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E-Eigenvalue Inclusion Theorems for Tensors

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Abstract. Two Z-eigenvalue inclusion theorems for tensors presented by Wang *et al.* (Discrete Cont. Dyn.-B, 2017, 22(1): 187–198) are first generalized to E-eigenvalue inclusion theorems. And then a tighter E-eigenvalue inclusion theorem for tensors is established. Based on the new set, a sharper upper bound for the Z-spectral radius of weakly symmetric nonnegative tensors is obtained. Finally, numerical examples are given to verify the theoretical results.

1. Introduction

For a positive integer n, $n \geq 2$, N denotes the set $\{1, 2, \cdots, n\}$. $\mathbb{C}$ ($\mathbb{R}$) denotes the set of all complex (real) numbers. We call $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ a real tensor of order m dimension n, denoted by $\mathcal{A} \in \mathbb{R}^{[m,n]}$, if

$$a_{i_1 i_2 \cdots i_m} \in \mathbb{R},$$

where $i_j \in N$ for $j = 1, 2, \cdots, m$. $\mathcal{A}$ is called nonnegative if $a_{i_1 i_2 \cdots i_m} \geq 0$. $\mathcal{A} = (a_{i_1 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ is called symmetric [1] if

$$a_{i_1 \cdots i_m} = a_{i_{\pi(1)} \cdots i_{\pi(m)}}, \ \forall \pi \in \Pi_m,$$

where Π_m is the permutation group of m indices. $\mathcal{A} = (a_{i_1 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ is called weakly symmetric [2] if the associated homogeneous polynomial

$$\mathcal{A}x^m = \sum_{i_1, \cdots, i_m \in N} a_{i_1 \cdots i_m} x_{i_1} \cdots x_{i_m}$$

satisfies $\nabla \mathcal{A}x^m = m\mathcal{A}x^{m-1}$, where $x = (x_1, x_2 \cdots, x_n)^T \in \mathbb{R}^n$, and $\mathcal{A}x^{m-1}$ is an n dimension vector whose ith component is

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2, \cdots, i_m \in N} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}.$$

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It is shown in [2] that a symmetric tensor is necessarily weakly symmetric, but the converse is not true in general.

Given a tensor $\mathcal{A} = (a_{i_1 \cdots i_m}) \in \mathbb{R}^{[m,n]}$, if there are $\lambda \in \mathbb{C}$ and $x = (x_1, x_2 \cdots, x_n)^T \in \mathbb{C}^n \backslash \{0\}$ such that

$$\mathcal{A}x^{m-1} = \lambda x \text{ and } x^T x = 1,$$

then λ is called an E-eigenvalue of $\mathcal{A}$ and x an E-eigenvector of $\mathcal{A}$ associated with λ. Particularly, if λ and x are all real, then λ is called a Z-eigenvalue of $\mathcal{A}$ and x a Z-eigenvector of $\mathcal{A}$ associated with λ; for details, see [1, 3]. Denote by $\sigma(\mathcal{A})$ (respectively, $E(\mathcal{A})$) the set of all Z-eigenvalues (respectively, E-eigenvalues) of $\mathcal{A}$. Assume $\sigma(\mathcal{A}) \neq 0$, then the Z-spectral radius [2] of $\mathcal{A}$, denoted $\varrho(\mathcal{A})$, is defined as

$$\varrho(\mathcal{A}) := \max\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}.$$

Note here that, Chang *et al.* in [2] demonstrated by an example that the Z-spectral radius $\varrho(\mathcal{A})$ of a nonnegative tensor $\mathcal{A}$ may not be itself a positive Z-eigenvalue of $\mathcal{A}$, and proved that if $\mathcal{A}$ is a weakly symmetric nonnegative tensor, then $\varrho(\mathcal{A})$ is a Z-eigenvalue of $\mathcal{A}$; see [2], for details.

The Z-eigenvalue problem plays a fundamental role in best rank-one approximation, which has numerous applications in engineering and higher order statistics [1, 4], and neural networks [5]. Recently, much literature has focused on locating all Z-eigenvalues of tensors and bounding the Z-spectral radius of nonnegative tensors in [6–20]. In 2017, Wang *et al.* [6] generalized Geršgorin eigenvalue inclusion theorem from matrices to tensors and established the following Geršgorin-type Z-eigenvalue inclusion theorem.

Theorem 1.1. [6, Theorem 3.1] *Let* $\mathcal{A} = (a_{i_1 \cdots i_m}) \in \mathbb{R}^{[m,n]}$. *Then*

$$\sigma(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) = \bigcup_{i \in N} \mathcal{K}_i(\mathcal{A}),$$

where

$$\mathcal{K}_i(\mathcal{A}) = \{z \in \mathbb{C} : |z| \leq R_i(\mathcal{A})\} \text{ and } R_i(\mathcal{A}) = \sum_{i_2, \cdots, i_m \in N} |a_{i i_2 \cdots i_m}|.$$

Based on the set $\mathcal{K}(\mathcal{A})$, the following upper bound for $\rho(\mathcal{A})$ presented in [7] is obtained easily.

Theorem 1.2. [7, Corollary 4.5] *Let* $\mathcal{A} \in \mathbb{R}^{[m,n]}$ *be nonnegative. Then*

$$\varrho(\mathcal{A}) \leq \max_{i \in N} R_i(\mathcal{A}).$$

To get a tighter Z-eigenvalue inclusion set than $\mathcal{K}(\mathcal{A})$, Wang *et al.* [6] obtained the following Brauer-type Z-eigenvalue inclusion theorem for tensors.

Theorem 1.3. [6, Theorem 3.3] *Let* $\mathcal{A} = (a_{i_1 \cdots i_m}) \in \mathbb{R}^{[m,n]}$. *Then*

$$\sigma(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A}) = \bigcup_{i,j \in N, i \neq j} \left(\mathcal{M}_{i,j}(\mathcal{A}) \cup \mathcal{H}_{i,j}(\mathcal{A})\right),$$

where

$$\mathcal{M}_{i,j}(\mathcal{A}) = \left\{z \in \mathbb{C} : \left(|z| - (R_i(\mathcal{A}) - |a_{ij\cdots j}|)\right)(|z| - P_j^i(\mathcal{A})) \leq |a_{ij\cdots j}|(R_j(\mathcal{A}) - P_j^i(\mathcal{A}))\right\},$$

$$\mathcal{H}_{i,j}(\mathcal{A}) = \left\{z \in \mathbb{C} : |z| < R_i(\mathcal{A}) - |a_{ij\cdots j}|, |z| < P_j^i(\mathcal{A})\right\},$$

and

$$P_j^i(\mathcal{A}) = \sum_{\substack{i_2, \cdots, i_m \in N, \\ i \notin \{i_2, \cdots, i_m\}}} |a_{j i_2 \cdots i_m}|.$$

Based on the set $\mathcal{M}(\mathcal{A})$, Wang *et al.* [6] obtained a better upper bound than that in Theorem 1.2.

Theorem 1.4. [6, Theorem 4.6] *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ be a weakly symmetric nonnegative tensor. Then*

$$\varrho(\mathcal{A}) \leq \Psi(\mathcal{A}) = \max_{i,j \in N, i \neq j} \left\{ \frac{1}{2}\Big(R_i(\mathcal{A}) - a_{ij\cdots j} + P_j^i(\mathcal{A}) + \Lambda_{i,j}^{\frac{1}{2}}(\mathcal{A})\Big), R_i(\mathcal{A}) - a_{ij\cdots j}, P_j^i(\mathcal{A}) \right\},$$

where

$$\Lambda_{i,j}(\mathcal{A}) = (R_i(\mathcal{A}) - a_{ij\cdots j} - P_j^i(\mathcal{A}))^2 + 4a_{ij\cdots j}(R_j(\mathcal{A}) - P_j^i(\mathcal{A})).$$

Due to various new and important applications of E-eigenvalue problem in numerical multilinear algebra [21], image processing [22], higher order Markov chains [23], spectral hypergraph theory, the study of quantum entanglement, and so on, some properties of E-eigenvalues have been studied systematically; see [8] for details. However, characterizations of inclusion set for E-eigenvalue are still underdeveloped. This stimulates us to establish some inclusion theorems to identify the distribution of E-eigenvalues.

In the sequel, we research on the E-eigenvalue localization problems for tensors and their applications. First, Theorems 1.1 and 1.3 are extended to E-eigenvalue inclusion theorems. Second, a new E-eigenvalue inclusion set for tensors is presented and proved to be tighter than those in Theorems 1.1 and 1.3. Finally, as an application of the new set, a new upper bound for the Z-spectral radius of weakly symmetric nonnegative tensors is given and proved to be sharper than those in Theorems 1.2 and 1.4.

2. *E*-eigenvalue inclusion sets for tensors

In this section, we first generalized those sets in Theorems 1.1 and 1.3 to E-eigenvalue inclusion sets. And then we present a new E-eigenvalue inclusion set for tensors and establish the comparison among these three sets. Firstly, similar to the proof of Theorems 3.1 and 3.3 of [6], the following theorem is obtained easily.

Theorem 2.1. *Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in \mathbb{R}^{[m,n]}$. Then*

$$E(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}), \text{ and } E(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A}).$$

Next, a new E-eigenvalue inclusion theorem for tensors is presented.

Theorem 2.2. *Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in \mathbb{R}^{[m,n]}$. Then*

$$E(\mathcal{A}) \subseteq \Omega(\mathcal{A}) = \bigcup_{i,j \in N, j \neq i} \Big(\hat{\Omega}_{i,j}(\mathcal{A}) \cup \Big(\tilde{\Omega}_{i,j}(\mathcal{A}) \cap \mathcal{K}_i(\mathcal{A})\Big)\Big),$$

where

$$\hat{\Omega}_{i,j}(\mathcal{A}) = \Big\{z \in \mathbb{C} : |z| < P_i^j(\mathcal{A}) \text{ and } |z| < P_j^i(\mathcal{A})\Big\}$$

and

$$\tilde{\Omega}_{i,j}(\mathcal{A}) = \Big\{z \in \mathbb{C} : \Big(|z| - P_i^j(\mathcal{A})\Big)\Big(|z| - P_j^i(\mathcal{A})\Big) \leq \Big(R_i(\mathcal{A}) - P_i^j(\mathcal{A})\Big)\Big(R_j(\mathcal{A}) - P_j^i(\mathcal{A})\Big)\Big\}.$$

Proof. Let λ be an E-eigenvalue of $\mathcal{A}$ with corresponding E-eigenvector $x = (x_1, \cdots, x_n)^T \in \mathbb{C}^n \backslash \{0\}$, i.e.,

$$\mathcal{A}x^{m-1} = \lambda x, \text{ and } \|x\|_2 = 1. \tag{1}$$

Let $|x_t| \geq |x_s| \geq \max_{i \in N, i \neq t,s} |x_i|$. Obviously, $0 < |x_t|^{m-1} \leq |x_t|^{m-2} \leq |x_t| \leq 1$. From (1), we have

$$\lambda x_t = \sum_{\substack{i_2,\cdots,i_m \in N, \\ s \in \{i_2,\cdots,i_m\}}} a_{ti_2\cdots i_m} x_{i_2} \cdots x_{i_m} + \sum_{\substack{i_2,\cdots,i_m \in N, \\ s \notin \{i_2,\cdots,i_m\}}} a_{ti_2\cdots i_m} x_{i_2} \cdots x_{i_m}.$$

Taking modulus in the above equation and using the triangle inequality give

$$\begin{aligned}
|\lambda||x_t| &\leq \sum_{\substack{i_2,\cdots,i_m\in N,\\ s\in\{i_2,\cdots,i_m\}}} |a_{ti_2\cdots i_m}||x_{i_2}|\cdots|x_{i_m}| + \sum_{\substack{i_2,\cdots,i_m\in N,\\ s\notin\{i_2,\cdots,i_m\}}} |a_{ti_2\cdots i_m}||x_{i_2}|\cdots|x_{i_m}| \\
&\leq \sum_{\substack{i_2,\cdots,i_m\in N,\\ s\in\{i_2,\cdots,i_m\}}} |a_{ti_2\cdots i_m}||x_s||x_t|^{m-2} + \sum_{\substack{i_2,\cdots,i_m\in N,\\ s\notin\{i_2,\cdots,i_m\}}} |a_{ti_2\cdots i_m}||x_t|^{m-1} \\
&\leq \sum_{\substack{i_2,\cdots,i_m\in N,\\ s\in\{i_2,\cdots,i_m\}}} |a_{ti_2\cdots i_m}||x_s| + \sum_{\substack{i_2,\cdots,i_m\in N,\\ s\notin\{i_2,\cdots,i_m\}}} |a_{ti_2\cdots i_m}||x_t| \\
&= (R_t(\mathcal{A}) - P_t^s(\mathcal{A}))|x_s| + P_t^s(\mathcal{A})|x_t|,
\end{aligned}$$

i.e.,

$$\left(|\lambda| - P_t^s(\mathcal{A})\right)|x_t| \leq (R_t(\mathcal{A}) - P_t^s(\mathcal{A}))|x_s|. \tag{2}$$

By (2), it is not difficult to see $|\lambda| \leq R_t(\mathcal{A})$, that is, $\lambda \in \mathcal{K}_t(\mathcal{A})$. If $|x_s| = 0$, then $|\lambda| - P_t^s(\mathcal{A}) \leq 0$ as $|x_t| > 0$. When $|\lambda| - P_t^s(\mathcal{A}) = 0$, obviously, $\lambda \in \left(\tilde{\Omega}_{t,s}(\mathcal{A}) \cap \mathcal{K}_t(\mathcal{A})\right) \subseteq \Omega(\mathcal{A})$. And when $|\lambda| - P_t^s(\mathcal{A}) < 0$, if $|\lambda| \geq P_s^t(\mathcal{A})$, then we have

$$\left(|\lambda| - P_t^s(\mathcal{A})\right)\left(|\lambda| - P_s^t(\mathcal{A})\right) \leq 0 \leq (R_t(\mathcal{A}) - P_t^s(\mathcal{A}))\left(R_s(\mathcal{A}) - P_s^t(\mathcal{A})\right),$$

which implies $\lambda \in \left(\tilde{\Omega}_{t,s}(\mathcal{A}) \cap \mathcal{K}_t(\mathcal{A})\right) \subseteq \Omega(\mathcal{A})$; if $|\lambda| < P_s^t(\mathcal{A})$, then we have $\lambda \in \hat{\Omega}_{t,s}(\mathcal{A}) \subseteq \Omega(\mathcal{A})$.

Otherwise, $|x_s| > 0$. By (1), we can get

$$\begin{aligned}
|\lambda||x_s| &\leq \sum_{\substack{i_2,\cdots,i_m\in N,\\ t\in\{i_2,\cdots,i_m\}}} |a_{si_2\cdots i_m}||x_{i_2}|\cdots|x_{i_m}| + \sum_{\substack{i_2,\cdots,i_m\in N,\\ t\notin\{i_2,\cdots,i_m\}}} |a_{si_2\cdots i_m}||x_{i_2}|\cdots|x_{i_m}| \\
&\leq \sum_{\substack{i_2,\cdots,i_m\in N,\\ t\in\{i_2,\cdots,i_m\}}} |a_{si_2\cdots i_m}||x_t|^{m-1} + \sum_{\substack{i_2,\cdots,i_m\in N,\\ t\notin\{i_2,\cdots,i_m\}}} |a_{si_2\cdots i_m}||x_s|^{m-1}, \\
&\leq \sum_{\substack{i_2,\cdots,i_m\in N,\\ t\in\{i_2,\cdots,i_m\}}} |a_{si_2\cdots i_m}||x_t| + \sum_{\substack{i_2,\cdots,i_m\in N,\\ t\notin\{i_2,\cdots,i_m\}}} |a_{si_2\cdots i_m}||x_s|,
\end{aligned}$$

i.e.,

$$\left(|\lambda| - P_s^t(\mathcal{A})\right)|x_s| \leq (R_s(\mathcal{A}) - P_s^t(\mathcal{A}))|x_t|. \tag{3}$$

When $|\lambda| \geq P_t^s(\mathcal{A})$ or $|\lambda| \geq P_s^t(\mathcal{A})$ holds, multiplying (2) with (3) and noting that $|x_t||x_s| > 0$, we have

$$\left(|\lambda| - P_t^s(\mathcal{A})\right)\left(|\lambda| - P_s^t(\mathcal{A})\right) \leq \left(R_t(\mathcal{A}) - P_t^s(\mathcal{A})\right)\left(R_s(\mathcal{A}) - P_s^t(\mathcal{A})\right),$$

which implies $\lambda \in \left(\tilde{\Omega}_{t,s}(\mathcal{A}) \cap \mathcal{K}_t(\mathcal{A})\right) \subseteq \Omega(\mathcal{A})$. And when $|\lambda| < P_t^s(\mathcal{A})$ and $|\lambda| < P_s^t(\mathcal{A})$ hold, we have $\lambda \in \hat{\Omega}_{t,s}(\mathcal{A}) \subseteq \Omega(\mathcal{A})$. Hence, the conclusion $\sigma(\mathcal{A}) \subseteq \Omega(\mathcal{A})$ follows immediately from what we have proved. □

Next, a comparison theorem is given for Theorems 2.1 and 2.2.

Theorem 2.3. *Let* $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$. *Then*

$$\Omega(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}).$$

Proof. By Corollary 3.2 in [6], $\mathcal{M}(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A})$ holds. Hence, we only prove $\Omega(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$. Let $z \in \Omega(\mathcal{A})$. Then there are $t, s \in N$ and $t \neq s$ such that $z \in \hat{\Omega}_{t,s}(\mathcal{A})$ or $z \in \left(\tilde{\Omega}_{t,s}(\mathcal{A}) \cap \mathcal{K}_t(\mathcal{A})\right)$. We divide the proof into two parts.

Case I: If $z \in \hat{\Omega}_{t,s}(\mathcal{A})$, that is, $|z| < P_t^s(\mathcal{A})$ and $|z| < P_s^t(\mathcal{A})$. Then, it is easily to see that

$$|z| < P_t^s(\mathcal{A}) \leq R_t(\mathcal{A}) - |a_{ts\cdots s}|,$$

which implies that $z \in \mathcal{H}_{t,s}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$, consequently, $\Omega(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$.

Case II: If $z \notin \hat{\Omega}_{t,s}(\mathcal{A})$, that is,

$$|z| \geq P_s^t(\mathcal{A}) \tag{4}$$

or

$$|z| \geq P_t^s(\mathcal{A}), \tag{5}$$

then $z \in \left(\tilde{\Omega}_{t,s}(\mathcal{A}) \cap \mathcal{K}_t(\mathcal{A})\right)$, i.e.,

$$|z| \leq R_t(\mathcal{A}) \tag{6}$$

and

$$\Big(|z| - P_t^s(\mathcal{A})\Big)\Big(|z| - P_s^t(\mathcal{A})\Big) \leq \Big(R_t(\mathcal{A}) - P_t^s(\mathcal{A})\Big)\Big(R_s(\mathcal{A}) - P_s^t(\mathcal{A})\Big). \tag{7}$$

(i) Assume $\Big(R_t(\mathcal{A}) - P_t^s(\mathcal{A})\Big)\Big(R_s(\mathcal{A}) - P_s^t(\mathcal{A})\Big) = 0$. When (4) holds, by (7), we have

$$\begin{aligned}\Big(|z| - (R_t(\mathcal{A}) - |a_{ts\cdots s}|)\Big)\Big(|z| - P_s^t(\mathcal{A})\Big) &\leq \Big(|z| - P_t^s(\mathcal{A})\Big)\Big(|z| - P_s^t(\mathcal{A})\Big)\\ &\leq \Big(R_t(\mathcal{A}) - P_t^s(\mathcal{A})\Big)\Big(R_s(\mathcal{A}) - P_s^t(\mathcal{A})\Big)\\ &= 0\\ &\leq |a_{ts\cdots s}|\Big(R_s(\mathcal{A}) - P_s^t(\mathcal{A})\Big),\end{aligned}$$

which implies that $z \in \mathcal{M}_{t,s}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$. On the other hand, when (5) holds, we only prove $z \in \mathcal{M}(\mathcal{A})$ under the case that $|z| < P_s^t(\mathcal{A})$. When

$$P_t^s(\mathcal{A}) \leq |z| < R_t(\mathcal{A}) - |a_{ts\cdots s}|, \tag{8}$$

we have $z \in \mathcal{H}_{t,s}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$. And when

$$R_t(\mathcal{A}) - |a_{ts\cdots s}| \leq |z| \leq R_t(\mathcal{A}), \tag{9}$$

from

$$\Big(|z| - (R_t(\mathcal{A}) - |a_{ts\cdots s}|)\Big)\Big(|z| - P_s^t(\mathcal{A})\Big) \leq 0 \leq |a_{ts\cdots s}|\Big(R_s(\mathcal{A}) - P_s^t(\mathcal{A})\Big), \tag{10}$$

we have $z \in \mathcal{M}_{t,s}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$.

(ii) Assume $\Big(R_t(\mathcal{A}) - P_t^s(\mathcal{A})\Big)\Big(R_s(\mathcal{A}) - P_s^t(\mathcal{A})\Big) > 0$. Then dividing both sides by $\Big(R_t(\mathcal{A}) - P_t^s(\mathcal{A})\Big)\Big(R_s(\mathcal{A}) - P_s^t(\mathcal{A})\Big)$ in (7), we have

$$\frac{|z| - P_t^s(\mathcal{A})}{R_t(\mathcal{A}) - P_t^s(\mathcal{A})}\frac{|z| - P_s^t(\mathcal{A})}{R_s(\mathcal{A}) - P_s^t(\mathcal{A})} \leq 1. \tag{11}$$

If $|a_{ts\cdots s}| > 0$, let $a = |z|, b = P_t^s(\mathcal{A}), c = R_t(\mathcal{A}) - |a_{ts\cdots s}| - P_t^s(\mathcal{A})$ and $d = |a_{ts\cdots s}|$, by (6) and Lemma 2.2 in [24], we have

$$\frac{|z| - (R_t(\mathcal{A}) - |a_{ts\cdots s}|)}{|a_{ts\cdots s}|} = \frac{a - (b + c)}{d} \leq \frac{a - b}{c + d} = \frac{|z| - P_t^s(\mathcal{A})}{R_t(\mathcal{A}) - P_t^s(\mathcal{A})}. \tag{12}$$

When (4) holds, by (11) and (12), we have

$$\frac{|z|-(R_t(\mathcal{A})-|a_{ts\cdots s}|)}{|a_{ts\cdots s}|}\frac{|z|-P_s^t(\mathcal{A})}{R_s(\mathcal{A})-P_s^t(\mathcal{A})}\leq\frac{|z|-P_t^s(\mathcal{A})}{R_t(\mathcal{A})-P_t^s(\mathcal{A})}\frac{|z|-P_s^t(\mathcal{A})}{R_s(\mathcal{A})-P_s^t(\mathcal{A})}\leq 1,$$

equivalently,

$$\Big(|z|-(R_t(\mathcal{A})-|a_{ts\cdots s}|)\Big)\Big(|z|-P_s^t(\mathcal{A})\Big)\leq|a_{ts\cdots s}|\Big(R_s(\mathcal{A})-P_s^t(\mathcal{A})\Big),$$

which implies that $z\in\mathcal{M}_{t,s}(\mathcal{A})\subseteq\mathcal{M}(\mathcal{A})$. On the other hand, when (5) holds, we only prove $z\in\mathcal{M}(\mathcal{A})$ under the case that $|z|<P_s^t(\mathcal{A})$. If (8) holds, then $z\in\mathcal{H}_{t,s}(\mathcal{A})\subseteq\mathcal{M}(\mathcal{A})$. And if (9) holds, by (10), we have $z\in\mathcal{M}_{t,s}(\mathcal{A})\subseteq\mathcal{M}(\mathcal{A})$.

If $|a_{ts\cdots s}|=0$, by $|z|\leq R_t(\mathcal{A})$, we have $|z|-(R_t(\mathcal{A})-|a_{ts\cdots s}|)\leq 0=|a_{ts\cdots s}|$. When (4) holds, we can obtain

$$\Big(|z|-(R_t(\mathcal{A})-|a_{ts\cdots s}|)\Big)\Big(|z|-P_s^t(\mathcal{A})\Big)\leq 0=|a_{ts\cdots s}|\Big(R_s(\mathcal{A})-P_s^t(\mathcal{A})\Big),\tag{13}$$

which implies that $z\in\mathcal{M}_{t,s}(\mathcal{A})\subseteq\mathcal{M}(\mathcal{A})$. On the other hand, when (5) holds, we only prove $z\in\mathcal{M}(\mathcal{A})$ under the case that $|z|<P_s^t(\mathcal{A})$. If (8) holds, then $z\in\mathcal{H}_{t,s}(\mathcal{A})\subseteq\mathcal{M}(\mathcal{A})$. And if (9) holds, by (13), we have $z\in\mathcal{M}_{t,s}(\mathcal{A})\subseteq\mathcal{M}(\mathcal{A})$. The conclusion follows from Case I and Case II. □

Remark 2.4. *Theorem 2.3 shows that the set $\Omega(\mathcal{A})$ in Theorem 2.2 is tighter than $\mathcal{K}(\mathcal{A})$ and $\mathcal{M}(\mathcal{A})$ in Theorem 2.1, that is, $\Omega(\mathcal{A})$ can capture all E-eigenvalues of $\mathcal{A}$ more precisely than $\mathcal{K}(\mathcal{A})$ and $\mathcal{M}(\mathcal{A})$.*

In the following, an example is given to verify Remark 2.4.

Example 2.5. *Let $\mathcal{A}=(a_{ijk})\in\mathbb{R}^{[3,3]}$ with entries defined as follows:*

$$A(:,:,1)=\begin{pmatrix}0&3&3\\2&1&1\\3&1&0\end{pmatrix},A(:,:,2)=\begin{pmatrix}2&0.5&1\\0&2&0\\1&0&0\end{pmatrix},A(:,:,3)=\begin{pmatrix}3&1&1\\1&1&0\\2&0&1\end{pmatrix}.$$

We now locate all E-eigenvalues of $\mathcal{A}$. By Theorem 2.1, we have

$$\mathcal{K}(\mathcal{A})=\{z\in\mathbb{C}:|z|\leq 14.5000\}\ \textit{and}\ \mathcal{M}(\mathcal{A})=\{z\in\mathbb{C}:|z|\leq 14.2228\}.$$

By Theorem 2.2, we have

$$\Omega(\mathcal{A})=\{z\in\mathbb{C}:|z|\leq 11.5000\}.$$

The E-eigenvalue inclusion sets $\mathcal{K}(\mathcal{A})$, $\mathcal{M}(\mathcal{A})$, $\Omega(\mathcal{A})$ and all E-eigenvalues $-6.3796, -3.2536, -1.8154, -0.8351, -0.7011-0.8430i, -0.7011+0.8430i, -0.4608, 0.4608, 0.7011-0.8430i, 0.7011+0.8430i, 0.8351, 1.8154, 3.2536, 6.3796$ *are drawn in Figure 1, where $\mathcal{K}(\mathcal{A})$, $\mathcal{M}(\mathcal{A})$, $\Omega(\mathcal{A})$ and the exact E-eigenvalues are represented by black solid boundary, blue dashed boundary, red solid boundary and black "+", respectively. It is easy to see that*

$$\sigma(\mathcal{A})\subseteq\Omega(\mathcal{A})\subset\mathcal{M}(\mathcal{A})\subset\mathcal{K}(\mathcal{A}),$$

that is, $\Omega(\mathcal{A})$ can capture all E-eigenvalues of $\mathcal{A}$ more precisely than $\mathcal{M}(\mathcal{A})$ and $\mathcal{K}(\mathcal{A})$.

3. A sharper upper bound for the Z-spectral radius of weakly symmetric nonnegative tensors

As an application of the set $\Omega(\mathcal{A})$ in Theorem 2.2, a new upper bound for the Z-spectral radius of weakly symmetric nonnegative tensors is given.

Theorem 3.1. *Let $\mathcal{A}=(a_{i_1\cdots i_m})\in\mathbb{R}^{[m,n]}$ be a weakly symmetric nonnegative tensor. Then*

$$\varrho(\mathcal{A})\leq\Omega_{max}=\max\Big\{\hat{\Omega}_{max},\tilde{\Omega}_{max}\Big\},$$

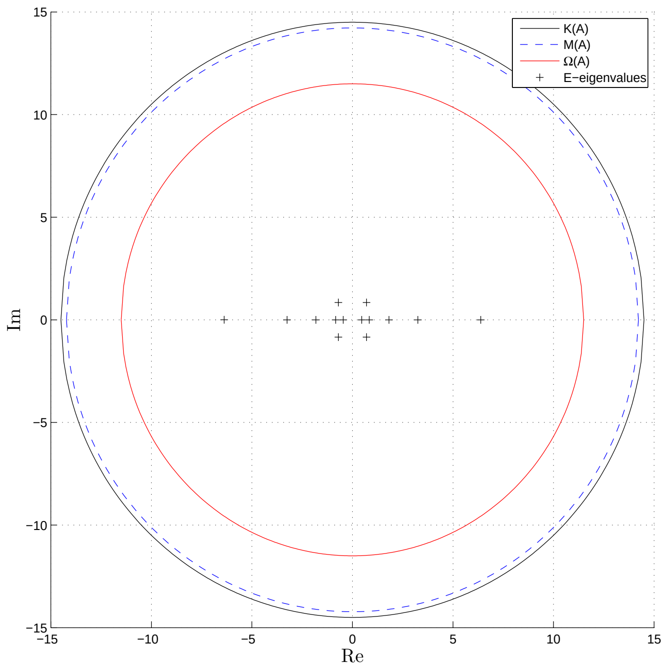

Figure 1: Comparisons of $\mathcal{K}(\mathcal{A})$, $\mathcal{M}(\mathcal{A})$ and $\Omega(\mathcal{A})$.

where

$$\hat{\Omega}_{max} = \max_{i,j\in N, j\neq i} \min\{P_i^j(\mathcal{A}), P_j^i(\mathcal{A})\},$$

$$\tilde{\Omega}_{max} = \max_{i,j\in N, j\neq i} \min\left\{R_i(\mathcal{A}), \Delta_{i,j}(\mathcal{A})\right\},$$

and

$$\Delta_{i,j}(\mathcal{A}) = \frac{1}{2}\left\{P_i^j(\mathcal{A}) + P_j^i(\mathcal{A}) + \sqrt{\left(P_i^j(\mathcal{A}) - P_j^i(\mathcal{A})\right)^2 + 4\left(R_i(\mathcal{A}) - P_i^j(\mathcal{A})\right)\left(R_j(\mathcal{A}) - P_j^i(\mathcal{A})\right)}\right\}.$$

Proof. As stated in Section 1, if $\mathcal{A}$ is weakly symmetric and nonnegative, then $\varrho(\mathcal{A})$ is the largest Z-eigenvalue of $\mathcal{A}$. Hence, by Theorem 2.2, we have

$$\varrho(\mathcal{A}) \in \bigcup_{i,j\in N, j\neq i} \left(\hat{\Omega}_{i,j}(\mathcal{A}) \cup \left(\tilde{\Omega}_{i,j}(\mathcal{A}) \cap \mathcal{K}_i(\mathcal{A})\right)\right),$$

that is, there are $t, s \in N, t \neq s$ such that $\varrho(\mathcal{A}) \in \hat{\Omega}_{t,s}(\mathcal{A})$ or $\varrho(\mathcal{A}) \in \left(\tilde{\Omega}_{t,s}(\mathcal{A}) \cap \mathcal{K}_t(\mathcal{A})\right)$. If $\varrho(\mathcal{A}) \in \hat{\Omega}_{t,s}(\mathcal{A})$, i.e., $\varrho(\mathcal{A}) < P_t^s(\mathcal{A})$ and $\varrho(\mathcal{A}) < P_s^t(\mathcal{A})$, we have $\varrho(\mathcal{A}) < \min\{P_t^s(\mathcal{A}), P_s^t(\mathcal{A})\}$. Furthermore, we have

$$\varrho(\mathcal{A}) \leq \max_{i,j\in N, j\neq i} \min\{P_i^j(\mathcal{A}), P_j^i(\mathcal{A})\}. \tag{14}$$

If $\varrho(\mathcal{A}) \in \left(\tilde{\Psi}_{t,s}(\mathcal{A}) \cap \mathcal{K}_t(\mathcal{A})\right)$, i.e., $\varrho(\mathcal{A}) \leq R_t(\mathcal{A})$ and

$$\left(\varrho(\mathcal{A}) - P_t^s(\mathcal{A})\right)\left(\varrho(\mathcal{A}) - P_s^t(\mathcal{A})\right) \leq \left(R_t(\mathcal{A}) - P_t^s(\mathcal{A})\right)\left(R_s(\mathcal{A}) - P_s^t(\mathcal{A})\right), \tag{15}$$

then solving $\varrho(\mathcal{A})$ in (15) gives

$$\varrho(\mathcal{A}) \leq \frac{1}{2}\left\{P_t^s(\mathcal{A}) + P_s^t(\mathcal{A}) + \sqrt{\left(P_t^s(\mathcal{A}) - P_s^t(\mathcal{A})\right)^2 + 4\left(R_t(\mathcal{A}) - P_t^s(\mathcal{A})\right)\left(R_s(\mathcal{A}) - P_s^t(\mathcal{A})\right)}\right\} = \Delta_{t,s}(\mathcal{A}),$$

and furthermore

$$\varrho(\mathcal{A}) \leq \min\{R_t(\mathcal{A}), \Delta_{t,s}(\mathcal{A})\} \leq \max_{i,j\in N, j\neq i} \min\left\{R_i(\mathcal{A}), \Delta_{i,j}(\mathcal{A})\right\}. \tag{16}$$

The conclusion follows from (14) and (16). □

By Theorem 2.3 and Corollary 4.2 in [6], the following comparison theorem can be derived easily.

Theorem 3.2. *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$ be a weakly symmetric nonnegative tensor. Then the upper bound in Theorem 3.1 is sharper than those in Theorems 1.2 and 1.4, that is,*

$$\varrho(\mathcal{A}) \leq \Omega_{max} \leq \Psi(\mathcal{A}) \leq \max_{i\in N} R_i(\mathcal{A}).$$

Finally, we show that in some cases the upper bound in Theorem 3.1 is sharper than those in [6, 7, 9–15] by an example.

Example 3.3. Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[4,2]}$ be a symmetric tensor defined by

$$a_{1111} = \frac{1}{2},\ a_{2222} = 3,\ a_{ijkl} = \frac{1}{3} \text{ elsewhere.}$$

By computation, we obtain $(\rho(\mathcal{A}), x) = (3.1092, (0.1632, 0.9866))$. By Corollary 4.5 of [7], we have

$$\varrho(\mathcal{A}) \leq 5.3333.$$

By Theorem 2.7 of [15], we have

$$\varrho(\mathcal{A}) \leq 5.2846.$$

By Theorem 3.3 of [11], we have

$$\varrho(\mathcal{A}) \leq 5.1935.$$

By Theorem 4.5, Theorem 4.6 and Theorem 4.7 of [6], we all have

$$\varrho(\mathcal{A}) \leq 5.1822.$$

By Theorem 3.5 of [12] and Theorem 6 of [13], we both have

$$\varrho(\mathcal{A}) \leq 5.1667.$$

By Theorem 7 of [9], we have

$$\varrho(\mathcal{A}) \leq 5.0437.$$

By Theorem 2.9 of [14], we have

$$\varrho(\mathcal{A}) \leq 4.5147.$$

By Theorem 5 of [10], we have

$$\varrho(\mathcal{A}) \leq 4.4768.$$

By Theorem 3.1, we obtain

$$\varrho(\mathcal{A}) \leq 4.3971,$$

which shows that this upper bound is better.

4. Conclusion

In this paper, we first generalize two Z-eigenvalue inclusion sets $\mathcal{K}(\mathcal{A})$ and $\mathcal{M}(\mathcal{A})$ presented by Wang *et al*. in [6] to E-eigenvalue localization sets. And then we establish a new E-eigenvalue localization set $\Omega(\mathcal{A})$ and prove that it is tighter than $\mathcal{K}(\mathcal{A})$ and $\mathcal{M}(\mathcal{A})$. Based on the set $\Omega(\mathcal{A})$, we obtain a new upper bound Ω_{max} for the Z-spectral radius of weakly symmetric nonnegative tensors and show that it is better than those in [6, 7, 9–15] in some cases by a numerical example.

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