{\partial y^{s-1}}, \tag{1.6}$$

$$\hat{P}_{SH} := \frac{\partial}{\partial y} \tag{1.7}$$

and

$$\hat{M}_{RH} := -D_x^{-1} + D_y^{-1} + sz\frac{\partial^{s-1}}{\partial y^{s-1}}, \tag{1.8}$$

$$\hat{P}_{RH} := -\frac{\partial}{\partial x}x\frac{\partial}{\partial x}, \tag{1.9}$$

respectively.

Consequently, $\hat{M}_{SH}$, $\hat{P}_{SH}$ and $\hat{M}_{RH}$, $\hat{P}_{SH}$ satisfy the following recurrence relations:

$$\hat{M}_{SH}\{{}_SH_n^{(s)}(x,y,z)\} = {}_SH_{n+1}^{(s)}(x,y,z), \tag{1.10}$$

$$\hat{P}_{SH}\{{}_SH_n^{(s)}(x,y,z)\} = n\; {}_SH_{n-1}^{(s)}(x,y,z) \tag{1.11}$$

and

$$\hat{M}_{RH}\left\{\frac{{}_RH_n^{(s)}(x,y,z)}{n!}\right\} = \frac{{}_RH_{n+1}^{(s)}(x,y,z)}{(n+1)!}, \tag{1.12}$$

$$\hat{P}_{RH}\left\{\frac{{}_RH_n^{(s)}(x,y,z)}{n!}\right\} = n\frac{{}_RH_{n-1}^{(s)}(x,y,z)}{(n-1)!}, \tag{1.13}$$

respectively, for all $n \in \mathbb{N}$.

In view of the monomiality principle equations

$$\hat{M}_{SH}\hat{P}_{SH}\{{}_SH_n^{(s)}(x,y,z)\} = n\ {}_SH_n^{(s)}(x,y,z), \tag{1.14}$$

$$\hat{M}_{RH}\hat{P}_{RH}\left\{\frac{{}_RH_n^{(s)}(x,y,z)}{n!}\right\} = n\frac{{}_RH_n^{(s)}(x,y,z)}{n!}, \tag{1.15}$$

the differential equations satisfied by ${}_SH_n^{(s)}(x,y,z)$ and $\frac{{}_RH_n^{(s)}(x,y,z)}{n!}$ are [21]:

$$\left(2\frac{\partial^2}{\partial y^2} + sz\frac{\partial^{s+1}}{\partial x\partial y^s} + y\frac{\partial^2}{\partial x\partial y} - n\frac{\partial}{\partial x}\right){}_SH_n^{(s)}(x,y,z) = 0 \tag{1.16}$$

and

$$\left(-\frac{\partial}{\partial y} + sz\frac{\partial^{s+1}}{\partial x\partial y^s} + (1-n)\frac{\partial}{\partial x}\right){}_RH_n^{(s)}(x,y,z) = 0, \tag{1.17}$$

respectively.

Also, ${}_SH_n^{(s)}(x,y,z)$ and $\frac{{}_RH_n^{(s)}(x,y,z)}{n!}$ can be explicitly constructed as:

$${}_SH_n^{(s)}(x,y,z) = \hat{M}_{SH}^n\{1\}, \quad {}_SH_0^{(s)}(x,y,z) = 1 \tag{1.18}$$

and

$$\frac{{}_RH_n^{(s)}(x,y,z)}{n!} = \hat{M}_{RH}^n\{1\}, \quad {}_RH_0^{(s)}(x,y,z) = 1, \tag{1.19}$$

respectively.

Identities (1.18) and (1.19) imply that the exponential functions of ${}_SH_n^{(s)}(x,y,z)$ and $\frac{{}_RH_n^{(s)}(x,y,z)}{n!}$ can be given in the forms:

$$exp\left(t\hat{M}_{SH}\right)\{1\} = \sum_{n=0}^{\infty} {}_SH_n^{(s)}(x,y,z)\frac{t^n}{n!}, \quad |t| < \infty \tag{1.20}$$

and

$$exp\left(t\hat{M}_{RH}\right)\{1\} = \sum_{n=0}^{\infty} \frac{{}_RH_n^{(s)}(x,y,z)}{n!}\frac{t^n}{n!}, \quad |t| < \infty, \tag{1.21}$$

respectively.

For suitable values of the indices and variables, the LeGHP ${}_SH_n^{(s)}(x,y,z)$ and $\frac{{}_RH_n^{(s)}(x,y,z)}{n!}$ give a number of other known special polynomials as special cases (see [21, Table 2.1]).

Costabile *et al.* [4] has given a new approach to Bernoulli polynomials based on a determinant definition. This approach is further extended to provide determinant definitions of the Appell and Sheffer polynomial sequences by Costabile and Longo in [5] and [6], respectively. The determinant approach is equivalent to the corresponding approach based on operational methods. However, the simplicity of this approach allows non-specialists to use its applications and it is also suitable for computation. The above mentioned research works of Costabile and Longo and the importance of operational methods in the theory of special functions motivated the authors to introduce and study the Legendre-Gould-Hopper based Appell polynomials by using operational techniques and determinant approach.

In this paper, the composition of Legendre-Gould-Hopper and Appell polynomials is considered to introduce a new family of special polynomials, namely the Legendre-Gould-Hopper based Appell family by using the concepts and the methods associated with monomiality principle. The important properties of this family are established. In Section 2, the generating function, series definition and determinant definition for the Legendre-Gould-Hopper based Appell polynomials are established. Further, these polynomials are framed within the context of the monomiality principle and their properties are derived. In Section 3, some operational representations for these polynomials are also established. In Section 4, certain results for the 2-variable Hermite Kampé de Fériet based Bernoulli polynomials are also obtained. Surface plot of this example is also considered.

2. Legendre-Gould-Hopper based Appell polynomials

The Legendre-Gould-Hopper based Appell polynomials (LeGHAP) denoted by ${}_{SH^{(s)}}A_n(x,y,z)$ and ${}_{RH^{(s)}}A_n(x,y,z)$ are introduced in this section by means of generating functions and series definitions. Determinant definitions of these polynomials are also established. In this connection, we first derive the generating functions for the Legendre-Gould-Hopper based Appell polynomials by proving the following result:

Theorem 2.1. *The following generating functions for the Legendre-Gould-Hopper based Appell polynomials (LeGHAP) ${}_{SH^{(s)}}A_n(x,y,z)$ and ${}_{RH^{(s)}}A_n(x,y,z)$ hold true:*

$$A(t)exp(yt + zt^s)\; C_0(-xt^2) = \sum_{n=0}^{\infty} {}_{SH^{(s)}}A_n(x,y,z)\frac{t^n}{n!}, \tag{2.1}$$

$$A(t)exp(zt^s)\; C_0(xt)\; C_0(-yt) = \sum_{n=0}^{\infty} {}_{RH^{(s)}}A_n(x,y,z)\frac{t^n}{n!}, \tag{2.2}$$

respectively.

Proof. Replacing x in the l.h.s. and r.h.s. of equation (1.2) by the multiplicative operator $\hat{M}_{SH}$ of the LeGHP ${}_SH_n^{(s)}(x,y,z)$, we have

$$A(t)exp\Big(\hat{M}_{SH}t\Big) = \sum_{n=0}^{\infty} A_n\Big(\hat{M}_{SH}\Big)\frac{t^n}{n!}. \tag{2.3}$$

Using the expression of $\hat{M}_{SH}$ given in equation (1.6) and then decoupling the exponential operator in the l.h.s. of the resultant equation by using the Crofton-type identity [8, p.12]:

$$f\Big(y + m\lambda\frac{d^{m-1}}{dy^{m-1}}\Big)\{1\} = exp\Big(\lambda\frac{d^m}{dx^m}\Big)\{f(y)\}, \tag{2.4}$$

we get

$$A(t)exp\Big(z\frac{\partial^s}{\partial y^s}\Big)exp\Big(\Big(y + 2D_x^{-1}\frac{\partial}{\partial y}\Big)t\Big) = \sum_{n=0}^{\infty} A_n\Big(y + 2D_x^{-1}\frac{\partial}{\partial y} + sz\frac{\partial^{s-1}}{\partial y^{s-1}}\Big)\frac{t^n}{n!}, \tag{2.5}$$

which on further use of identity (2.4) in the l.h.s. becomes

$$A(t)exp\Big(z\frac{\partial^s}{\partial y^s}\Big)exp\Big(D_x^{-1}\frac{\partial^2}{\partial y^2}\Big)exp(yt) = \sum_{n=0}^{\infty} A_n\Big(y + 2D_x^{-1}\frac{\partial}{\partial y} + sz\frac{\partial^{s-1}}{\partial y^{s-1}}\Big)\frac{t^n}{n!}. \tag{2.6}$$

Now, expanding the second exponential in the l.h.s. of equation (2.6) and using definition (1.5), we find

$$A(t)C_0\Big(-xt^2\Big)exp\Big(z\frac{\partial^s}{\partial y^s}\Big)exp(yt) = \sum_{n=0}^{\infty} A_n\Big(y + 2D_x^{-1}\frac{\partial}{\partial y} + zs\frac{\partial^{s-1}}{\partial y^{s-1}}\Big)\frac{t^n}{n!}. \tag{2.7}$$

Again expanding the first exponential in the l.h.s. of equation (2.7) and denoting the resultant LeGHAP in the r.h.s. by ${}_{sH^{(s)}}A_n(x, y, z)$, that is

$${}_{sH^{(s)}}A_n(x, y, z) = A_n\Big(\hat{M}_{SH}\Big) = A_n\Big(y + 2D_x^{-1}\frac{\partial}{\partial y} + zs\frac{\partial^{s-1}}{\partial y^{s-1}}\Big), \tag{2.8}$$

we get assertion (2.1). Moreover, making use of (1.8) and using a similar argument as in the above proof of (2.1), we can obtain assertion (2.2). □

Theorem 2.2. *The Legendre-Gould-Hopper based Appell polynomials (LeGHAP) ${}_{sH^{(s)}}A_n(x, y, z)$ and ${}_{RH^{(s)}}A_n(x, y, z)$ are defined by the series:*

$${}_{sH^{(s)}}A_n(x, y, z) = \sum_{k=0}^{n} \binom{n}{k} A_k \; {}_SH^{(s)}_{n-k}(x, y, z) \tag{2.9}$$

and

$${}_{RH^{(s)}}A_n(x, y, z) = \sum_{k=0}^{n} \binom{n}{k} A_k \frac{{}_RH^{(s)}_{n-k}(x, y, z)}{(n-k)!}, \tag{2.10}$$

respectively, where A_k is given by equation (1.1).

Proof. In view of equations (1.20) and (2.8), equation (2.3) can be written as

$$A(t)\sum_{n=0}^{\infty} {}_SH^{(s)}_n(x, y, z)\frac{t^n}{n!} = \sum_{n=0}^{\infty} {}_{sH^{(s)}}A_n(x, y, z)\frac{t^n}{n!}. \tag{2.11}$$

Now, using expansion (1.1) of $A(t)$ in the l.h.s. of equation (2.11), simplifying and then equating the coefficients of like powers of t on both sides of the resultant equation, we get assertion (2.9). Similarly, we can get assertion (2.10). □

By using a similar approach given in [22] and in view of equations (1.18) and (2.8), the following determinant form for ${}_{sH^{(s)}}A_n(x, y, z)$ is obtained:

Definition 2.3. *The Legendre-Gould-Hopper based Appell polynomials ${}_{sH^{(s)}}A_n(x, y, z)$ of degree n are defined by*

$${}_{sH^{(s)}}A_0(x, y, z) = \frac{1}{\beta_0}, \qquad \beta_0 = \frac{1}{A_0}, \tag{2.12}$$

$${}_{sH^{(s)}}A_n(x, y, z) = \frac{(-1)^n}{(\beta_0)^{n+1}} \begin{vmatrix} 1 & {}_SH^{(s)}_1(x,y,z) & {}_SH^{(s)}_2(x,y,z) & \cdots & {}_SH^{(s)}_{n-1}(x,y,z) & {}_SH^{(s)}_n(x,y,z) \\ \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix}, \tag{2.13}$$

$$\beta_n = -\frac{1}{A_0}\Big(\sum_{k=1}^{n} \binom{n}{k} A_k \beta_{n-k}\Big), \qquad n = 1, 2, 3, \ldots,$$

where $\beta_0, \beta_1, ..., \beta_n \in \mathbb{R}, \beta_0 \neq 0$ *and* ${}_SH_n^{(s)}(x,y,z)(n=0,1,2,...,)$ *are the Legendre-Gould-Hopper polynomials defined by equation (1.3).*

Similarly, the determinant form for ${}_{_RH^{(s)}}A_n(x,y,z)$ can be obtained:

Definition 2.4. *The Legendre-Gould-Hopper based Appell polynomials* ${}_{_RH^{(s)}}A_n(x,y,z)$ *of degree n are defined by*

$$ {}_{_RH^{(s)}}A_0(x,y,z) = \frac{1}{\beta_0}, \quad \beta_0 = \frac{1}{A_0}, \tag{2.14}$$

$$ {}_{_RH^{(s)}}A_n(x,y,z) = \frac{(-1)^n}{(\beta_0)^{n+1}} \begin{vmatrix} 1 & {}_RH_1^{(s)}(x,y,z) & \frac{{}_RH_2^{(s)}(x,y,z)}{2!} & \cdots & \frac{{}_RH_{n-1}^{(s)}(x,y,z)}{(n-1)!} & \frac{{}_RH_n^{(s)}(x,y,z)}{n!} \\ \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix}, \tag{2.15}$$

$$\beta_n = -\frac{1}{A_0}\Big(\sum_{k=1}^{n}\binom{n}{k}A_k\beta_{n-k}\Big), \qquad n=1,2,3,...,$$

where $\beta_0, \beta_1, ..., \beta_n \in \mathbb{R}, \beta_0 \neq 0$ *and* $\frac{{}_RH_n^{(s)}(x,y,z)}{n!}(n=0,1,2,...,)$ *are the Legendre-Gould-Hopper polynomials defined by equation (1.4).*

The Appell and Legendre-Gould-Hopper polynomials are quasi-monomial. In order to show that the LeGHAP ${}_{_SH^{(s)}}A_n(x,y,z)$ and ${}_{_RH^{(s)}}A_n(x,y,z)$ are quasi-monomial, we prove the following results:

Theorem 2.5. *The Legendre-Gould-Hopper based Appell polynomials* ${}_{_SH^{(s)}}A_n(x,y,z)$ *and* ${}_{_RH^{(s)}}A_n(x,y,z)$ *are quasi-monomial with respect to the following multiplicative and derivative operators:*

$$\hat{M}_{SHA} := y + 2D_x^{-1}D_y + szD_y^{s-1} + \frac{A'(D_y)}{A(D_y)}, \tag{2.16}$$

$$\hat{P}_{SHA} := D_y \tag{2.17}$$

and

$$\hat{M}_{RHA} := -D_x^{-1} + D_y^{-1} + szD_y^{s-1} + \frac{A'(D_yyD_y)}{A(D_yyD_y)}, \tag{2.18}$$

$$\hat{P}_{RHA} := D_yyD_y, \tag{2.19}$$

respectively.

Proof. Consider the following identity:

$$D_y\Big\{exp\Big(yt+zt^s\Big)\Big\} = t\,exp\Big(yt+zt^s\Big). \tag{2.20}$$

Differentiating equation (2.3) partially with respect to t and in view of relation (2.8), we find

$$\Big(\hat{M}_{SH} + \frac{A'(t)}{A(t)}\Big)A(t)exp(\hat{M}_{SH}t) = \sum_{n=0}^{\infty} {}_{_SH^{(s)}}A_{n+1}(x,y,z)\frac{t^n}{n!}, \tag{2.21}$$

which on using equations (1.3) and (1.20) gives

$$\Big(\hat{M}_{SH} + \frac{A'(t)}{A(t)}\Big)A(t)C_0(-xt^2)exp\Big(yt + zt^s\Big) = \sum_{n=0}^{\infty} {}_{sH^{(s)}}A_{n+1}(x,y,z)\frac{t^n}{n!}. \tag{2.22}$$

Now, in view of relation (2.20) and generating function (2.1), the above equation becomes

$$\Big(\hat{M}_{SH} + \frac{A'(D_y)}{A(D_y)}\Big)\Big\{\sum_{n=0}^{\infty} {}_{sH^{(s)}}A_n(x,y,z)\frac{t^n}{n!}\Big\} = \sum_{n=0}^{\infty} {}_{sH^{(s)}}A_{n+1}(x,y,z)\frac{t^n}{n!}. \tag{2.23}$$

Adjusting the summation in the l.h.s. of equation (2.23) and then equating the coefficients of like powers of t, we find

$$\Big(\hat{M}_{SH} + \frac{A'(D_y)}{A(D_y)}\Big)\Big\{{}_{sH^{(s)}}A_n(x,y,z)\Big\} = {}_{sH^{(s)}}A_{n+1}(x,y,z), \tag{2.24}$$

which, in view of equation (1.10) shows that the corresponding multiplicative operator for ${}_{sH^{(s)}}A_n(x,y,z)$ is given as:

$$\hat{M}_{SHA} = \hat{M}_{SH} + \frac{A'(D_y)}{A(D_y)}. \tag{2.25}$$

Finally, using equation (1.6) in the r.h.s. of above equation, we get assertion (2.16).

Next, consider the following identity

$$(D_y y D_y)C_0(-yt) = t\, C_0(-yt) \tag{2.26}$$

and use a similar argument as in the above proof, with the help of equation (1.8) we obtain assertion (2.18).

Again, in view of identity (2.20), we have

$$D_y\{A(t)C_0(-xt^2)exp\Big(yt + zt^s\Big)\} = t\, A(t)C_0(-xt^2)exp\Big(yt + zt^s\Big), \tag{2.27}$$

which on using generating function (2.1) becomes

$$D_y\Big\{\sum_{n=0}^{\infty} {}_{sH^{(s)}}A_n(x,y,z)\frac{t^n}{n!}\Big\} = t\,\sum_{n=0}^{\infty} {}_{sH^{(s)}}A_{n-1}(x,y,z)\frac{t^n}{(n-1)!}. \tag{2.28}$$

Adjusting the summation in the l.h.s. of the above equation and then equating the coefficients of like powers of t, we get

$$D_y\Big\{{}_{sH^{(s)}}A_n(x,y,z)\Big\} = n\, {}_{sH^{(s)}}A_{n-1}(x,y,z), \qquad n \geq 1, \tag{2.29}$$

which in view of equation (1.11) yields assertion (2.17). Similarly, we can obtain the assertion (2.19). □

Theorem 2.6. *The Legendre-Gould-Hopper based Appell polynomials ${}_{sH^{(s)}}A_n(x,y,z)$ and ${}_{RH^{(s)}}A_n(x,y,z)$ are the solutions of the following differential equations:*

$$\Big(yD_y + 2D_x^{-1}D_y^2 + szD_y^s + \frac{A'(D_y)}{A(D_y)}D_y - n\Big)_{sH^{(s)}}A_n(x,y,z) = 0 \tag{2.30}$$

and

$$\Big(-D_x^{-1}\frac{\partial}{\partial D_y^{-1}} + D_y^{-1}\frac{\partial}{\partial D_y^{-1}} + sz\frac{\partial^s}{\partial y^{s-1}\partial D_y^{-1}} + \frac{A'(D_y y D_y)}{A(D_y y D_y)}\frac{\partial}{\partial D_y^{-1}} - n\Big)_{RH^{(s)}}A_n(x,y,z) = 0, \tag{2.31}$$

respectively.

Proof. Using equations (2.16) and (2.17) in the corresponding equation (1.14) for the LeGHAP ${}_{sH^{(s)}}A_n(x,y,z)$, we get assertion (2.30). Also, using equations (2.18) and (2.19) in the corresponding equation (1.15) for the LeGHAP ${}_{RH^{(s)}}A_n(x,y,z)$, we get assertion (2.31). □

The special cases of the LeGHP ${}_SH_n^{(s)}(x,y,z)$ and $\frac{{}_RH_n^{(s)}(x,y,z)}{n!}$ are given in [21, Table 2.1]. Now, for the same choice of the variables and indices the LeGHAP ${}_{sH^{(s)}}A_n(x,y,z)$ and ${}_{RH^{(s)}}A_n(x,y,z)$ reduce to the corresponding special cases. We mention these known and new special polynomials related to the Appell sequences in Table 1.

Table 1: Special cases of LeGHAP ${}_{sH^{(s)}}A_n(x,y,z)$ and ${}_{RH^{(s)}}A_n(x,y,z)$

S. No.	**Values of the indices and variables**	**Relation between LeGHAP ${}_{sH^{(s)}}A_n(x,y,z)$, ${}_{RH^{(s)}}A_n(x,y,z)$ and their special cases**	**Name of the special polynomials**
I.	$x=0$	${}_{sH^{(s)}}A_n(0,y,z) = {}_{H^{(s)}}A_n(y,z)$	Gould-Hopper based Appell polynomials (GHAP) [11]
II.	$z=0$	${}_{sH^{(s)}}A_n(x,y,0) = {}_{2L}A_n(x,y)$	2-Variable Legendre based Appell polynomials (2VLeAP)
III.	i. $s=m; x=0,$ $y\to -D_x^{-1}, z\to y$ ii. $s=m; y=0, z\to y$	${}_{sH^{(m)}}A_n(0,-D_x^{-1},y) = {}_{[m]L}A_n(x,y)$ ${}_{RH^{(m)}}A_n(x,0,y) = {}_{[m]L}A_n(x,y)$	2-Variable Generalized Laguerre type based Appell polynomials (2VGLTAP) [12]
IV.	$s=m-1;\ x=0,$ $y\to x, z\to y$	${}_{sH^{(m-1)}}A_n(0,x,y) = {}_{U^{(m)}}A_n(x,y)$	Generalized Chebyshev based Appell polynomials (GCAP) [12]
V.	i. $s=1; x=0, z\to -D_x^{-1}$ ii. $s=1; y=0, z\to y$	${}_{sH^{(1)}}A_n(0,y,-D_x^{-1}) = {}_LA_n(x,y)$ ${}_{RH^{(1)}}A_n(x,o,y) = {}_LA_n(x,y)$	2-Variable Laguerre based Appell polynomials (2VLAP) [10]
VI.	$z=0$	${}_{RH^{(s)}}A_n(x,y,0) = {}_RA_n(x,y)$	2-Variable Legendre based Appell polynomials (2VLeAP)
VII.	$x=0,\ y\to x, z\to yD_yy$	${}_{sH^{(s)}}A_n(0,x,yD_yy) = {}_{e^{(s)}}A_n(x,y)$	2-Variable truncated based Appell polynomials of order s (2VTAP)
VIII.	$s=2; x=0$	${}_{sH^{(2)}}A_n(0,y,z) = {}_HA_n(y,z)$	2-Variable Hermite-Kamp de Fériet é based Appell polynomials (2VHKFAP) [12]
IX.	i. $s=2; x=0,$ $y\to D_x^{-1}, z\to y$ ii. $s=2; x=0,$ $y\to x, z\to y$	${}_{sH^{(2)}}A_n(0,D_x^{-1},y) = {}_GA_n(x,y)$ ${}_{RH^{(2)}}A_n(0,x,y) = {}_GA_n(x,y)$	Hermite type based Appell polynomials (HTAP) [12]
X.	i. $x\to(\frac{x^2-1}{4}),$ $y\to x, z=0$ ii. $s=1; x\to(\frac{1-x}{2}),$ $y\to(\frac{1+x}{2}), z=0$	${}_{sH^{(s)}}A_n(\frac{x^2-1}{4},x,0) = {}_PA_n(x)$ ${}_{RH^{(1)}}A_n(\frac{1-x}{2},\frac{1+x}{2},0) = {}_PA_n(x)$	Legendre based Appell polynomials (LeAP) [12]
XI.	$s=3; x\to zD_z\,z,$ $y\to x, z\to y$	${}_{sH^{(3)}}A_n(zD_zz,x,y) = {}_{H^{(3,2)}}A_n(x,y,z)$	Bell-type based Appell polynomials (BTAP) [12]

Remark: In view of the special cases mentioned in Table 1, the results for the special polynomials related to the Appell sequences can be obtained.

Next, we derive certain operational representations for the LeGHAP ${}_{sH^{(s)}}A_n(x,y,z)$ and ${}_{RH^{(s)}}A_n(x,y,z)$.

3. Operational representations

To establish the operational representation for the LeGHAP ${}_{sH^{(s)}}A_n(x,y,z)$ and ${}_{RH^{(s)}}A_n(x,y,z)$, we prove the following results:

Theorem 3.1. *The following operational representation between the LeGHAP ${}_{sH^{(s)}}A_n(x,y,z)$, ${}_{RH^{(s)}}A_n(x,y,z)$ and the Appell polynomials $A_n(x)$ hold true:*

$$ {}_{sH^{(s)}}A_n(x,y,z) = exp\Big(D_x^{-1}\frac{\partial^2}{\partial y^2} + z\frac{\partial^s}{\partial y^s}\Big)A_n(y) \tag{3.1}$$

and

$$ {}_{RH^{(s)}}A_n(x,y,z) = exp\Big(z\frac{\partial^s}{\partial y^s}\Big)A_n\Big(-D_x^{-1} + D_y^{-1}\Big), \tag{3.2}$$

respectively.

Proof. In view of equation (2.8), the proof is direct use of identity (2.4). □

Theorem 3.2. *The following operational representation between the LeGHAP ${}_{RH^{(s)}}A_n(x,y,z)$ and the 2VLeAP ${}_RA_n(x,y)$ holds true:*

$$ {}_{RH^{(s)}}A_n(x,y,z) = exp\Big(z\frac{\partial^s}{\partial D_y^{-s}}\Big){}_RA_n(x,y), \tag{3.3}$$

or, equivalently

$$ {}_{RH^{(s)}}A_n(x,y,z) = exp\Big((-1)^s z\frac{\partial^s}{\partial D_x^{-s}}\Big){}_RA_n(x,y). \tag{3.4}$$

Proof. From equation (2.2), we have

$$ \frac{\partial^s}{\partial D_y^{-s}}\ {}_{RH^{(s)}}A_n(x,y,z) = \frac{\partial}{\partial z}\ {}_{RH^{(s)}}A_n(x,y,z). \tag{3.5}$$

Also, from Table 1(VI), we have

$$ {}_{RH^{(s)}}A_n(x,y,0) = {}_RA_n(x,y). \tag{3.6}$$

Now, solving equation (3.5) subject to initial condition (3.6), we get assertion (3.3). Again using a similar argument as in the above proof of (3.3), we establish the assertion (3.4). □

Theorem 3.3. *The following operational representation between the LeGHAP ${}_{sH^{(s)}}A_n(x,y,z)$ and the 2VLeTAP ${}_{2L}A_n(x,y)$ holds true:*

$$ {}_{sH^{(s)}}A_n(x,y,z) = exp\Big(z\frac{\partial^s}{\partial y^s}\Big){}_{2L}A_n(x,y). \tag{3.7}$$

Proof. Using a similar argument as in the above proof of Theorem 3.2, we establish the assertion (3.7) of the Theorem 3.3. □

Theorem 3.4. *The following operational representation between the LeGHAP ${}_{sH^{(s)}}A_n(x,y,z)$ and the GHAP ${}_{H^{(s)}}A_n(y,z)$ hold true:*

$$ {}_{sH^{(s)}}A_n(x,y,z) = exp\Big(D_x^{-1}\frac{\partial^2}{\partial y^2}\Big){}_{H^{(s)}}A_n(y,z). \tag{3.8}$$

Proof. From equations (1.5) and (2.1), we have

$$ \frac{\partial^2}{\partial y^2}\ {}_{sH^{(s)}}A_n(x,y,z) = \frac{\partial}{\partial D_x^{-1}}\ {}_{sH^{(s)}}A_n(x,y,z). \tag{3.9}$$

Also, from Table 1(I), we have

$$ {}_{sH^{(s)}}A_n(0,y,z) = {}_{H^{(s)}}A_n(y,z). \tag{3.10}$$

Solving equation (3.9) subject to initial condition (3.10), we get assertion (3.8). □

In the next section, we introduce 2-variable Hermite Kampé de Fériet based Bernoulli polynomials (2VHKdFBP) ${}_HB_n(y,z)$ as an example of the Legendre-Gould-Hopper based Appell family.

4. Appendix

Since, for $A(t) = \frac{t}{e^t-1}$, the AP $A_n(x)$ reduce to the Bernoulli polynomials (BP) $B_n(x)$ [15] and for $s = 2, x = 0$, the LeGHP ${}_SH_n^{(s)}(x, y, z)$ reduces to the 2VHKdFP $H_n(y, z)$ [21, Table 2.1(VIII)]. Therefore, for the same choices, the LeGHAP ${}_{SH^{(s)}}A_n(x, y, z)$ reduce to the 2-variable Hermite Kampé de Fériet based Bernoulli polynomials (2VHKdFBP) ${}_HB_n(y, z)$. Thus, by using these substitutions in equations (2.1), (2.16), (2.17), (2.30), (3.1) and (3.8), we can obtain the following results for 2VHKdFBP ${}_HB_n(y, z)$:

Table 2: Results for the 2VHKdFBP ${}_HB_n(y, z)$

I.	Generating functions	$\frac{t}{e^t-1}\, exp(yt + zt^s) = \sum_{n=0}^{\infty} {}_HB_n(y, z)\frac{t^n}{n!}$
II.	Multiplicative and derivative operators	$\hat{M}_{SHB} := y + 2zD_y + \frac{e^{D_y}(1-e^{D_y})-1}{D_y(e^{D_y}-1)},\ \ \hat{P}_{SHB} := D_y$
III.	Differential equations	$\Big(yD_y + 2zD_y^2 + \frac{e^{D_y}(1-e^{D_y})-1}{(e^{D_y}-1)} - n\Big){}_HB_n(y, z) = 0$
IV.	Operational representations	${}_HB_n(y, z) = exp\Big(z\frac{\partial^2}{\partial y^2}\Big)B_n(y)$

The series definition for the 2VHKdFBP ${}_HB_n(y, z)$ can be given as:

$$ {}_HB_n(y, z) = n! \sum_{k=0}^{[\frac{n}{2}]} \frac{z^k\, B_{n-2k}(y)}{k!\,(n-2k)!}. \tag{4.1}$$

Further, it has been shown in [5] that for $\beta_0 = 1$ and $\beta_i = \frac{1}{i+1}, (i = 1, 2, 3, ..., n)$ the determinant definition of the Appell polynomials $A_n(x)$ reduces to the determinant form of Bernoulli polynomials $B_n(x)$ (see [4, 5]). Therefore, in view of equations (2.12) and (2.13), the determinant definition of the 2VHKdFBP ${}_HB_n(y, z)$ can be given as:

Definition 4.1. *The 2VHKdFBP ${}_HB_n(y, z)$ of degree n are defined by*

$$ {}_HB_0(y, z) = 1, \tag{4.2}$$

$$ {}_HB_n(y, z) = \frac{(-1)^n}{(\beta_0)^{n+1}} \begin{vmatrix} 1 & H_1(y,z) & H_2(y,z) & \cdots & H_{n-1}(y,z) & H_n(y,z) \\ 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} & \frac{1}{n+1} \\ 0 & 1 & \binom{2}{1}\frac{1}{2} & \cdots & \binom{n-1}{1}\frac{1}{n-1} & \binom{n}{1}\frac{1}{n} \\ 0 & 0 & 1 & \cdots & \binom{n-1}{2}\frac{1}{n-2} & \binom{n}{2}\frac{1}{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \binom{n}{n-1}\frac{1}{2} \end{vmatrix},\quad n = 1, 2, 3, ..., \tag{4.3}$$

where $H_n(y, z)(n = 0, 1, 2, ...,)$ are the 2-variable Hermite Kampé de Fériet polynomials of degree n.

Now, we draw the surface plot of the 2VHKdFBP ${}_HB_n(y, z)$. To draw the surface plot of 2VHKdFBP ${}_HB_n(y, z)$, we consider the values of the first six Bernoulli polynomials $B_n(x)$ given in Table 3.

Table 3: First six expressions of Bernoulli polynomials

n	0	1	2	3	4	5
$B_n(x)$	1	$x - \frac{1}{2}$	$x^2 - x + \frac{1}{6}$	$x^3 - \frac{3}{2}x^2 + \frac{x}{2}$	$x^4 - 2x^3 + x^2 - \frac{1}{30}$	$x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{x}{6}$

Set $n = 5$ in the series definition of the 2VHKdFBP ${}_HB_n(y, z)$ (4.1), we have

$$ {}_HB_5(y, z) = B_5(y) + 20zB_3(y) + 60z^2B_1(y) \tag{4.4}$$

Using the particular values of $B_n(y)$ from Table 3 in equation (4.4), we find

$$ {}_HB_5(y,z) = y^5 - \frac{5}{2}y^4 + \frac{5}{3}y^3 - \frac{1}{6}y + 20zy^3 - 30zy^2 + 10zy + 60z^2y - 30z^2. \tag{4.5} $$

In view of equation (4.5) and with the help of Matlab, we get the following surface plot of ${}_HB_5(y,z)$:

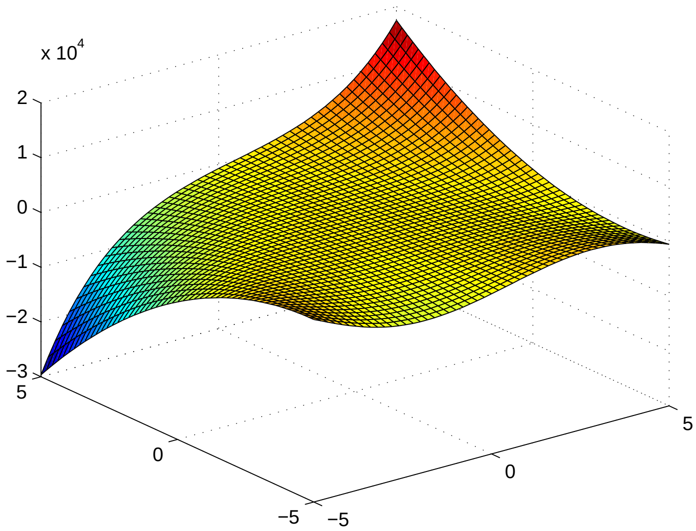

Figure 1: Surface plot of ${}_HB_5(y,z)$

Also, by giving suitable values to the variables and indices, we can find many important results for the members belonging to Legendre-Gould-Hopper based Appell family.

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