Filomat 33:12 (2019), 3855–3867
https://doi.org/10.2298/FIL1912855B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The Extended Odd Family of Probability Distributions with Practice to a Submodel

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Abstract. In this paper, we introduce a new family of distributions extending the odd family of distributions. A new tuning parameter is introduced, with some connections to the well-known transmuted transformation. Some mathematical results are obtained, including moments, generating function and order statistics. Then, we study a special case dealing with the standard loglogistic distribution and the modified Weibull distribution. Its main features are to have densities with flexible shapes where skewness, kurtosis, heavy tails and modality can be observed, and increasing-decreasing-increasing, unimodal and bathtub shaped hazard rate functions. Estimation of the related parameters is investigated by the maximum likelihood method. We illustrate the usefulness of our extended odd family of distributions with applications to two practical data sets.

1. Introduction with motivations

Over the last decades, many efforts have been devoted to increase chances of modeling data of various types. These efforts resulted in the development of powerful methods to generate new families of (probability) distributions. The T-X transformation introduced by [1] is the most popular one. This transformation has generated plethora families of distributions with significant applications in many areas (engineering, sciences, economics, biological studies, environmental sciences . . .). One of the most useful of these families is the (generalized) odd family of distributions described below. Let $F(x)$ be a baseline cumulative distribution function (cdf) of a continuous probability distribution, its complement is $\overline{F}(x) = 1 - F(x)$, and $G(x)$ be a cdf of a continuous probability distribution with support on $(0, \infty)$. The odd family of distributions is characterized by the following cdf:

$$H(x) = G\left(\frac{F(x)}{\overline{F}(x)}\right), \qquad x \in \mathbb{R}. \tag{1}$$

With suitable choices for $F(x)$ and $G(x)$, the odd family of distributions provides good statistical models for a wide variety of real-life data. Recent advances and applications of this family can be found in [2], [3], [4], [5] and [6].

2010 *Mathematics Subject Classification*. 60E05; 62E15
Keywords. Odd family of distributions, Maximum likelihood estimation, Data analysis
Received: 09 March 2018; Revised: 16 May 2018; Accepted: 13 August 2018
Communicated by Biljana Popović
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In this paper, motivated by the usefulness of the odd family of distributions, we propose a new extension, also using the T-X transformation developed by [1]. Among other, it has the features to be both simple ; only a new additive term is considered, and to have a deep connection with another successful family of distributions, namely the transmuted family of distributions. To be more specific, our new family of distributions is described by the following function :

$$H(x;\lambda) = G\left(\frac{F(x)}{\overline{F}(x)} + \lambda F(x)\right), \qquad x \in \mathbb{R}, \tag{2}$$

which is a cdf for $\lambda \geq -1$ (observe that λ can be negative and positive, the critical value -1 is to ensure that $H(x,\lambda)$ satisfies all the properties of a cdf, see Proposition 2.1 below). Thus defined $H(x;\lambda)$ is a flexible version of the cdf $H(x)$ given by (1), with the advantage of a clear location compared to $H(x)$ according to the sign of λ : Since $G(x)$ is increasing, if $\lambda \leq 0$ we have $H(x;\lambda) \leq H(x;0) = H(x)$ and if $\lambda \geq 0$, the reverse holds: $H(x;\lambda) \geq H(x;0) = H(x)$. This opens the door to the construction of statistical models not perfectly fit with the former odd family of distributions. On the other hand, the flexibility of $H(x;\lambda)$ can be also highlighted by considering the transmuted version of $F(x)$ proposed by [17]: $F_\lambda(x) = (1+\lambda)F(x) - \lambda[F(x)]^2 = F(x)(1+\lambda\overline{F}(x))$. Indeed, from (2), we can write : $H(x;\lambda) = G\left(F_\lambda(x)/\overline{F}(x)\right)$. The great ability of the transmuted family of distributions to construct good statistical models is now established; we refer to [7], [8] and [9], and the references therein. The definition of $H(x;\lambda)$ injects this feature into the former definition of the odd family of distributions. In this sense, our new family of distributions can be viewed as a hybrid version of the odd and transmuted families of distributions. We name the corresponding new family of distributions by the EO family of distributions (for Extension of the Odd family of distributions). The preceding motivations lead us to study theoretical and practical aspects of the EO family of distributions, with a special focus on the construction of new statistical models. In particular, we show that it yields a "competitive fit" in certain practical situations.

The rest of this paper is organized as follows. In Section 2, we prove that $H(x;\lambda)$ has the properties of a cdf for $\lambda \geq 1$, determine the probability density function (pdf) and the hazard rate function (hrf) corresponding to (2). We then explore the analytical properties of the shapes of these two functions. Some statistical characteristics of the EO family are studied, including simulated method, useful expansion for $H(x;\lambda)$ and $h(x;\lambda)$, general expressions for moments, generating function, incomplete moments and order statistics. Section 3 is devoted to a special case of the EO family. It has four parameters and is defined with the cdf of standard loglogistic distribution for $G(x)$ and the cdf of the modified Weibull distribution introduced by [16] for $F(x)$. The main features of this special case is to have pdfs and hrfs with remarkable flexibility properties. This aspect is illustrated via several graphics, with various values for the parameters. Then, some of its mathematical properties are presented. Estimation of the four parameter is performed via the maximum likelihood method. Two practical data sets are considered and fair comparisons with existing recent distributions are performed in terms of goodness of fit. The first one concerns the first failure of generators and the second one about remissions times of cancer patients.

2. Some properties of the new family

2.1. A key result

First of all, let us present the mathematical result at the genesis of the EO family.

Proposition 2.1. *Let $F(x)$ be a baseline cdf of a certain continuous distribution and $G(x)$ be a cdf of an another continuous distribution with support on $(0,\infty)$. Then, for any $\lambda \geq -1$, the following function is a cdf of a continuous distribution:*

$$H(x;\lambda) = G\left(\frac{F(x)}{\overline{F}(x)} + \lambda F(x)\right), \qquad x \in \mathbb{R}.$$

Let us now prove Proposition 2.1. Since $F(x) \in [0,1]$ and $\lambda \geq -1$, we have $(F(x)/\overline{F}(x)) + \lambda F(x) = F(x)(1/\overline{F}(x) + \lambda) \geq F(x)(1+\lambda) \geq 0$, so $(F(x)/\overline{F}(x)) + \lambda F(x)$ is in the support of $G(x)$. The function $H(x;\lambda)$ is continuous on $\mathbb{R}$ by composition of continuous functions on $\mathbb{R}$. We have $\lim_{x\to-\infty} H(x;\lambda) = G(0) = 0$ and $\lim_{x\to+\infty} H(x;\lambda) = \lim_{x\to+\infty} G(x) = 1$. Let $f(x)$ be the pdf corresponding to $F(x)$ and $g(x)$ be the pdf corresponding to $G(x)$. Since $F(x) \in [0,1]$, $g(x) \geq 0$, $f(x) \geq 0$ and $\lambda \geq -1$, we have, almost everywhere,

$$H'(x;\lambda) = g\left(\frac{F(x)}{\overline{F}(x)} + \lambda F(x)\right) f(x)\left[\frac{1}{[\overline{F}(x)]^2} + \lambda\right] \geq g\left(\frac{F(x)}{\overline{F}(x)} + \lambda F(x)\right) f(x)(1+\lambda) \geq 0.$$

Hence $H(x;\lambda)$ is increasing. The proof of Proposition 2.1 is completed.

2.2. *On shapes of the probability density and hazard rate functions*

Since it is at the heart of the definition of $H(x;\lambda)$, Table 1 lists the function $(F(x)/\overline{F}(x)) + \lambda F(x)$ and the corresponding parameters for some well-known distributions of the literature. All these functions can be used to construct special cases of the EO family of distributions. An extension of the one using the Weibull distribution will be at the heart of our coming study.

Table 1: Distributions and corresponding $(F(x)/\overline{F}(x)) + \lambda F(x)$ functions

Distributions	Support	$F(x)/\overline{F}(x) + \lambda F(x)$	Parameters
Uniform	$(0,\theta)$	$x/(\theta - x) + \lambda x/\theta$	(θ,λ)
Exponential	$(0,+\infty)$	$e^{\theta x} - \lambda e^{-\theta x} + \lambda - 1$	(θ,λ)
Weibull	$(0,+\infty)$	$e^{\theta x^{\alpha}} - \lambda e^{-\theta x^{\alpha}} + \lambda - 1$	(θ,α,λ)
Fréchet	$(0,+\infty)$	$\left(e^{\theta x^{\alpha}} - 1\right)^{-1} + \lambda e^{-\theta x^{-\alpha}}$	(θ,α,λ)
Half-logistic	$(0,+\infty)$	$(e^{x}-1)/2 + \lambda(e^{x}-1)/(e^{x}+1)$	λ
Power function	$(0,1/\theta)$	$\left[(\theta x)^{-k} - 1\right]^{-1} + \lambda(\theta x)^{k}$	(θ,k,λ)
Pareto	$(\theta,+\infty)$	$(x/\theta)^{k} - \lambda(\theta/x)^{k} + \lambda - 1$	(θ,k,λ)
Burr XII	$(0,+\infty)$	$[1+(x/s)^{c}]^{k} - \lambda\,[1+(x/s)^{c}]^{-k} + \lambda - 1$	(s,k,c,λ)
Log-logistic	$(0,+\infty)$	$[1+(x/s)^{c}] - \lambda\,[1+(x/s)^{c}]^{-1} + \lambda - 1$	(s,c,λ)
Lomax	$(0,+\infty)$	$[1+x/s]^{k} - \lambda\,[1+x/s]^{-k} + \lambda - 1$	(s,k,λ)
Gumbel	$\mathbb{R}$	$\left[\exp\left(e^{-(x-\mu)/\sigma}\right) - 1\right]^{-1} + \lambda\exp\left(-e^{-(x-\mu)/\sigma}\right)$	(μ,σ,λ)
Kumaraswamy	$(0,1)$	$(1-x^{a})^{-b} - \lambda(1-x^{a})^{b} + \lambda - 1$	(a,b,λ)
Normal	$\mathbb{R}$	$\Phi((x-\mu)/\sigma)\,[1/(1-\Phi((x-\mu)/\sigma)) + \lambda]$	(μ,σ,λ)

The pdf corresponding to (2) is given by

$$h(x;\lambda) = g\left(\frac{F(x)}{\overline{F}(x)} + \lambda F(x)\right) f(x)\left[\frac{1}{[\overline{F}(x)]^2} + \lambda\right], \qquad x \in \mathbb{R}. \tag{3}$$

Let $r(x)$ be the hrf associated to $G(x)$: $r(x) = g(x)/\overline{G}(x)$, with $\overline{G}(x) = 1 - G(x)$. The hrf associated to $H(x;\lambda)$ is given by

$$s(x;\lambda) = \frac{h(x;\lambda)}{1 - H(x;\lambda)} = r\left(\frac{F(x)}{\overline{F}(x)} + \lambda F(x)\right) f(x)\left[\frac{1}{[\overline{F}(x)]^2} + \lambda\right], \qquad x \in \mathbb{R}. \tag{4}$$

Analytic descriptions for the shapes of $h(x;\lambda)$ and $s(x;\lambda)$ can be performed. Let us first focus on the critical points of $h(x;\lambda)$. We have

$$\log[h(x;\lambda)] = \log\left[g\left(\frac{F(x)}{\overline{F}(x)} + \lambda F(x)\right)\right] + \log[f(x)] + \log\left[1 + \lambda[\overline{F}(x)]^2\right] - 2\log\left[\overline{F}(x)\right],$$

implying that

$$\frac{d}{dx}\log\left[h(x;\lambda)\right]=\frac{g'\left((F(x)/\overline{F}(x))+\lambda F(x)\right)}{g\left((F(x)/\overline{F}(x))+\lambda F(x)\right)}f(x)\left[\frac{1}{[\overline{F}(x)]^2}+\lambda\right]+\frac{f'(x)}{f(x)}-2\frac{\lambda\overline{F}(x)f(x)}{1+\lambda[\overline{F}(x)]^2}+2\frac{f(x)}{\overline{F}(x)}.$$

The critical points of $h(x;\lambda)$ are the roots of the equation $d\log\left[h(x;\lambda)\right]/dx=0$. More than one root may be obtained. The nature of a root x_0 depends on the sign of $\eta(x)=d^2\log\left[h(x;\lambda)\right]/dx$ taking with $x=x_0$; if $\eta(x_0)<0$, it is a local maximum, if $\eta(x_0)>0$, it is a local minimum and if $\eta(x_0)=0$, it is a point of inflection.

Similar arguments hold for the critical points of the hrf $s(x;\lambda)$. We have

$$\frac{d}{dx}\log\left[s(x;\lambda)\right]=\frac{r'\left((F(x)/\overline{F}(x))+\lambda F(x)\right)}{r\left((F(x)/\overline{F}(x))+\lambda F(x)\right)}f(x)\left[\frac{1}{[\overline{F}(x)]^2}+\lambda\right]+\frac{f'(x)}{f(x)}-2\frac{\lambda\overline{F}(x)f(x)}{1+\lambda[\overline{F}(x)]^2}+2\frac{f(x)}{\overline{F}(x)}.$$

The critical points of $s(x;\lambda)$ are the roots of the equation $d\log\left[s(x;\lambda)\right]/dx=0$. As for the pdf, more than one root may be obtained. The nature of a root x_0 depends on the sign of $\upsilon(x)=d^2\log\left[s(x;\lambda)\right]/dx$ taking with $x=x_0$; if $\upsilon(x_0)<0$, it is a local maximum, if $\upsilon(x_0)>0$, it is a local minimum and if $\upsilon(x_0)=0$, it is a point of inflection.

2.3. Some mathematical properties of the family

The quantile function of the family is given by, for $\lambda\neq 0$,

$$Q(y;\lambda)=F^{-1}\left(\frac{1+G^{-1}(y)+\lambda-\sqrt{(1+G^{-1}(y)+\lambda)^2-4\lambda G^{-1}(y)}}{2\lambda}\right),\qquad y\in(0,1).$$

The EO family of distributions can be simulated using this function; let U be a random variable having the uniform distribution on the interval $[0,1]$, then, for $\lambda\neq 0$, $X=Q(U;\lambda)$ has the pdf $h(x;\lambda)$ given by (3).

Let us now set some useful expansions for the cdf $H(x;\lambda)$ given by (2) and the pdf $h(x;\lambda)$ given by (3). Suppose that $G(x)$ has a series expansion of the form: $G(x)=\sum_{k=0}^{+\infty}a_kx^k$, $x>0$, where $(a_k)_{k\in\mathbb{N}}$ denotes a positive sequence of real numbers. Using this representation and binomial series, the cdf $H(x;\lambda)$ given by (2) can be expressed as

$$\begin{aligned}H(x;\lambda)&=\sum_{k=0}^{+\infty}a_k[F(x)]^k\left(\frac{1}{\overline{F}(x)}+\lambda\right)^k=\sum_{k=0}^{+\infty}\sum_{\ell=0}^{k}\binom{k}{\ell}\lambda^{k-\ell}a_k[F(x)]^k\frac{1}{[\overline{F}(x)]^\ell}\\&=\sum_{k,m=0}^{+\infty}\sum_{\ell=0}^{k}\binom{k}{\ell}\binom{-\ell}{m}(-1)^m\lambda^{k-\ell}a_k[F(x)]^{k+m}.\end{aligned}$$

The corresponding pdf can be expressed as

$$h(x;\lambda)=f(x)\sum_{\substack{k,m=0\\k+m>0}}^{+\infty}\sum_{\ell=0}^{k}\binom{k}{\ell}\binom{-\ell}{m}(-1)^m\lambda^{k-\ell}a_k(k+m)[F(x)]^{k+m-1}.\tag{5}$$

One can remark that both $H(x;\lambda)$ and $h(x;\lambda)$ can be expressed as infinite linear combination of exponentiated F cumulative distribution and density functions. Structural properties of these functions are now well-known for various distributions. See, for instance, [13] and [15].

Let us now focus our attention on crucial parameters for the EO family of distributions. For the sake of brevity, in the next, we set $b_{k,\ell,m}=\binom{k}{\ell}\binom{-\ell}{m}(-1)^m\lambda^{k-\ell}a_k(k+m)$ and we suppose that all the introduced quantities exist. Let X be a random variable having the cdf $H(x;\lambda)$. For any $r \geq 0$, the r-th moment of X is given by

$$\begin{aligned} E(X^r) &= \int_{-\infty}^{+\infty} x^r h(x;\lambda)dx = \int_{-\infty}^{+\infty} x^r g\left(\frac{F(x)}{\overline{F}(x)} + \lambda F(x)\right) f(x) \left[\frac{1}{[\overline{F}(x)]^2} + \lambda\right] dx \\ &= \int_0^1 [F^{-1}(x)]^r g\left(\frac{x}{1-x} + \lambda x\right)\left[\frac{1}{(1-x)^2} + \lambda\right] dx. \end{aligned}$$

The change of variable $x = F^{-1}(y)$ has been operated for the last line. On the other hand, a series expression comes from (5):

$$E(X^r) = \sum_{\substack{k,m=0 \\ k+m>0}}^{+\infty} \sum_{\ell=0}^{k} b_{k,\ell,m} \int_{-\infty}^{+\infty} x^r [F(x)]^{k+m-1} f(x)dx = \sum_{\substack{k,m=0 \\ k+m>0}}^{+\infty} \sum_{\ell=0}^{k} b_{k,\ell,m} \int_0^1 [F^{-1}(x)]^r x^{k+m-1} dx.$$

The moment generating function of X is given by

$$\begin{aligned} M(t) &= E(\mathrm{e}^{tX}) = \int_{-\infty}^{+\infty} \mathrm{e}^{tx} h(x;\lambda)dx = \int_{-\infty}^{+\infty} \mathrm{e}^{tx} g\left(\frac{F(x)}{\overline{F}(x)} + \lambda F(x)\right) f(x) \left[\frac{1}{[\overline{F}(x)]^2} + \lambda\right] dx \\ &= \int_0^1 \mathrm{e}^{tF^{-1}(x)} g\left(\frac{x}{1-x} + \lambda x\right)\left[\frac{1}{(1-x)^2} + \lambda\right] dx. \end{aligned}$$

On the other hand, a series expansion can be deduced from (5):

$$M(t) = \sum_{\substack{k,m=0 \\ k+m>0}}^{+\infty} \sum_{\ell=0}^{k} b_{k,\ell,m} \int_{-\infty}^{+\infty} \mathrm{e}^{tx} [F(x)]^{k+m-1} f(x)dx = \sum_{\substack{k,m=0 \\ k+m>0}}^{+\infty} \sum_{\ell=0}^{k} b_{k,\ell,m} \int_0^1 \mathrm{e}^{tF^{-1}(x)} x^{k+m-1} dx.$$

Another important integral term is $\int_{-\infty}^{t} x^r h(x;\lambda)dx$, $r > 0$, $t \in \mathbb{R}$. For instance, it plays a crucial role in the determination of the r-th incomplete moment of X: $E(X^r \mid X < t)$, the mean deviation of X about the mean $\mu = E(X)$: $\delta_1 = E(|X - E(X)|)$ or about the median: $\delta_2 = E(|X - M|)$, where M denotes the median. We have

$$\begin{aligned} \int_{-\infty}^{t} x^r h(x;\lambda)dx &= \int_{-\infty}^{t} x^r g\left(\frac{F(x)}{\overline{F}(x)} + \lambda F(x)\right) f(x) \left[\frac{1}{[\overline{F}(x)]^2} + \lambda\right] dx \\ &= \int_0^{F(t)} [F^{-1}(x)]^r g\left(\frac{x}{1-x} + \lambda x\right)\left[\frac{1}{(1-x)^2} + \lambda\right] dx. \end{aligned}$$

Using (5), an expansion is given by

$$\int_{-\infty}^{t} x^r h(x;\lambda)dx = \sum_{\substack{k,m=0 \\ k+m>0}}^{+\infty} \sum_{\ell=0}^{k} b_{k,\ell,m} \int_{-\infty}^{t} x^r [F(x)]^{k+m-1} f(x)dx = \sum_{\substack{k,m=0 \\ k+m>0}}^{+\infty} \sum_{\ell=0}^{k} b_{k,\ell,m} \int_0^{F(t)} [F^{-1}(x)]^r x^{k+m-1} dx.$$

Let us now discuss some properties of the order statistics for the EO family of distributions.

Let $X_1, \ldots, X_n$ be a random sample of length n from the EO family of distributions. Then, the pdf $h_{i:n}(x;\lambda)$ of the i-th order statistic, say $X_{i:n}$, is given by, with $K = n!/[(i-1)!(n-i)!]$,

$$\begin{aligned} &h_{i:n}(x;\lambda) \\ &= Kh(x;\lambda)[H(x;\lambda)]^{i-1}[1 - H(x;\lambda)]^{n-i} = Kh(x;\lambda)\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}[H(x;\lambda)]^{j+i-1} \\ &= Kg\left(\frac{F(x)}{\overline{F}(x)} + \lambda F(x)\right) f(x)\left[\frac{1}{[\overline{F}(x)]^2} + \lambda\right]\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}\left[G\left(\frac{F(x)}{\overline{F}(x)} + \lambda F(x)\right)\right]^{j+i-1}. \end{aligned}$$

For any $r \geq 0$, the r-th moment of $X_{i:n}$ is given by

$$
\begin{aligned}
E(X_{i:n}^r) &= \int_{-\infty}^{+\infty} x^r h_{i:n}(x;\lambda)dx \\
&= K\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}\int_{-\infty}^{+\infty} x^r g\left(\frac{F(x)}{\bar{F}(x)}+\lambda F(x)\right)f(x)\left[\frac{1}{[\bar{F}(x)]^2}+\lambda\right]\left[G\left(\frac{F(x)}{\bar{F}(x)}+\lambda F(x)\right)\right]^{j+i-1}dx \\
&= K\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}\int_0^1 [F^{-1}(x)]^r g\left(\frac{x}{1-x}+\lambda x\right)\left[\frac{1}{(1-x)^2}+\lambda\right]\left[G\left(\frac{x}{1-x}+\lambda x\right)\right]^{j+i-1}dx.
\end{aligned}
$$

In many cases, for given $F(x)$ and $G(x)$, analytic formula of all the mathematical terms above can be easily handled by using symbolic computation software such as Mathematica or Matlab.

3. The Extended Odd Loglogistic Modified Weibull Distribution

In this section we study a special EO distribution called an extended odd loglogistic modified Weibull (EOLLMW) distribution based on well-known distributions characterized by $F(x)$ and $G(x)$.

3.1. Behavior shapes of the probability density and hazard rate functions

Let us consider the cdf $G(x)$ associated to the standard loglogistic distribution: $G(x) = 1-(1+x^\alpha)^{-1}$, x, $\alpha > 0$, and the cdf $F(x)$ associated to the modified Weibull (MW) distribution [16]: $F(x) = 1-e^{-\theta x-\beta x^k}$, x, θ, β, $k > 0$, then the cdf of EOLLMW distribution depending on the four parameters $(\beta, k, \alpha, \lambda)$ is given by

$$
H(x;\beta,k,\alpha,\lambda) = 1-\left[1+\left\{e^{\beta x+\beta x^k}-1+\lambda\left(1-e^{-\beta x-\beta x^k}\right)\right\}^\alpha\right]^{-1} \quad x>0. \tag{6}
$$

The corresponding pdf is given by

$$
\begin{aligned}
h(x;\beta,k,\alpha,\lambda) = {}& \alpha\beta\left(1+kx^{k-1}\right)e^{-\beta x-\beta x^k}\left(\lambda+e^{2\beta x+2\beta x^k}\right)\left\{e^{\beta x+\beta x^k}-1+\lambda\left(1-e^{-\beta x-\beta x^k}\right)\right\}^{\alpha-1} \\
&\times\left[1+\left\{e^{\beta x+\beta x^k}-1+\lambda\left(1-e^{-\beta x-\beta x^k}\right)\right\}^\alpha\right]^{-2}.
\end{aligned} \tag{7}
$$

It is worth mentioning here that we replaced the scale parameter θ in the MW distribution with β so that we have only two scale parameters in new model.

Figures 1-2 illustrate some possible shapes of the pdfs and hrfs of the EOLLMW model for the selected parameter values. Figure 1 shows flexibility of the pdf shapes where skewness, kurtosis, heavy tails and modality can be observed. Figure 2 represents increasing-decreasing-increasing, unimodal and bathtub shaped hrfs.

3.2. The EOLLMW distribution case

Due to the complexity of the cdf $H(x;\beta,k,\alpha,\lambda)$ of the EOLLMW distribution given by (6), we have not an analytic expression for $F^{-1}(x)$ with $F(x) = 1-e^{-\beta(x+x^k)}$ for arbitrary values for k (the simple cases are $k=1$ with $F^{-1}(x) = -(1/(2\beta))\log(1-x)$ and $k=2$ where $F^{-1}(x) = \sqrt{-(1/\beta)\log(1-x)+1/4}-1/2$) and the series expansion of $G(x)$ is not of the general form : $G(x) = \sum_{\ell=0}^{+\infty} a_\ell x^\ell$ uniformly for $x>0$. For this reason, we now investigate some of mathematical properties of the EOLLMW distribution separately, using a specific methodology. Here, we suppose that $\lambda \in (-1,1)$.

Introducing the indicator function $\mathbf{1}_A$ equal to 1 if $x \in A$ and 0 otherwise, it follows from the geometric formula that

$$
G(x) = \frac{x^\alpha}{1+x^\alpha} = \sum_{\ell=0}^{+\infty}(-1)^\ell x^{\alpha(\ell+1)}\mathbf{1}_{\{0<x<1\}} + \frac{1}{2}\mathbf{1}_{\{x=1\}} + \sum_{\ell=0}^{+\infty}(-1)^\ell x^{-\alpha\ell}\mathbf{1}_{\{x>1\}}.
$$

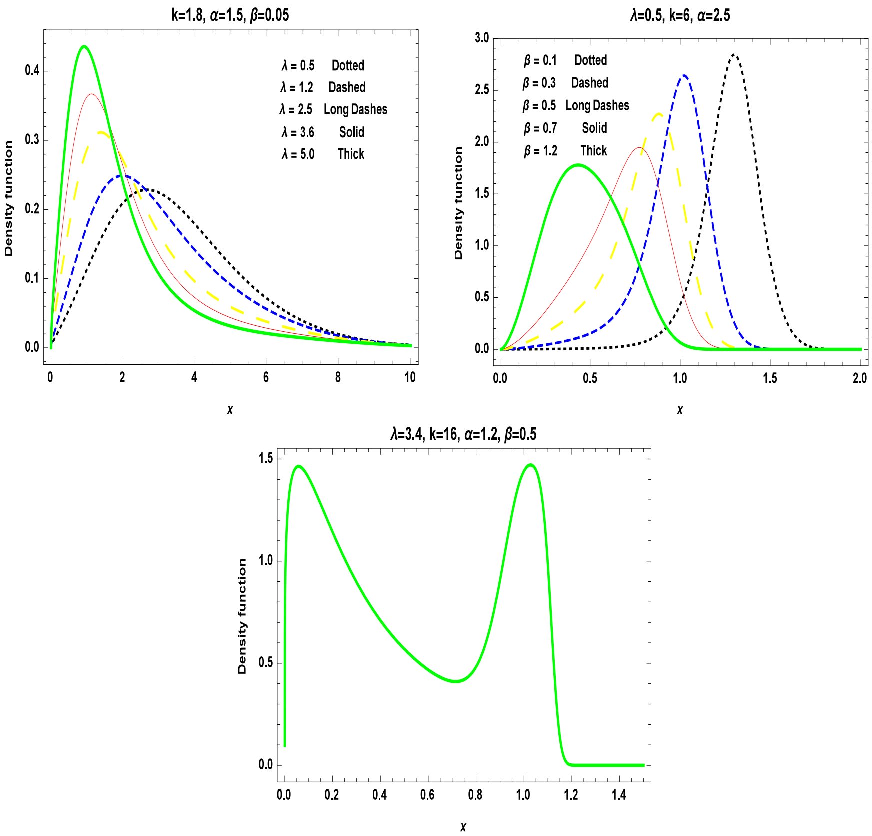

Figure 1: Plots of the EOLLMW pdf.

Let x_* be the unique positive real number such that $F(x_*)/\overline{F}(x_*) + \lambda F(x_*) = 1$. Then, using the fact that $F(x)/\overline{F}(x)+\lambda F(x)$ is increasing according to x and the equality $F(x)/\overline{F}(x)+\lambda F(x) = (1/\overline{F}(x))(1-\overline{F}(x))(1+\lambda\overline{F}(x))$, we have

$$\begin{aligned} H(x;\beta,k,\alpha,\lambda) &= \sum_{\ell=0}^{+\infty}(-1)^{\ell}\left(\frac{1}{\overline{F}(x)}\right)^{\alpha(\ell+1)}\left(1-\overline{F}(x)\right)^{\alpha(\ell+1)}\left(1+\lambda\overline{F}(x)\right)^{\alpha(\ell+1)}\mathbf{1}_{\{0<x<x_*\}}+\frac{1}{2}\mathbf{1}_{\{x=x_*\}} \\ &+ \sum_{\ell=0}^{+\infty}(-1)^{\ell}\left(\frac{1}{\overline{F}(x)}\right)^{-\alpha\ell}\left(1-\overline{F}(x)\right)^{-\alpha\ell}\left(1+\lambda\overline{F}(x)\right)^{-\alpha\ell}\mathbf{1}_{\{x>x_*\}}. \end{aligned}$$

Using the general binomial formula and $\overline{F}(x) = \mathrm{e}^{-\beta(x+x^k)}$, we have

$$H(x;\beta,k,\alpha,\lambda) = \sum_{i,j,\ell=0}^{+\infty} a_{i,j,\ell}\mathrm{e}^{-\beta(i+j-\alpha(\ell+1))(x+x^k)}\mathbf{1}_{\{0<x<x_*\}}+\frac{1}{2}\mathbf{1}_{\{x=x_*\}}+\sum_{i,j,\ell=0}^{+\infty} b_{i,j,\ell}\mathrm{e}^{-\beta(i+j+\alpha\ell)(x+x^k)}\mathbf{1}_{\{x>x_*\}},$$

with $a_{i,j,\ell} = (-1)^{\ell+i}\lambda^j\binom{\alpha(\ell+1)}{i}\binom{\alpha(\ell+1)}{j}$ and $b_{i,j,\ell} = (-1)^{\ell+i}\lambda^j\binom{-\alpha\ell}{i}\binom{-\alpha\ell}{j}$. Therefore, almost everywhere, by derivation, we have an expansion for $h(x;\beta,k,\alpha,\lambda)$ given by (7):

$$h(x;\beta,k,\alpha,\lambda) = \sum_{i,j,\ell=0}^{+\infty} a^*_{i,j,\ell}(1+k\,x^{k-1})\mathrm{e}^{-\beta(i+j-\alpha(\ell+1))(x+x^k)}\mathbf{1}_{\{0<x<x_*\}}+\sum_{\substack{i,j,\ell=0\\ i+j+\ell>0}}^{+\infty} b^*_{i,j,\ell}(1+k\,x^{k-1})\mathrm{e}^{-\beta(i+j+\alpha\ell)(x+x^k)}\mathbf{1}_{\{x>x_*\}},$$

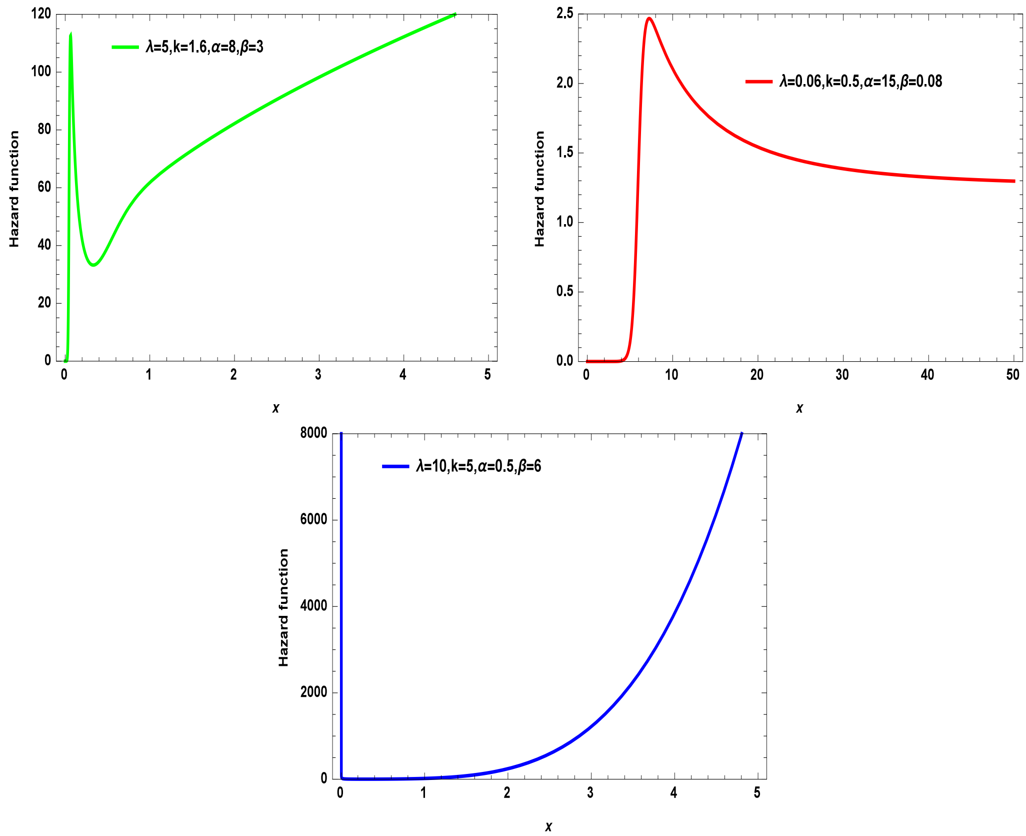

Figure 2: Plots of the EOLLMW hrf.

with $a^*_{i,j,\ell} = -\beta(i + j - \alpha(\ell + 1))a_{i,j,\ell}$ and $b^*_{i,j,\ell} = -\beta(i + j + \alpha\ell)b_{i,j,\ell}$. Let X be a random variable having the cdf $H(x; \lambda)$ of the EOLLMW distribution. For any $r \geq 0$, the r-th moment of X is given by

$$E(X^r) = \sum_{i,j,\ell=0}^{+\infty} a^*_{i,j,\ell} \int_0^{x_*} x^r(1 + k\,x^{k-1})e^{-\beta(i+j-\alpha(\ell+1))(x+x^k)}dx + \sum_{\substack{i,j,\ell=0\\ i+j+\ell>0}}^{+\infty} b^*_{i,j,\ell} \int_{x_*}^{+\infty} x^r(1 + k\,x^{k-1})e^{-\beta(i+j+\alpha\ell)(x+x^k)}dx.$$

The moment generating function of X is given by

$$M(t) = \sum_{i,j,\ell=0}^{+\infty} a^*_{i,j,\ell} \int_0^{x_*} (1 + k\,x^{k-1})e^{tx-\beta(i+j-\alpha(\ell+1))(x+x^k)}dx + \sum_{\substack{i,j,\ell=0\\ i+j+\ell>0}}^{+\infty} b^*_{i,j,\ell} \int_{x_*}^{+\infty} (1 + k\,x^{k-1})e^{tx-\beta(i+j+\alpha\ell)(x+x^k)}dx.$$

Some properties of the order statistics for the EOLLMW distribution can be set in a similar manner to the general case.

3.3. Maximum likelihood estimation

Among the estimation methods proposed in the literature, the maximum likelihood method is the most commonly used. The aim of this section is to investigate the maximum likelihood estimators (MLEs) of the parameters of the EOLLMW distribution. Let $\xi = (k, \alpha, \beta)$ and $\Theta = (\lambda, \xi)^T$ be the vector of all the unknown parameters. Let $x_1, \ldots, x_n$ be n observed values from EOLLMW distribution with vector parameters Θ. The log-likelihood function for Θ is given by

$$\ell_n(\Theta) = n\left(\log[\alpha] + \log[\beta]\right) - 2\sum_{i=1}^{n} \log\left[1 + \left\{e^{\beta(x_i+x_i^k)} - 1 + \lambda\left(1 - e^{-\beta(x_i+x_i^k)}\right)\right\}^{\alpha}\right] + \sum_{i=1}^{n} \log\left[\lambda + e^{2\beta(x_i+x_i^k)}\right]$$
$$+ (\alpha - 1)\sum_{i=1}^{n} \log\left[e^{\beta(x_i+x_i^k)} - 1 + \lambda\left(1 - e^{-\beta(x_i+x_i^k)}\right)\right] + \sum_{i=1}^{n} \log\left[1 + kx_i^{k-1}\right] - \beta\sum_{i=1}^{n}\left(x_i + x_i^k\right).$$

Then, we have

$$\frac{\partial}{\partial\lambda}\ell_n(\Theta) = \sum_{i=1}^{n} \frac{1}{e^{2\beta(x_i+x_i^k)} + \lambda} - 2\sum_{i=1}^{n} \frac{\left(1 - e^{-\beta(x_i+x_i^k)}\right)\alpha\left(-1 + e^{\beta(x_i+x_i^k)} + \left(1 - e^{-\beta(x_i+x_i^k)}\right)\lambda\right)^{-1+\alpha}}{1 + \left(-1 + e^{\beta(x_i+x_i^k)} + \left(1 - e^{-\beta(x_i+x_i^k)}\right)\lambda\right)^{\alpha}}$$
$$+ (\alpha - 1)\sum_{i=1}^{n} \frac{1 - e^{-\beta(x_i+x_i^k)}}{-1 + e^{\beta(x_i+x_i^k)} + \left(1 - e^{-\beta(x_i+x_i^k)}\right)\lambda}$$

$$\frac{\partial}{\partial k}\ell_n(\Theta) = -\sum_{i=1}^{n} \beta\log[x_i]\, x_i^k + \sum_{i=1}^{n} \frac{2e^{2\beta(x_i+x_i^k)}\beta\log[x_i]\, x_i^k}{e^{2\beta(x_i+x_i^k)} + \lambda} + \sum_{i=1}^{n} \frac{x_i^{-1+k} + k\log[x_i]\, x_i^{-1+k}}{1 + kx_i^{-1+k}}$$
$$+ (\alpha - 1)\sum_{i=1}^{n} \frac{e^{\beta(x_i+x_i^k)}\beta\log[x_i]\, x_i^k + e^{-\beta(x_i+x_i^k)}\beta\lambda\log[x_i]\, x_i^k}{-1 + e^{\beta(x_i+x_i^k)} + \left(1 - e^{-\beta(x_i+x_i^k)}\right)\lambda}$$
$$- 2\sum_{i=1}^{n} \frac{\alpha\left(-1 + e^{\beta(x_i+x_i^k)} + \left(1 - e^{-\beta(x_i+x_i^k)}\right)\lambda\right)^{-1+\alpha}\left(e^{\beta(x_i+x_i^k)}\beta\log[x_i]\, x_i^k + e^{-\beta(x_i+x_i^k)}\beta\lambda\log[x_i]\, x_i^k\right)}{1 + \left(-1 + e^{\beta(x_i+x_i^k)} + \left(1 - e^{-\beta(x_i+x_i^k)}\right)\lambda\right)^{\alpha}}$$

$$\frac{\partial}{\partial\alpha}\ell_n(\Theta) = \frac{n}{\alpha} + \sum_{i=1}^{n} \log\left[-1 + e^{\beta(x_i+x_i^k)} + \left(1 - e^{-\beta(x_i+x_i^k)}\right)\lambda\right]$$
$$- 2\sum_{i=1}^{n} \frac{\left(-1 + e^{\beta(x_i+x_i^k)} + \left(1 - e^{-\beta(x_i+x_i^k)}\right)\lambda\right)^{\alpha}\log\left[-1 + e^{\beta(x_i+x_i^k)} + \left(1 - e^{-\beta(x_i+x_i^k)}\right)\lambda\right]}{1 + \left(-1 + e^{\beta(x_i+x_i^k)} + \left(1 - e^{-\beta(x_i+x_i^k)}\right)\lambda\right)^{\alpha}}$$

$$\frac{\partial}{\partial\beta}\ell_n(\Theta) = \frac{n}{\beta} - \sum_{i=1}^{n}\left(x_i + x_i^k\right) + \sum_{i=1}^{n} \frac{2e^{2\beta(x_i+x_i^k)}\left(x_i + x_i^k\right)}{e^{2\beta(x_i+x_i^k)} + \lambda} + (\alpha - 1)\sum_{i=1}^{n} \frac{e^{-\beta(x_i+x_i^k)}\lambda\left(x_i + x_i^k\right) + e^{\beta(x_i+x_i^k)}\left(x_i + x_i^k\right)}{-1 + e^{\beta(x_i+x_i^k)} + \left(1 - e^{-\beta(x_i+x_i^k)}\right)\lambda}$$
$$- 2\sum_{i=1}^{n} \frac{\alpha\left(-1 + e^{\beta(x_i+x_i^k)} + \left(1 - e^{-\beta(x_i+x_i^k)}\right)\lambda\right)^{-1+\alpha}\left(e^{-\beta(x_i+x_i^k)}\lambda\left(x_i + x_i^k\right) + e^{\beta(x_i+x_i^k)}\left(x_i + x_i^k\right)\right)}{1 + \left(-1 + e^{\beta(x_i+x_i^k)} + \left(1 - e^{-\beta(x_i+x_i^k)}\right)\lambda\right)^{\alpha}}.$$

The MLEs can be obtained numerically by solving the non-linear equations simultaneously: $\partial\ell_n(\Theta)/\partial\lambda = 0$ and $\partial\ell_n(\Theta)/\partial\xi = 0$. Iterative methods can be used such as the Newton-Raphson type algorithms.

With the objective of showing the likelihood equations have a unique solution in the parameters, we plot the profile log-likelihood functions of the parameters of EOLLMW distribution for the two data sets considered in application section of this paper. Figures 3-4 confirm the uniqueness in the support of the parameters of the mentioned submodel.

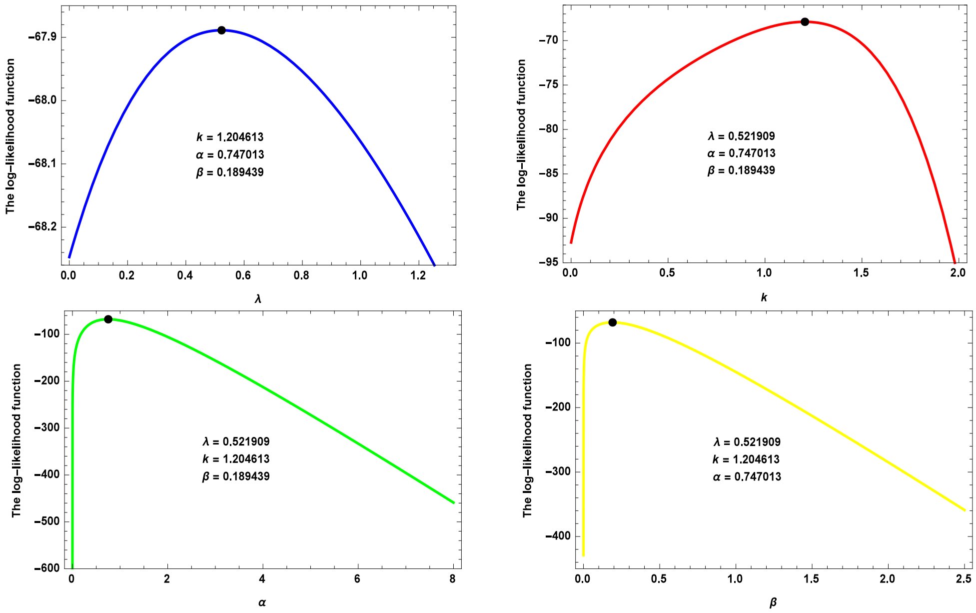

Figure 3: Generators data: The log-likelihood as a function of parameters for EOLLMW.

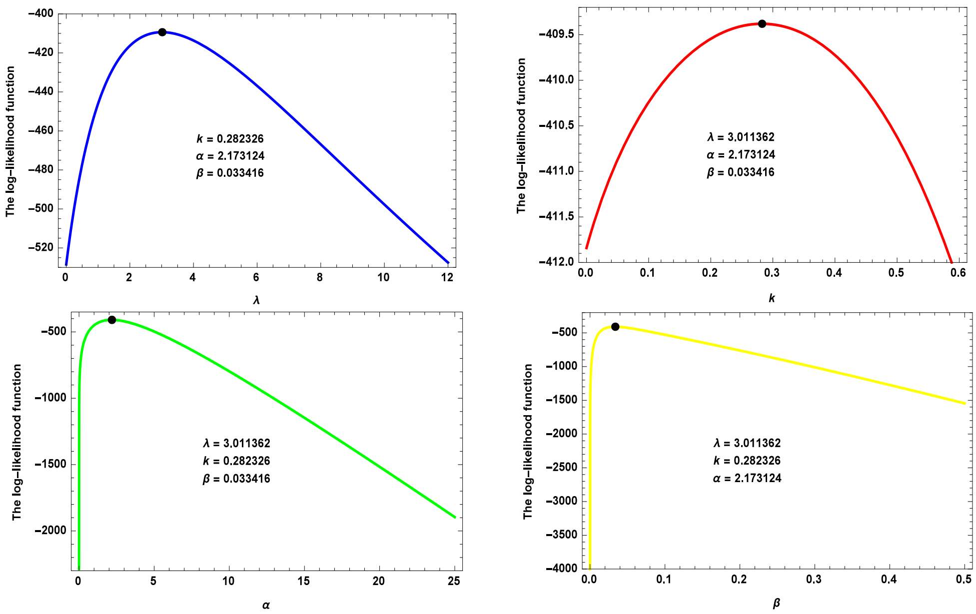

Figure 4: Cancer data: The log-likelihood as a function of parameters for EOLLMW.

3.4. *Practice to real-life data*

In this subsection, we evaluate the performance of the EO family of distributions by fitting a sub-model of this family, namely EOLLMW to two practical data sets. We compare the fit of the model with generalized modified Weibull (GMW) introduced by [10] and odd generalized exponential Weibull (OGEW) introduced by [18]. We estimate the model parameters by using the maximum likelihood method. We compare the goodness-of-fit of the models using Anderson-Darling (A^*) and Cramér–von Mises (W^*) statistics, which are described in detail by [11]. In general, the smaller the values of these statistics, the better fit to the data. The pdfs of the compared models are, respectively, given by:

$$GMW : f_{GMW}(x; a, \alpha, \gamma, \lambda) = a\,\alpha\, x^{\gamma-1}(\gamma + \lambda x)\, e^{\lambda x - \alpha x^{\gamma} e^{\lambda x}} \left(1 - e^{-\alpha x^{\gamma} e^{\lambda x}}\right)^{a-1}, \quad x, a, \alpha, \gamma, \lambda > 0,$$

$$OGEW : f_{OGEW}(x; \theta, \lambda, \alpha, \beta) = \lambda\,\theta\,\alpha\,\beta\, x^{\beta-1}\, e^{\theta x^{\beta}}\, e^{-\lambda\left(e^{\theta x^{\beta}}-1\right)} \left\{1 - e^{-\lambda\left(e^{\theta x^{\beta}}-1\right)}\right\}^{\alpha-1}, \ x, \theta, \lambda, \alpha, \beta > 0.$$

3.5. Data sets description

In this subsection, we present the details of the used data sets.

3.5.1. Data set 1: Generators data

The first data set as reported in [12] represents the time to first failure of 500MW generators.

3.5.2. Data set 2: Cancer patients data

The second uncensored data set was previously studied by Lee and Wang [14] and represents the remission times (in months) of a random sample of 128 bladder cancer patients. Bladder cancer is a disease in which certain cells in the bladder become abnormal and multiply without control or order in the bladder. The bladder is a hollow, muscular organ in the lower abdomen that stores urine until it is ready to be excreted from the body. The most common type of bladder cancer begins in cells lining the inside of the bladder and is called transitional cell carcinoma.

Graphical measure for the data analysis The total time test (TTT) plot due to Aarset [19] is an important graphical method to verify whether the data can be applied to a specific distribution or not. According to Aarset [19], the empirical version of the TTT plot is given by $G(r/n) = \left[\left(\sum_{i=1}^{r} Y_{i:n}\right) - (n-r)\,Y_{r:n}\right] / \left(\sum_{i=1}^{r} Y_{i:n}\right)$, where $r = 1, 2, \ldots, n$ and $Y_{i:n}$ represents the order statistics of the sample. Aarset [19] showed that the hrf is constant if the TTT plot is graphically presented as a straight diagonal, the hrf is increasing (or decreasing) if the TTT plot is concave (or convex). The hrf is U-shaped if the TTT plot is convex and then concave, if not, the hrf is unimodal. The TTT plot for two data sets in Figure 5 indicates a bathtub shaped failure rate for the generators data and unimodal shaped failure rate function for the cancer data set. Based on the features of the EOLLMW distribution (Section 3), we use it for fitting and modeling the mentioned data sets with comparing to some recent distributions.

Tables 2-3 show the estimates and standard error (in parentheses) from the MLE for the two data sets using the command NMAXIMIZE of the software MATHEMATICA. It can be observed that the EOLLMW model performs better than the other widely used three parameter models in literature based on the statistics A^* and W^*. Further, Figure 6 presents the estimated pdfs of the EOLLMW distribution on histogram of the two data sets and this confirms the fitting of the EOLLMW distribution for such data. Further, Figure 7 presents the estimated hrf shapes which are in agreement to the shapes given in Figure 5.

Table 2: Estimates of the parameters(Standard errors in parenthesis) for generators data

Distributions	Estimates				A^*	W^*
GMW(a, α, γ, λ)	249.999267	5.438424	0.074460	0.033953	0.484094	0.072220
	(1015.415822)	(4.027689)	(0.062946)	(0.020414)		
OGEW(θ, λ, α, β)	0.176081	2.302129	0.938840	0.728762	0.497267	0.068333
	(0.568325)	(5.155527)	(1.183679)	(0.777558)		
EOLLMW(λ, k, α, β)	0.521909	1.204613	0.747012	0.189439	0.458956	0.067090
	(1.046796)	(0.384762)	(0.207792)	(0.075697)		

Table 3: Estimates of the parameters(Standard errors in parenthesis) for cancer data

Distributions	Estimates				A^*	W^*
GMW(a, α, γ, λ)	2.796005	0.453691	0.654409	0.000000	0.271984	0.040500
	(1.857717)	(0.371825)	(0.248111)	(0.006280)		
OGEW(θ, λ, α, β)	4.515492	0.010403	5.707575	0.091299	0.260553	0.039128
	(10.154768)	(0.115954)	(6.606806)	(0.141472)		
EOLLMW(λ, k, α, β)	3.011362	0.282326	2.173124	0.033416	0.082265	0.012006
	(1.586482)	(0.220316)	(0.213037)	(0.011793)		

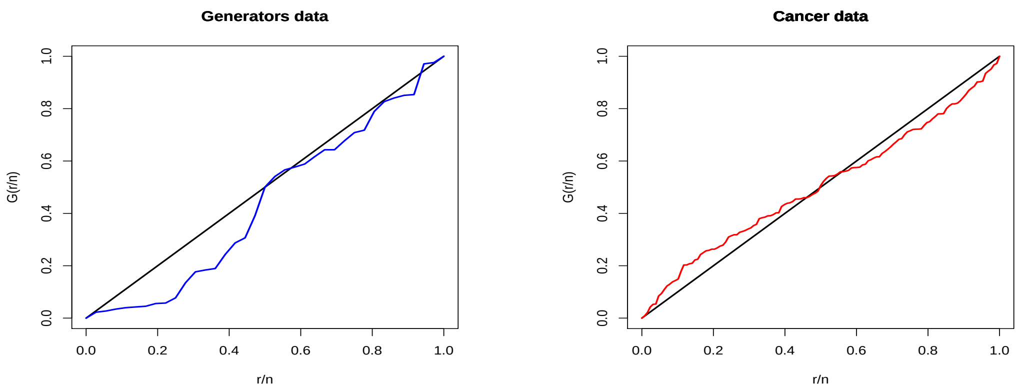

Figure 5: TTT plots of the data sets.

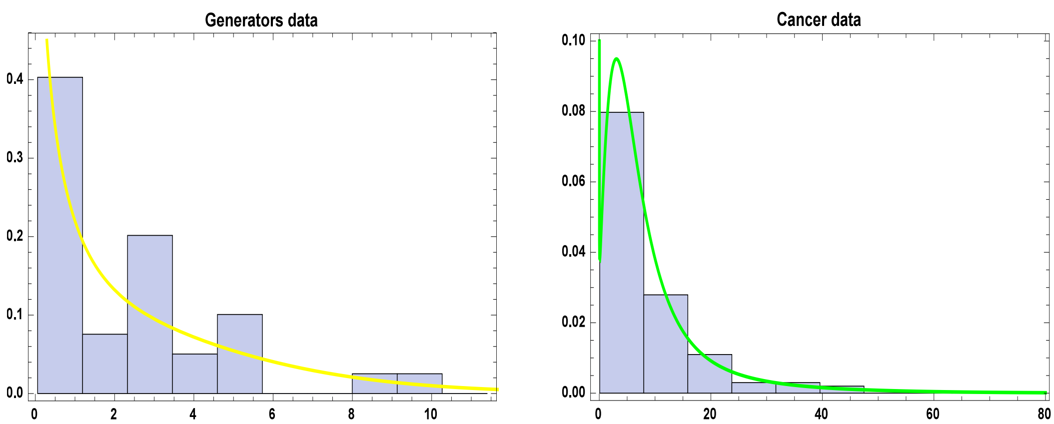

Figure 6: The estimated densities superimposed on the histogram of data sets.

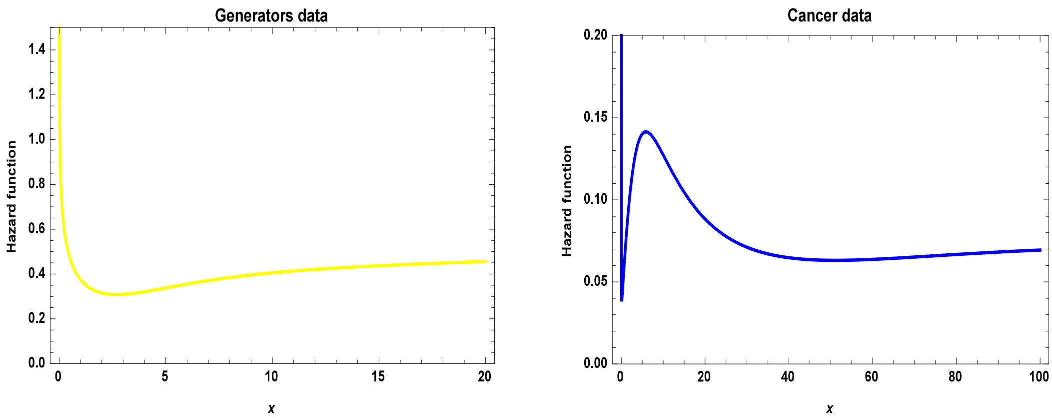

Figure 7: The estimated hrfs of the EOLLMW distribution.

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