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Approximation of the Block Numerical Range of Block Operator Matrices

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Abstract. In this paper we obtain an approximation of the block numerical range of bounded and unbounded block operator matrices by projection methods.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{H}$. The spectra of linear operators play quite a relevant role in many branches of mathematics and in numerous applications. The classical tool to enclosed the spectrum of a linear operator $\mathcal{A} \in \mathcal{B}(\mathcal{H})$ is the numerical range (see [1, 2]). In [4, 7], the notion of quadratic numerical range was introduced and it may give a better localization of the spectrum than the usual numerical range. In [5], the quadratic numerical range of a (finite) block matrix was approximated by projection methods.

This concept was generalized to block numerical range in [8]. Using the refinement of the decomposition of the space, it was shown that there exists a decreasing sequence of compact sets $\{\overline{W^k(\mathcal{A})}\}_{k=1}^{\infty}$, such that $\sigma(\mathcal{A}) \subseteq \bigcap_{k=1}^{\infty} \overline{W^k(\mathcal{A})}$ (see [8]). A total decomposition of $\mathcal{H}$ and an estimable decomposition of $\mathcal{H}$ for $\sigma(\mathcal{A})$ were introduced in [6]. By an estimable decomposition, one can approximate the spectrum of $\mathcal{A}$ by block numerical ranges of $\mathcal{A}$, i.e., there exist a decreasing sequence $\{\overline{W^k(\mathcal{A})}\}_{k=1}^{\infty}$, such that $\sigma(\mathcal{A}) = \bigcap_{k=1}^{\infty} \overline{W^k(\mathcal{A})}$. But, the existence on the estimable decomposition is, in general, hard to obtain and numerical approximations for the spectra may not be reliable, in particular, if the operator is not self-adjoint or normal. This paper arose from an attempt to gain a better understanding of the block numerical range. In contrast with the quadratic numerical range, we consider how to compute $W^n(\mathcal{A})$ by projection methods, which reduce the problem to that of computing the block numerical range of a (finite) block matrix. When $\mathcal{A}$ is unbounded, we do assume either that $\mathcal{A}$ is diagonally dominant or off-diagonally dominant.

The organization of this paper is as follows: In section 2 we are going to introduce the related definition and lemma. In section 3 we will give an approximation of the block numerical range of bounded block

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operator matrices. In section 4 we obtain the approximations of the block numerical range of unbounded block operator matrices which are diagonally (off-diagonally) dominant.

2. Preliminaries

The following notion of block numerical range for the bounded block operator matrix is due to Tretter and Wagenhofer [8].

Let $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$, where $\mathcal{H}_1, \ldots, \mathcal{H}_n$ are Hilbert spaces. With respect to this decomposition, the bounded linear operator $\mathcal{A}$ on $\mathcal{H}$ has a block operator matrix representation:

$$\mathcal{A} := \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix}, \tag{1}$$

where $A_{ij} \in \mathcal{B}(\mathcal{H}_j, \mathcal{H}_i)$, $i, j = 1, \ldots, n$.

Definition 2.1. *Let $S^n := \{(x_1, \ldots, x_n)^t \in \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n : \|x_1\| = \cdots = \|x_n\| = 1\}$. For $x = (x_1, \ldots, x_n)^t \in S^n$, define the $n \times n$ matrix $\mathcal{A}_x$ as follows:*

$$\mathcal{A}_x := \begin{bmatrix} (A_{11}x_1, x_1) & \cdots & (A_{1n}x_n, x_1) \\ \vdots & \ddots & \vdots \\ (A_{n1}x_1, x_n) & \cdots & (A_{nn}x_n, x_n) \end{bmatrix}. \tag{2}$$

Let

$$W^n(\mathcal{A}) := \{\lambda \in \mathbb{C} : \lambda \in \sigma(\mathcal{A}_x), x \in S^n\}$$

be block numerical range of the block operator matrix $\mathcal{A}$, which is defined by (1).

Remark 2.2. *For $n = 1$, the block numerical range is just the usual numerical range, for $n = 2$, it is the quadratic numerical range.*

In the following Lemma we state some properties for block numerical range of the bounded block operator matrix. (For details see [7, 8].)

Lemma 2.3. *Let $\mathcal{A}$ as in (1) be a block operator matrix on $\mathcal{H}$. Then*

(1) $\sigma_p(\mathcal{A}) \subseteq W^n(\mathcal{A})$, where $\sigma_p(\mathcal{A})$ is the point spectrum of $\mathcal{A}$.

(2) $\sigma(\mathcal{A}) \subseteq \overline{W^n(\mathcal{A})}$, where $\sigma(\mathcal{A})$ is the spectrum of $\mathcal{A}$.

(3) $W^n(\mathcal{A}) \subseteq W(\mathcal{A})$.

(4) $W^n(\mathcal{A}^*) := \{\lambda : \bar{\lambda} \in W^n(\mathcal{A})\}$.

(5) $W^{\hat{n}}(\mathcal{A}) \subseteq W^n(\mathcal{A})$, where $\widehat{\mathcal{H}}_1 \oplus \cdots \oplus \widehat{\mathcal{H}}_{\hat{n}}$ is a refinement (see [7], Definition 1.11.12) of $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$.

3. Convergence Theorems for Bounded Operator

Theorem 3.1 (For bounded operator). *Let*

$$\mathcal{A} := \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix},$$

be a bounded operator in $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$. For $i = 1, \ldots, n$, let $(U^i_{k_i})^{\infty}_{k_i=1}$ be nested families of space in $\mathcal{H}_i$, given by $U^i_{k_i} := span\{\alpha^i_1, \ldots, \alpha^i_{k_i}\}$, where $(\alpha^i_k)^{\infty}_{k=1}$ is orthnormal. Let $\mathbb{N}_+ := \{1, 2, 3, \cdots\}$, and multi-index $k := (k_1, \ldots, k_n) \in \mathbb{N}^n_+$. Consider

$$\mathbb{A}_k := \begin{bmatrix} A_{k_1\times k_1} & \cdots & A_{k_1\times k_n} \\ \vdots & \ddots & \vdots \\ A_{k_n\times k_1} & \cdots & A_{k_n\times k_n} \end{bmatrix}, \tag{3}$$

where $(A_{k_p\times k_q})_{st} = (A_{pq}\alpha^q_t, \alpha^p_s), s = 1, \ldots, k_p; t = 1, \ldots, k_q; p, q = 1, \ldots, n$. Then $W^n(\mathbb{A}_k) \subseteq W^n(\mathcal{A})$.

Proof. Let $\lambda \in W^n(\mathbb{A}_k)$, there then exists $\beta := (\beta_1, \ldots, \beta_n)^t$, where $\beta_i \in \mathbb{C}^{k_i}$, with $\|\beta_i\| = 1, i = 1, \ldots, n$, such that λ is an eignvalue of

$$(\mathbb{A}_k)_\beta := \begin{bmatrix} (A_{k_1\times k_1}\beta_1, \beta_1) & \cdots & (A_{k_1\times k_n}\beta_n, \beta_1) \\ \vdots & \ddots & \vdots \\ (A_{k_n\times k_1}\beta_1, \beta_n) & \cdots & (A_{k_n\times k_n}\beta_n, \beta_n) \end{bmatrix}. \tag{4}$$

Define isometries $\pi^i_{k_i} : U^i_{k_i} \to \mathbb{C}^{k_i}$, by $\pi^i_{k_i}(\beta^i_1\alpha^i_1 + \cdots + \beta^i_{k_i}\alpha^i_{k_i}) := (\beta^i_1, \ldots, \beta^i_{k_i})^t := \beta_i$, for $i = 1, \ldots, n$.

Choose $x = (x_1, \ldots, x_n)^t$, where $x_i \in U^i_{k_i}$, such that $\pi^i_{k_i}(x_i) = \beta_i$, $\|x_i\| = 1$, for $i = 1, \ldots, n$. By a simple calculation, it then follows that $(\mathbb{A}_k)_\beta = \mathcal{A}_x$. Hence $\lambda \in W^n(\mathcal{A})$. □

Lemma 3.2. *Let $(U^i_{k_i})^{\infty}_{k_i=1}$ and $\mathbb{A}_k$ be as in Theorem 3.1. Suppose that $\widehat{k}, k \in \mathbb{N}^n_+$ and $\widehat{k} \geq k$, in the sense that, $\widehat{k_i} \geq k_i$, for all $i = 1, \ldots, n$. Then $W^n(\mathbb{A}_k) \subseteq W^n(\mathbb{A}_{\widehat{k}})$.*

Proof. This result is an immediate consequence of the fact that $\mathbb{C}^{k_i}$ is a subspace of $\mathbb{C}^{\widehat{k_i}}$ for $\widehat{k_i} \geq k_i$, $i = 1, \ldots, n$. In detail: suppose $\widehat{k_i} \geq k_i$, for $i = 1, \ldots, n$, and also $\lambda \in W^n(\mathbb{A}_k)$. There then exists $\beta := (\beta_1, \ldots, \beta_n)^t$, where $\beta_i \in \mathbb{C}^{k_i}$, with $\|\beta_i\| = 1, i = 1, \ldots, n$, such that in the notation of (4), λ is an eigenvalue of $(\mathbb{A}_k)_\beta$. For all $i = 1, \ldots, n$, choose $\widehat{\beta_i} \in \mathbb{C}^{\widehat{k_i}}$ by setting $\widehat{\beta_i} := (\beta^i_1, \ldots, \beta^i_{k_i}, 0, \ldots, 0)^t$. By a simple calculation, it then follows that $W^n(\mathbb{A}_k)_\beta = W^n(\mathbb{A}_{\widehat{k}})_{\widehat{\beta}}$, where $\widehat{\beta} = (\widehat{\beta_1}, \ldots, \widehat{\beta_n})^t$, and hence $\lambda \in W^n(\mathbb{A}_{\widehat{k}})$. □

Remark 3.3. *In the proof of Theorem 3.1 and Lemma 3.2, the boundedness of operators is less important than one might expect. In fact, the same results also hold, if $\mathcal{A}$ is an unbounded operator in $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$, and let $(U^i_{k_i})^{\infty}_{k_i=1}$ be nested families of space in $\mathcal{D}_i := \bigcap^n_{j=1} \mathcal{D}_{ji}$, where $\mathcal{D}_{ji}$ is the domain of A_{ji}, for $i, j = 1, \ldots, n$.*

Roughly speaking, the proof of Theorem 3.1 and Lemma 3.2 also yield for unbounded operators.

Theorem 3.4. *Let $\mathcal{A}$, $\mathbb{A}_k$ and $(U^i_{k_i})^{\infty}_{k_i=1}$ be as in Theorem 3.1. Suppose that $(\alpha^i_k)^{\infty}_{k=1}$ is orthnormal basis of $\mathcal{H}_i$, for $i = 1, \ldots, n$. Then*

$$\overline{\bigcup_{k\in\mathbb{N}^n_+} W^n(\mathbb{A}_k)} = \overline{\bigcup_{m^n\in\mathbb{N}^n_+} W^n(\mathbb{A}_{m^n})} = \overline{W^n(\mathcal{A})},$$

where $m^n := (m, \ldots, m) \in \mathbb{N}^n_+$.

Proof. By Lemma 3.2, it is immediate that

$$\bigcup_{k\in\mathbb{N}^n_+} W^n(\mathbb{A}_k) \subseteq \bigcup_{m^n\in\mathbb{N}^n_+} W^n(\mathbb{A}_{m^n}),$$

where $m := \max\{k_1, \ldots, k_n\}$. To see the other inclusion, consider $m := \min\{k_1, \ldots, k_n\}$. And hence proves that

$$\overline{\bigcup_{k\in\mathbb{N}^n_+} W^n(\mathbb{A}_k)} = \overline{\bigcup_{m^n\in\mathbb{N}^n_+} W^n(\mathbb{A}_{m^n})}.$$

To complete this proof, it therefore now remains to prove that $W^n(\mathcal{A}) \subseteq \overline{\bigcup_{k\in\mathbb{N}_+^n} W^n(\mathbb{A}_k)}$.

Let $\lambda \in W^n(\mathcal{A})$. There then exists $x \in S^n$, such that λ is an eigenvalue of $\mathcal{A}_x$ as defined in (2). Since $(\alpha_k^i)_{k=1}^{\infty}$ is orthnormal basis of $\mathcal{H}_i$, $i = 1, \dots, n$, there exists a sequence $(x_k^i)_{k=1}^{\infty}$, with each $x_k^i \in span\{\alpha_1^i, \dots, \alpha_{k_i}^i\}$ for some $k_i > 0$, such that $\|x^i - x_k^i\| \to 0$, and $\|A_{ji}x^i - A_{ji}x_k^i\| \to 0$, as $k \to \infty$, where x^i denotes the i-th component of x and $j = 1, \dots, n$. Let $x_k = (x_k^1, \dots, x_k^n)^t$, by a simple calculation, we then obtain that $\|\mathcal{A}_{x_k} - \mathcal{A}_x\| \to 0$, as $k \to \infty$.

Fix x_k as above. Let $\pi_{k_i}^i : U_{k_i}^i \to \mathbb{C}^{k_i}$ be the isometries in the proof of Theorem 3.1. Define $\beta_i \in \mathbb{C}^{k_i}, i = 1, \dots, n$, by $\beta_i = \pi_{k_i}^i(x_k^i)/\|\pi_{k_i}^i(x_k^i)\|$. Consider the matrix

$$M_k := \begin{bmatrix} (A_{k_1\times k_1}\beta_1, \beta_1) & \cdots & (A_{k_1\times k_n}\beta_n, \beta_1) \\ \vdots & \ddots & \vdots \\ (A_{k_n\times k_1}\beta_1, \beta_n) & \cdots & (A_{k_n\times k_n}\beta_n, \beta_n) \end{bmatrix}.$$

A simple calculation yields that $M_k = \mathcal{A}_{x_k}$. Since $\|\mathcal{A}_{x_k} - \mathcal{A}_x\| \to 0$ as $k \to \infty$, this entails that $\|M_k - \mathcal{A}_x\| \to 0$, as $k \to \infty$. Obviously, the eigenvalues of M_k are elements of $W^n(\mathbb{A}_k)$, where $k := (k_1, \dots, k_n) \in \mathbb{N}_+^n$. There hence exists $\lambda_k \in W^n(\mathbb{A}_k)$ such that $\lambda_k \to \lambda$, as $k \to \infty$. It then follows from Lemma 3.2 that $\lambda \in \overline{\bigcup_{k\in\mathbb{N}_+^n} W^n(\mathbb{A}_k)}$. $\square$

4. Convergence Theorems for Unbounded Operator

For a unbounded linear operator $\mathcal{A}$ in $\mathcal{H}$ which admits a so-called block operator matrix representation:

$$\mathcal{A} := \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix}, \tag{5}$$

where $A_{ij} : \mathcal{H}_j \to \mathcal{H}_i$, is closable operators with dense domains $\mathcal{D}_{ij} \in \mathcal{H}_j$, for $i, j = 1, \dots, n$. We always suppose that $\mathcal{A}$ with its natural domain $\mathcal{D}(\mathcal{A}) := \mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_n$, where $\mathcal{D}_j := \bigcap_{i=1}^n \mathcal{D}_{ij} \in \mathcal{H}_j$, is also densely defined for $i, j = 1, \dots, n$.

Remark 4.1. *It should be noted that, unlike bounded operators, unbounded linear operators, in general, do not admit a matrix representation (5), with respect to a given decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$.*

Definition 4.2. *The block operator matrix $\mathcal{A}$ in (5) is called*
(1) diagonally dominant if A_{ij} is A_{jj}-bounded (see [7], Definition 2.1.2),
(2) off-diagonally dominant if A_{ij} is $A_{n+1-j,j}$-bounded, where $i, j = 1, \dots, n$.

The definition of the block numerical range for bounded linear operators (see [7], Definition 1.11.12) generalizes as follows to unbounded block operator matrices $\mathcal{A}$ of the form (1) with dense domain $\mathcal{D}(\mathcal{A})$.

Definition 4.3. *Let $S^n := \{(x_1, \dots, x_n)^t \in \mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_n : \|x_1\| = \cdots = \|x_n\| = 1\}$. For $x = (x_1, \dots, x_n)^t \in S^n$, define the $n \times n$ matrix $\mathcal{A}_x$ as follows:*

$$\mathcal{A}_x := \begin{bmatrix} (A_{11}x_1, x_1) & \cdots & (A_{1n}x_n, x_1) \\ \vdots & \ddots & \vdots \\ (A_{n1}x_1, x_n) & \cdots & (A_{nn}x_n, x_n) \end{bmatrix}.$$

Let

$$W^n(\mathcal{A}) := \{\lambda \in \mathbb{C} : \lambda \in \sigma(\mathcal{A}_x), x \in S^n\}$$

be block numerical range of the unbounded block operator matrix $\mathcal{A}$, which is defined by (5).

Remark 4.4. *For $n = 1$, the block numerical range is just the usual numerical range, for $n = 2$, it is the quadratic numerical range, as the bounded case.*

The following result shows some important properties of the block numerical range of the unbounded block operator matrix.

Proposition 4.5. *For an unbounded block operator matrix $\mathcal{A}$, we have*
(1) $\sigma_p(\mathcal{A}) \subseteq W^n(\mathcal{A})$, where $\sigma_p(\mathcal{A})$ is the point spectrum of $\mathcal{A}$.
(2) $W^n(\mathcal{A}) \subseteq W(\mathcal{A})$.
(3) $W^{\hat{n}}(\mathcal{A}) \subseteq W^n(\mathcal{A})$, where $\widehat{\mathcal{D}}_1 \oplus \cdots \oplus \widehat{\mathcal{D}}_{\hat{n}}$ is a refinement (see [7], Definition 1.11.12) of $\mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_n$.

Proof. The proofs are completely analogous to the proofs of the bounded case (see [7]) if we take $x = (x_1, ..., x_n)^t \in S^n$. $\square$

In the following result we describe a property of convergence for unbounded operator.

Theorem 4.6 (For unbounded operator). *Let*

$$\mathcal{A} := \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix}$$

be an unbounded operator in $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$. For $i = 1, \ldots, n$, let $(U^i_{k_i})^{\infty}_{k_i=1}$ be nested families of space in $\mathcal{D}_i$, given by $U^i_{k_i} := span\{\alpha^i_1, \ldots, \alpha^i_{k_i}\}$, where $(\alpha^i_k)^{\infty}_{k=1}$ is orthnormal. And $\mathbb{A}_k$ denotes as in Theorem 3.1. Suppose that $\mathcal{A}$ is diagonally dominant, and $(U^i_{k_i})^{\infty}_{k_i=1}$ is a core (see [3],Section III.3) of $A_{ii}, i = 1, \ldots, n$. Then $\overline{\bigcup_{k\in\mathbb{N}^n_+} W^n(\mathbb{A}_k)} = \overline{\bigcup_{m^n\in\mathbb{N}^n_+} W^n(\mathbb{A}_{m^n})} = \overline{W^n(\mathcal{A})}$, where $m^n := (m, \ldots, m) \in \mathbb{N}^n_+$.

Proof. Since $(U^i_{k_i})^{\infty}_{k_i=1}$ is a core of A_{ii}, for $i = 1, \ldots, n$, there exists a sequence $(x^i_k)^{\infty}_{k=1}$, with each $x^i_k \in span\{\alpha^i_1, \ldots, \alpha^i_{k_i}\}$ for some $k_i > 0$, such that $\|x^i - x^i_k\| \to 0$, and $\|A_{ii}x^i - A_{ii}x^i_k\| \to 0$. Because A_{ji} is A_{ii}-bounded for $j = 1, \ldots, n$, we have $\|A_{ji}x^i - A_{ji}x^i_k\| \to 0$, as $k \to \infty$. The rest of proof is completely analogous to the proof of Theorem 3.4. $\square$

Remark 4.7. *The same result holds if $\mathcal{A}$ is off-diagonally dominant with $(U^i_{k_i})^{\infty}_{k_i=1}$ being a core of $A_{n+1-i,i}$, for $i = 1, \ldots, n$.*

Remark 4.8. *Note that, the result of Theorem 2.3 in [5], is the $n = 2$ case of Theorem 4.6.*

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References

[1] K. Gustafson, K. M. Rao, Numerical Range, Springer-Verlag, New York, 1997.
[2] R. Horn, C. Johnson, Topics in Matrix Analysis, Cambridge University Press, New York, 1991.
[3] T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, Berlin,Heideibeerg, New York, 1980.
[4] H. Langer, C. Tretter, Spectral decomposition of some nonselfadjoint block operator matrices, J. Oper. Theory 39 (1998) 339–359.
[5] A. Muhammad, M. Marletta, Approximation of the quadratic numerical range of block operator matrices, Integr. Equ. Oper. Theory 74 (2012) 151–162.
[6] A. Salemi, Total decomposotion and block numerical range, Banach J. Math. Anal. 5 (2011) 51–55.
[7] C. Tretter, Spectral Theory of Block Operators Matrices and Aapplications. Imperial College Press, 2008.
[8] C. Tretter, M. Wagenhofer, The block numerical range of an $n \times n$ block operator matrix, SIAM J. Matrix Anal. Appl. 22 (2003) 1003–1017.