Filomat 33:12 (2019), 3845–3853
https://doi.org/10.2298/FIL1912845M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Computing Bifurcations Behavior of Mixed Type Singular Time-Fractional Partial Integrodifferential Equations of Dirichlet Functions Types in Hilbert Space with Error Analysis

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Abstract. In this article, we propose and analyze a computational method for the numerical solutions of mixed type singular time-fractional partial integrodifferential equations of Dirichlet functions types. The method provide appropriate representation of the solutions in infinite series formula with accurately computable structures. By interrupting the n-term of exact solutions, numerical solutions of linear and nonlinear singular time-fractional equations of nonhomogeneous function type are studied from mathematical viewpoint. The utilized results show that the present method and simulated annealing provide a good scheduling methodology to such singular integrodifferential equations.

1. Preface

Fractional-order derivatives and integrals embed the description of the memory and hereditary properties of different substances. Accordingly, the field of time-fractional partial integrodifferential equations (PIDEs) has attracted interest of researchers in several important phenomenons in chemistry, hydrology, fluid mechanic, physics, gas dynamics, and signal processing (see, for instance, [1–17] and the references therein). Usually, it is too complicated to solve exactly this class of equations for most cases because, generally, the solution cannot be exhibited in a closed form even when it exists. Therefore, the development of analytical and numerical methods for the solutions of time-fractional PIDEs is of current importance.

In this study, a general technique based on the reproducing kernel theory is proposed for solving a class of singular time-fractional PIDEs in the appropriate reproducing kernel Hilbert space (RKHS). More specifically, we consider the following time-fractional PIDE:

$$\begin{aligned}\kappa_1(x,t)\,\partial_{t^\alpha}^{\alpha}u(x,t)+\kappa_2(x,t)\,\partial_x u(x,t)+\kappa_3(x,t)\,\partial_{x^2}^{2}u(x,t)\\ +\int_0^1 K_1(x,t,s)\,\partial_{x^2}^{2}u(x,s)ds+\int_0^t K_2(x,t,s)\,\partial_{x^2}^{2}u(x,s)ds=f(x,t,u(x,t)),\end{aligned} \tag{1}$$

2010 *Mathematics Subject Classification*. Primary 35R11; Secondary 47B32, 34K28, 34K37
Keywords. Reproducing kernel Hilbert space method, Fractional calculus theory, Singular partial integrodifferential equation
Received: 20 February 2018; Revised: 06 May 2018; Accepted: 12 August 2018
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subject to the following conditions:

$$\begin{cases} u(x,0) = \omega(x), \\ u(0,t) = \upsilon_1(t), \\ u(1,t) = \upsilon_2(t). \end{cases} \tag{2}$$

Throughout this paper, $0 \le x, t \le 1$, $u = u(x,t)$ is sought to be determined, $\kappa_1(x,t)$, $\kappa_2(x,t)$, and $\kappa_3(x,t)$ are analytical real-valued functions over the square $[0,1]^2$ and may take the values $\kappa_j(x_\lambda, t_\lambda) = 0$ for some $(x_\lambda, t_\lambda) \in [0,1]^2$ and some $j \in 1,2,3$ which make Eqs. 1 and 2 to be singular at $(x,t) = (x_\lambda, t_\lambda)$. Further,

$$\partial_{t^\alpha}^{\alpha} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \partial_\tau u(x,\tau)\, d\tau,\ 0 < \tau < t,\ 0 < \alpha < 1. \tag{3}$$

The main purpose of the present paper, is to construct a computational reproducing kernel Hilbert space method (RKHSM) to solve the time-fractional PIDEs of Eqs. 1 and 2. Historically, the reproducing Kernel theory was used first at 1907 to solve harmonic and biharmonic Dirichlet problems [18]. In 1950, it was formalized by knitting it with the reproducing kernel functions [19]. This theory, which is proxy in the RKHSM, has been used in diverse application in applied mathematics and engineering modeling [20–23]. Recently, a broad range of researches have applied the RKHSM for the solutions of several integral and differential operators alongside with their theories [24–54].

The RKHSM is a numerical, as well as, analytical technique for solving a large variety of ordinary and partial differential equations associated to different kind of initial conditions, and usually provides the solutions in term of rapidly convergent series with components that can be elegantly computed. The main idea is to construct the direct sum of the RKHSs that satisfying the initial conditions of the given systems in order to determining their exact and their numerical solutions. The exact and the numerical solutions are represented in the form of series through the functions value at the right-hand side of the corresponding differential and algebraic equations. The advantages of the utilized approach lie in the following main advantages; firstly, it can produce good globally smooth numerical solutions, and with ability to solve many differential systems with complex constraint conditions, which are difficult to solve; secondly, the numerical solutions and their derivatives are converge uniformly to the exact solutions and their derivatives, respectively; thirdly, the method is mesh-free, easily implemented and capable in treating various differential systems and various initial conditions; fourthly, since the method needs no time discretization, there is no matter, in which time the numerical solutions is computed, from the both elapsed time and stability problem, point of views.

2. Reproducing kernel theory

A Hilbert space which possesses a reproducing kernel is called a reproducing kernel Hilbert space (RKHS). Through this section, we denote $\|z\|_{\blacksquare}^2 = \langle z(*), z(*)\rangle_{\blacksquare}$, where $z \in \blacksquare$, $* \in [0,1]$, and $\blacksquare \in \left\{W_2^1, \hat{W}_2^1, W_2^2, W_2^3\right\}$.

- $W_2^1[0,1] = \{z = z(t) : z$ is absolutely continuous function on $[0,1]\}$. Here,

$$\langle z_1(t), z_2(t)\rangle_{W_2^1} = z_1(0)\, z_1(0) + \int_0^1 z_1'(t)\, z_2'(t)\, dt. \tag{4}$$

- $W_2^1[0,1]$ is a complete RK with

$$R_s^{\{1\}}(t) = 1 + \min(s,t). \tag{5}$$

Similarly, for $\hat{W}_2^1[0,1]$; $\langle z_1(x), z_2(x)\rangle_{\hat{W}_2^1} = z_1(0)\, z_2(0) + \int_0^1 z_1'(x)\, z_2'(x)\, dx$ and $\hat{R}_y^{\{1\}}(x) = 1 + \min(x,y)$.

- $W_2^2[0,1] = \{z = z(t) : z, z'$ are absolutely continuous functions on $[0,1]$ and $z(0) = 0\}$. Here,

$$\langle z_1(x), z_2(x)\rangle_{W_2^2} = \sum_{i=0}^{1} z_1^{(i)}(0) z_2^{(i)}(0) + \int_0^1 z_1''(t) z_2''(t)\, dt. \tag{6}$$

- $W_2^2[0,1]$ is a complete RK with

$$R_s^{\{2\}}(t) = \begin{cases} st + \frac{1}{2}st^2 - \frac{1}{6}t^3, & t \le s, \\ st + \frac{1}{2}s^2 t - \frac{1}{6}s^3, & t > s. \end{cases} \tag{7}$$

- $W_2^3[0,1] = \{z = z(x) : z, z', z''$ are absolutely continuous functions on $[0,1]$ and $z(0) = z(1) = 0\}$. Here,

$$\langle z_1(x), z_2(x)\rangle_{W_2^3} = \sum_{i=0}^{1} z_1^{(i)}(0) z_2^{(i)}(0) + z_1(1) z_2(1) + \int_0^1 z_1'''(x) z_2'''(x)\, dx. \tag{8}$$

- $W_2^3[0,1]$ is a complete RK with

$$R_y^{\{3\}}(x) = \begin{cases} \frac{1}{120}\left(\Delta_1(x,y) + \Delta_2(x,y) + \Delta_3(x,y)\right), & x \le y, \\ \frac{1}{120}\left(\Delta_1(y,x) + \Delta_2(y,x) + \Delta_3(y,x)\right), & x > y, \end{cases} \tag{9}$$

in which

$$\begin{aligned} \Delta_1(x,y) &= x^2 y^2 \left(126 - x^3 - y^3\right), \\ \Delta_2(x,y) &= y\left(y\left(y^3 - 10x^3\right) - 5x\left(-24 + y^3\right)\right), \\ \Delta_3(x,y) &= 5xy\left(y\left(x^3 - 24\right) + x\left(y^3 - 24\right)\right). \end{aligned} \tag{10}$$

Henceforth, we denote $\Omega = [0,1] \otimes [0,1]$, $\partial_{x^i t^j}^{i+j} = \left(\partial^i/\partial x^i\right)\left(\partial^j/\partial t^j\right)$, whenever $i, j = 1, 2$ and $\|u\|_{\blacksquare}^2 = \langle u(*,\circ), u(*,\circ)\rangle_{\blacksquare}$, where $u \in \blacksquare$, $*, \circ \in \Omega$, $\blacksquare \in \{H, W\}$

- $W(\Omega) = \{u = u(x,t) : \partial_{x^2}^2 \partial_{t^2}^2 u$ is continuous function in Ω and $u(x,0) = u(0,t) = u(1,t) = 0\}$. Here

$$\begin{aligned} \langle u_1(x,t), u_2(x,t)\rangle_W = {} & \textstyle\sum_{j=0}^{1} \left\langle \partial_{t^j}^j u_1(x,0), \partial_{t^j}^j u_2(x,0)\right\rangle W_2^3 \\ & + \textstyle\int_0^1 \left[\sum_{j=0}^{1} \partial_{t^2}^2 \partial_{x^j}^j u_1(0,t)\, \partial_{t^2}^2 \partial_{x^j}^j u_2(0,t) + \partial_{t^2}^2 u_1(1,t)\, \partial_{t^2}^2 u_2(1,t)\right] dt \\ & + \textstyle\int_0^1 \int_0^1 \partial_{x^3}^3 \partial_{t^2}^2 u_1(x,t)\, \partial_{x^3}^3 \partial_{t^2}^2 u_2(x,t)\, dx dt. \end{aligned} \tag{11}$$

- $W(\Omega)$ is a complete RK with

$$R_{(y,s)}(x,t) = R_y^{\{3\}}(x) R_s^{\{2\}}(t), \tag{12}$$

such that for any $u(x,t) \in W(\Omega)$, we have $\left\langle u(x,t), R_{(y,s)}(x,t)\right\rangle_W = u(y,s)$ and $R_{(y,s)}(x,t) = R_{(x,t)}(y,s)$, where $R_y^{\{3\}}(x)$ and $R_s^{\{2\}}(t)$ are the RK functions of the spaces $W_2^3[0,1]$ and $W_2^2[0,1]$, respectively.

- $H(\Omega) = \{u = u(x,t) : u$ is continuous function in $\Omega\}$. Here,

$$\begin{aligned} \langle u_1(x,t), u_2(x,t)\rangle_H = {} & \langle u_1(x,0), u_2(x,0)\rangle_{\hat{W}_2^1} \\ & + \textstyle\int_0^1 \partial_t u_1(0,t)\, \partial_t u_2(0,t)\, dt + \int_0^1 \int_0^1 \partial_{xt}^2 u_1(x,t)\, \partial_{xt}^2 u_2(x,t)\, dx dt. \end{aligned} \tag{13}$$

- $H(\Omega)$ is a complete RK with

$$r_{(y,s)}(x,t) = \hat{R}_y^{\{1\}}(x) R_s^{\{1\}}(t), \tag{14}$$

such that for any $u(x,t) \in H(\Omega)$, we have $\left\langle u(x,t), r_{(y,s)}(x,t)\right\rangle_H = u(y,s)$ and $r_{(y,s)}(x,t) = r_{(x,t)}(y,s)$, where $\hat{R}_y^{\{1\}}(x)$ and $R_s^{\{1\}}(t)$ are the RK functions of spaces $\hat{W}_2^1[0,1]$ and $W_2^1[0,1]$, respectively.

3. The numerical solution

Through the remainder sections, we will use the following markers:
$P = P(x, t, u(x, t))$, $P_k = P(x_k, t_k, u(x_k, t_k))$, and $P_k^n = P(x_k, t_k, u_n(x_k, t_k))$ whenever $k = 1, 2, 3, \ldots, \infty$.

To apply the RKHSM, we must homogenized the nonhomogeneous constraints conditions by suitable transformations, for the convenience, we still denote the solution of the new equation by $u(x, t)$. So, let

$$\begin{aligned}\kappa_1(x,t)\,\partial_{t^\alpha}^{\alpha} u(x,t) + \kappa_2(x,t)\,\partial_x u(x,t) + \kappa_3(x,t)\,\partial_{x^2}^2 u(x,t) + \int_0^1 K_1(x,t,s)\,\partial_{x^2}^2 u(x,s)ds \\ + \int_0^t K_2(x,t,s)\,\partial_{x^2}^2 u(x,s)ds = P(x,t,u(x,t)),\end{aligned} \tag{15}$$

subject to the following conditions:

$$\begin{cases} u(x,0) = 0, \\ u(0,t) = 0, \\ u(1,t) = 0. \end{cases} \tag{16}$$

For the conduct of proceedings, we define the fractional differential linear operator $\Pi: W(\Omega) \to H(\Omega)$ such that

$$\begin{aligned}\Pi u(x,t) := \kappa_1(x,t)\,\partial_{t^\alpha}^{\alpha} u(x,t) + \kappa_2(x,t)\,\partial_x u(x,t) + \kappa_3(x,t)\,\partial_{x^2}^2 u(x,t) \\ + \int_0^1 K_1(x,t,s)\,\partial_{x^2}^2 u(x,s)ds + \int_0^t K_2(x,t,s)\,\partial_{x^2}^2 u(x,s)ds.\end{aligned} \tag{17}$$

Thus, the time-fractional PIDEs to be solved is governed by the following equivalent functional equation:

$$\Pi u(x,t) = P(x,t,u(x,t)). \tag{18}$$

To build an orthogonal function systems of the space $W(\Omega)$, we choose a countable dense subset $\{(x_i, t_i)\}_{i=1}^{\infty}$ in Ω, define $\varphi_i(x,t) = r_{(x_i,t_i)}(x,t)$ and $\psi_i(x,t) = \Pi^* \varphi_i(x,t)$, where $\Pi^* : H(\Omega) \to W(\Omega)$ is the adjoint operator of Π and is uniquely determined.

The normalized orthonormal function systems $\left\{\overline{\psi}_i(x,t)\right\}_{i=1}^{\infty}$ of $W(\Omega)$ is usually constructed from the process of the Gram-Schmidt orthogonalization of $\{\psi_i(x,t)\}_{i=1}^{\infty}$ as

$$\overline{\psi}_i(x,t) = \sum_{k=1}^{i} \mu_{ik} \psi_k(x,t). \tag{19}$$

To apply the RKHSM, we divide the finite domain Ω into a $p \times q$ mesh point with the space step size $\Delta x = \frac{1}{p}$ in the x direction of $[0,1]$ and the time step size $\Delta t = \frac{1}{q}$ in the t direction of $[0,1]$, respectively, in which p and q are positive integers. Anyhow the grid points (x_l, t_m) in the space-time domain Ω are defined simultaneously as

$$(x_l, t_m) = (l\Delta x, m\Delta t),\ l = 0,1,...,p,\ m = 0,1,...,q. \tag{20}$$

At first, depending on the Schwarz inequality it is easy to see that $\Pi : W(\Omega) \to H(\Omega)$ is a bounded linear operator, that is $\|\Pi u(x,t)\|^2_{W_2^1} \le M\|u\|^2_W$ with $M > 0$.

Lemma 3.1. *The sequence $\{\psi_i(x,t)\}_{i=1}^{\infty}$ is a complete function system in $W(\Omega)$ with*

$$\psi_i(x,t) = \Pi_{(y,s)} R(x,t)\Big|_{(y,s)=(x_i,t_i)}. \tag{21}$$

Proof. Here, $\Pi_{(y,s)}$ indicates that the operator Π applies to the function of (y,s). Indeed

$$\begin{aligned}\psi_i(x,t) &= \Pi^*\varphi_i(x,t)\\ &= \left\langle \Pi^*\varphi_i(y,s), R_{(x,t)}(y,s)\right\rangle_W\\ &= \left\langle \varphi_i(y,s), \Pi_{(y,s)}R_{(x,t)}(y,s)\right\rangle_H\\ &= \Pi_{(y,s)}R_{(x,t)}(y,s)\Big|_{(y,s)=(x_i,t_i)}\\ &= \Pi_{(y,s)}R_{(y,s)}(x,t)\Big|_{(y,s)=(x_i,t_i)} \in W(\Omega).\end{aligned} \tag{22}$$

Now, for each fixed $u \in W(\Omega)$, let $\langle u(x,t), \psi_i(x,t)\rangle_W = 0, i = 1,2,\ldots$. Then, $\langle u(x,t), \psi_i(x,t)\rangle_W = \langle u(x,t), \Pi^*\varphi_i(x,t)\rangle_W = \langle \Pi u(x,t), \varphi_i(t)\rangle_H = \Pi u(x_i,t_i) = 0$. Whilst, $\{(x_i,t_i)\}_{i=1}^{\infty}$ is dense on Ω, we must have $\Pi u(x,t) = 0$ from the existence of Π^{-1}, it follows that $u = 0$. □

Theorem 3.2. *The sequence $\left\{R_{(x_i,t_i)}(x,t)\right\}_{i=1}^{\infty}$ is a linearly independent in $W(\Omega)$.*

Proof. It is adequate to show that $\left\{R_{(x_i,t_i)}(x,t)\right\}_{i=1}^{m}$ is a linearly independent for each $m \geq 1$. In fact, if $\{c_i\}_{i=1}^{m}$ satisfies $\sum_{i=1}^{m} c_i R_{(x_i,t_i)}(x,t) = 0$, taking $h_k(x,t) \in W(\Omega)$ such that $h_k(x_l,t_l) = \delta_{l,k}$ for each $l = 1,2,\ldots,m$. Then

$$\begin{aligned}0 &= \left\langle h_k(x,t) \textstyle\sum_{i=1}^{m} c_i R_{(x_i,t_i)}(x,t)\right\rangle_W\\ &= \textstyle\sum_{i=1}^{m} c_i \left\langle h_k(x,t), R_{(x_i,t_i)}(x,t)\right\rangle_W\\ &= \textstyle\sum_{i=1}^{m} c_i h_k(x_i,t_i)\\ &= c_k.\end{aligned} \tag{23}$$

Thus $c_k = 0$ for $k = 1,2,\ldots,m$. □

Theorem 3.3. *Suppose that $A_i = \sum_{k=1}^{i} \mu_{ik} P_k$. If $u \in W(\Omega)$ is the solution of Eqs.* (20) *and* (18), *then*

$$u(x,t) = \sum_{i=1}^{\infty} A_i \overline{\psi}_i(x,t). \tag{24}$$

Proof. Since, $\langle u(x,t), \varphi_i(x,t)\rangle_W = u(x_i,t_i)$ for each $u \in W(\Omega)$, whilst, $\sum_{i=1}^{\infty} A_i \overline{\psi}_i(x,t)$ is the Fourier series expansion about $\left\{\overline{\psi}_i(x,t)\right\}_{i=1}^{\infty}$, then it is a convergent in the sense of $\|\cdot\|_W$. Thus,

$$\begin{aligned}u(x,t) &= \textstyle\sum_{i=1}^{\infty}\sum_{k=1}^{i} \left\langle u(x,t), \overline{\psi}_i(x,t)\right\rangle_W \overline{\psi}_i(x,t)\\ &= \textstyle\sum_{i=1}^{\infty} \left\langle u(x,t), \sum_{k=1}^{i} \mu_{ik}\psi_k(x,t)\right\rangle_W \overline{\psi}_i(x,t)\\ &= \textstyle\sum_{i=1}^{\infty}\sum_{k=1}^{i} \mu_{ik} \langle u(x,t), \Pi^*\varphi_k(x,t)\rangle_W \overline{\psi}_i(x,t)\\ &= \textstyle\sum_{i=1}^{\infty}\sum_{k=1}^{i} \mu_{ik} \Pi u(x_k,t_k)\, \overline{\psi}_i(x,t)\\ &= \textstyle\sum_{i=1}^{\infty} A_i \overline{\psi}_i(x,t).\end{aligned} \tag{25}$$

In other words, $\sum_{i=1}^{\infty} A_i \overline{\psi}_i(x,t)$ is the exact solution of Eqs. 20 and 18. □

For numerical computations, put $(x_1,t_1) = (0,0)$, then from the constraints conditions of Eq. 18, the value of $u(x_1,t_1)$ is known. Set $u_0(x_1,t_1) = u(x_1,t_1)$ and define the n-term numerical solution of $u(x,t)$ using the truncating version as

$$u_n(x,t) = \sum_{i=1}^{n} A_i \overline{\psi}_i(x,t). \tag{26}$$

In order that $W(\Omega)$ is a Hilbert space, then the series $\sum_{i=1}^{\infty} A_i \overline{\psi}_i(x,t) < \infty$. Thus, we can guarantee that the numerical solution $u_n(x,t)$ satisfies the constraints conditions of Eq. 18.

Theorem 3.4. *The partial derivatives of the numerical solution $\partial^i_{x^i}\partial^j_{t^j}u_n(x,t)$ are converging uniformly to the partial derivatives of the exact solution $\partial^i_{x^i}\partial^j_{t^j}u(x,t)$, whenever $i=0,1,2$, $j=0,1$ as $n\to\infty$.*

Proof. Since $W(\Omega)$ is a Hilbert space, from Eq. 26, it is follows that, $\|u-u_n\|_W\to 0$ as $n\to\infty$. Again, since

$$\begin{aligned}\left|\partial^i_{x^i}\partial^j_{t^j}u(x,t)-\partial^i_{x^i}\partial^j_{t^j}u_n(x,t)\right| &= \left|\left\langle u(y,s)-u_n(y,s),\partial^i_{x^i}\partial^j_{t^j}\Pi R_{(x,t)}(y,s)\right\rangle_W\right| \\ &\le \|u-u_n\|_W\left\|\partial^i_{x^i}\partial^j_{t^j}\Pi R_{(x,t)}(y,s)\right\|_W \\ &\le M_{i,j}\|u-u_n\|_W.\end{aligned} \tag{27}$$

Thus, $\left|\partial^i_{x^i}\partial^j_{t^j}u(x,t)-\partial^i_{x^i}\partial^j_{t^j}u_n(x,t)\right|\to 0$ as $n\to\infty$. □

4. Numerical results

This section presents the numerical solutions for two different time-fractional PIDEs using the RKHSM. The results reveal that the algorithm is highly accurate, rapidly converge, and convenient to handle various physical problems in fractional calculus.

Example 4.1. *Consider the linear singular PIDE:*

$$\begin{aligned}&\tfrac{1}{t}\partial^\alpha_{t^\alpha}u(x,t)+xu(x,t)-\tfrac{1}{x-t}\partial_x u(x,t)+\tfrac{x^2}{t}\partial^2_{x^2}u(x,t)\\ &\quad+\int_0^1 tse^{x+t}\partial^2_{x^2}u(x,s)ds+\int_0^t e^{x-t}s^{\alpha+1}\partial^2_{x^2}u(x,s)ds=g(x,t),\end{aligned} \tag{28}$$

subject to the following conditions:

$$\begin{cases}u(x,0)=0,\\ u(0,t)=\tanh(1)t^{2\alpha}-t^\alpha,\\ u(1,t)=0,\end{cases} \tag{29}$$

where $0\le x,t\le 1$ and $0<\alpha\le 1$. Here, the exact solution is

$$u(x,t)=\tanh(1-x)\,t^{2\alpha}+(1-x)\,t^\alpha. \tag{30}$$

Example 4.2. *Consider the nonlinear singular PIDE:*

$$\begin{aligned}&\tfrac{1}{\sin(x-t)}\partial^\alpha_{t^\alpha}u(x,t)+u^3(x,t)+u^2(x,t)+\tfrac{x^2}{t}\partial_x u(x,s)\\ &\quad-\tfrac{1}{xt}\partial^2_{x^2}u(x,t)\int_0^1 s(x+t)\partial^2_{x^2}u(x,s)ds\int_0^t\sin(xt)\,\partial^2_{x^2}u(x,s)ds=g(x,t),\end{aligned} \tag{31}$$

subject to the following conditions:

$$\begin{cases}u(x,0)=0,\\ u(0,t)=0,\\ u(1,t)=0.25\left(t^2+t^{3\alpha}\right),\end{cases} \tag{32}$$

where $0\le x,t\le 1$ and $0<\alpha\le 1$. Here, the exact solution is

$$u(x,t)=0.25t\left(t+t^{3\alpha-1}\right)\sin^2(1.5\pi x). \tag{33}$$

With a view to demonstrate the agreement between the exact and the RKHSM approximate solutions, Tables 1 and 2 show the absolute error of approximate solution of Examples 1 and 2, respectively, obtained at various (x,t) in Ω when $\alpha\in\{0.25,0.5,0.75,1\}$.

Table 1: Absolute errors in Example 1.

x	t	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$
0.25	0.25	3.430833×10^{-3}	7.995964×10^{-4}	5.144964×10^{-4}	3.528566×10^{-5}
	0.5	9.299361×10^{-3}	3.710936×10^{-4}	2.886701×10^{-4}	7.253275×10^{-5}
	0.75	3.327637×10^{-3}	5.334014×10^{-4}	8.719411×10^{-4}	4.567442×10^{-5}
	1	6.967151×10^{-3}	8.609413×10^{-4}	3.509236×10^{-4}	3.690029×10^{-5}
0.5	0.25	9.795586×10^{-3}	3.554394×10^{-3}	6.240844×10^{-4}	3.910356×10^{-4}
	0.5	9.094017×10^{-3}	4.706146×10^{-3}	3.604104×10^{-4}	6.215311×10^{-4}
	0.75	4.197428×10^{-3}	2.188326×10^{-3}	1.934253×10^{-4}	7.339235×10^{-4}
	1	2.912611×10^{-3}	4.358614×10^{-3}	9.790957×10^{-4}	1.510181×10^{-4}
0.75	0.25	7.738202×10^{-3}	5.992349×10^{-4}	9.630874×10^{-4}	5.627229×10^{-5}
	0.5	5.031783×10^{-3}	1.829352×10^{-4}	1.797683×10^{-4}	2.907608×10^{-5}
	0.75	5.881743×10^{-3}	4.346278×10^{-4}	4.965781×10^{-4}	3.236452×10^{-5}
	1	7.388693×10^{-3}	2.021375×10^{-4}	5.511058×10^{-4}	7.157298×10^{-5}

Table 2: Absolute errors in Example 2.

x	t	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$
0.25	0.25	4.973814×10^{-3}	3.534328×10^{-4}	6.523242×10^{-4}	7.383310×10^{-5}
	0.5	7.614758×10^{-3}	1.456721×10^{-4}	8.712349×10^{-4}	1.784613×10^{-5}
	0.75	1.271543×10^{-3}	2.540384×10^{-4}	2.974959×10^{-4}	9.747171×10^{-5}
	1	8.214294×10^{-3}	7.306606×10^{-4}	1.257665×10^{-4}	5.302995×10^{-5}
0.5	0.25	6.605308×10^{-3}	2.637898×10^{-3}	7.790407×10^{-4}	1.065586×10^{-4}
	0.5	5.056966×10^{-3}	8.531553×10^{-3}	9.034364×10^{-4}	1.373669×10^{-4}
	0.75	3.247963×10^{-3}	9.096351×10^{-3}	4.473915×10^{-4}	6.156406×10^{-4}
	1	6.728873×10^{-3}	5.839846×10^{-3}	2.844205×10^{-4}	6.587978×10^{-4}
0.75	0.25	2.795225×10^{-3}	9.816987×10^{-4}	7.441276×10^{-4}	7.234314×10^{-5}
	0.5	7.905326×10^{-3}	8.978685×10^{-4}	1.643969×10^{-4}	3.839786×10^{-5}
	0.75	6.107781×10^{-3}	5.401769×10^{-4}	5.090348×10^{-4}	8.601375×10^{-5}
	1	3.227506×10^{-3}	4.534633×10^{-4}	3.020299×10^{-4}	3.491774×10^{-5}

Note that, the reduction in the step size, $n = pq = \frac{1}{\Delta x}\frac{1}{\Delta t}$, of Ω results in a reduction in the error and correspondingly an improvement in the accuracy of the obtained solution. This goes in agreement with the known fact that the error is monotone decreasing where more accurate solutions are achieved using a reduction in the step size, whilst, the cost to be paid while going in this direction is the rapid increase in the number of iterations required for convergence.

5. Conclusion

The fundamental significance of the proposed algorithm lies in its ability to, efficiently and reliably, handle the major challenges associated with the singular time-fractional PIDEs in terms of highly nonlinearity, nonhomogeneity, fractional level characteristics, and the nature of Dirichlet conditions may appear. It is observed that the calculated solutions bifurcate and produce similar patterns when $\alpha \in (0, 1]$ and the patterns coincides when α is close to 1. The comparative studies based on the absolute natural error function sense shows that the RKHSM approximate values are more acceptable in terms of accuracy and stability.

Acknowledgment: The authors would like to acknowledge the University of Jordan for funding this research study.

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