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Globally Asymptotic Stability of a Stochastic Mutualism Model with Saturated Response

Jingliang Lv[a,*], Heng Liu[a], Yifeng Zhang[a]

[a] *Department of Mathematics, Harbin Institute of Technology at Weihai, Shandong, 264209, P.R.China*

Abstract. A two-species stochastic mutualism model with saturated response is proposed and investigated in this paper. We demonstrate that there exists a unique positive solution to the model for any positive initial value. Under some conditions, we show that the stochastic model is globally asymptotically stable. Finally, we work out some figures to illustrate our results.

1. Introduction

Generally speaking, competition, predator-prey and mutualism are three basic relationships between species. There exist many successful results on competition and predator-prey interactions, but mutualism models are not understood theoretically [1]. Classic theory on mutualisms, however, suggested that mutualisms were highly destabilizing [2] and the population sizes of species increase infinitely causing divergence [3]. Therefore it is necessary and important to introduce appropriate models to investigate essential features of mutualisms.

Lattice gas models in mean-field theory may provide a way to consider mutualisms [3]. Recently Lattice gas models have drawn growing attention, see [3-9], among others. Especially, Wang and Wu [4] studied the following model

$$dx = r_1 x\Big[-d_1 + \Big(1 + \frac{a_1 y}{1 + b_1 y}\Big)(1 - x - y)\Big]dt, \quad (1)$$

$$dy = r_2 y\Big[-d_2 + \Big(1 + \frac{a_2 x}{1 + b_2 x}\Big)(1 - x - y)\Big]dt, \quad (2)$$

where $x(t)$ and $y(t)$ represent two species densities at time t respectively, $\frac{a_1}{b_1}$, $\frac{a_2}{b_2}$ represent the saturation levels of $x(t)$ and $y(t)$, $d_1 = \frac{D_1}{r_1}$, $d_2 = \frac{D_2}{r_2}$ are positive parameters that D_1, D_2 stand for death rates of species $x(t)$ and $y(t)$, r_1, r_2 are birth rates of $x(t)$ and $y(t)$. For biological representation of each coefficient in the population dynamics, we refer the reader to [3-4].

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*Corresponding author: Jingliang Lv
Email address: ljl3188@163.com (Jingliang Lv)

On the other hand, population dynamics is inevitably subjected to environmental noise (see e.g. [10-11]), which is an important component in an ecosystem. R.M.May [12] pointed out the fact that due to environmental noise, the birth rates, carrying capacity, competition coefficients and other parameters involved in the system exhibit random fluctuation to a greater or lesser extent. Many authors considered the corresponding stochastic models to reveal the effect of environmental variability on the dynamics in mathematical ecology; see e.g. [13-22]. These important results reveal the significant effect of the environmental noise to the population system. In this paper, taking into account the environmental noise, the stochastic system has the following form:

$$dx = x\Big[-d_1+\Big(1+\frac{a_1y}{1+b_1y}\Big)(1-x-y)\Big][r_1dt+\sigma_1dB_1(t)], \tag{3}$$

$$dy = y\Big[-d_2+\Big(1+\frac{a_2x}{1+b_2x}\Big)(1-x-y)\Big][r_2dt+\sigma_2dB_2(t)]. \tag{4}$$

The model, consisting of (3)-(4), together with the initial conditions $x(0)=x_0>0$ and $y(0)=y_0>0$ will be referred to as model (SMM).

A basic problem in the study of population dynamic is the coexistence of species. When time t is sufficiently large and the solutions of a stochastic model go to a positive equilibrium of the stochastic system, we can equate the state with coexistence of species mathematically. However, generally speaking, there is no positive equilibrium to a stochastic model. [23] and [24] make attempts to consider the positive equilibria of stochastic models, and conclude the models are globally asymptotically stable. However, so far as we know, there is no work has been done with the stability of stochastic mutualism model with saturated response (SMM). Motivated by the above ideas, we consider the globally asymptotic stability of its positive equilibrium of the stochastic system (SMM).

2. Main results

As $x(t)$ and $y(t)$ in model (SMM) are population size of the prey and the predator respectively, it should be non-negative. So for further study, we must firstly consider the system (SMM) has a globally positive solution.

Theorem 2.1. *Consider model (SMM), for any given initial value $(x_0,y_0)\in R^2_+$, there is an unique solution (x(t),y(t)) on $t\geq 0$ and the solution will remain in R^2_+ with probability 1, where $R^2_+=\{x\in R^2_+|x_i>0, i=1,2\}$.*

Proof. Define a C^2-function V: $R^2_+\to R_+$ by

$$V(x,y)=\sqrt{x}-1-\frac{1}{2}\ln x+\sqrt{y}-1-\frac{1}{2}\ln y.$$

The non-negativity of this function can be observed from $u-1-\ln u\geq 0$ on $u>0$. If $(x(t),y(t))\in R^2_+$, we compute

$$\begin{aligned} LV(x,y) &= \frac{r_1}{2}(\sqrt{x}-1)\Big[-d_1+\Big(1+\frac{a_1y}{1+b_1y}\Big)(1-x-y)\Big]+\frac{r_2}{2}(\sqrt{y}-1)\Big[-d_2+\Big(1+\frac{a_2x}{1+b_2x}\Big)(1-x-y)\Big] \\ &\quad +\frac{\sigma_1^2}{8}(2-\sqrt{x})\Big[-d_1+\Big(1+\frac{a_1y}{1+b_1y}\Big)(1-x-y)\Big]^2+\frac{\sigma_2^2}{8}(2-\sqrt{y})\Big[-d_2+\Big(1+\frac{a_2x}{1+b_2x}\Big)(1-x-y)\Big]^2 \\ &= -\frac{r_1}{2}\Big(1+\frac{a_1y}{1+b_1y}\Big)x^{\frac{3}{2}}-\frac{r_1}{2}\Big(1+\frac{a_1y}{1+b_1y}\Big)\sqrt{x}y-\frac{r_1}{2}\Big[-d_1+\Big(1+\frac{a_1y}{1+b_1y}\Big)(1-x-y)\Big] \\ &\quad +\frac{r_1}{2}\Big[-d_1+\Big(1+\frac{a_1y}{1+b_1y}\Big)\Big]\sqrt{x}-\frac{r_2}{2}\Big(1+\frac{a_2x}{1+b_2x}\Big)y^{\frac{3}{2}}-\frac{r_2}{2}\Big(1+\frac{a_2x}{1+b_2x}\Big)x\sqrt{y} \\ &\quad -\frac{r_2}{2}\Big[-d_2+\Big(1+\frac{a_2y}{1+b_2y}\Big)(1-x-y)\Big]+\frac{r_2}{2}\Big[-d_2+\Big(1+\frac{a_2y}{1+b_2y}\Big)\Big]\sqrt{y} \end{aligned}$$

$$
\begin{aligned}
& -\frac{\sigma_1^2}{8}\Big(1+\frac{a_1y}{1+b_1y}\Big)^2x^{\frac{5}{2}}+\frac{\sigma_1^2}{4}\Big(1+\frac{a_1y}{1+b_1y}\Big)^2(-x^{\frac{3}{2}}+2x)y+\frac{\sigma_1^2}{8}\Big(1+\frac{a_1y}{1+b_1y}\Big)^2(-y^2+2y)\sqrt{x}\\
& -\frac{\sigma_1^2}{4}d_1\Big(1+\frac{a_1y}{1+b_1y}\Big)\sqrt{x}y-\frac{\sigma_1^2}{2}\Big(1+\frac{a_1y}{1+b_1y}\Big)^2y+\frac{\sigma_1^2}{2}d_1\Big(1+\frac{a_1y}{1+b_1y}\Big)y\\
& +\frac{\sigma_1^2}{8}(2-\sqrt{x})\Big[d_1^2+\Big(1+\frac{a_1y}{1+b_1y}\Big)^2(1-2x)-2d_1\Big(1+\frac{a_1y}{1+b_1y}\Big)(1-x)\Big]-\frac{\sigma_1^2}{4}\Big(1+\frac{a_1y}{1+b_1y}\Big)\sqrt{x}y\\
& -\frac{\sigma_2^2}{8}\Big(1+\frac{a_2x}{1+b_2x}\Big)^2y^{\frac{5}{2}}+\frac{\sigma_2^2}{4}\Big(1+\frac{a_2x}{1+b_2x}\Big)^2(-y^{\frac{3}{2}}+2y)x+\frac{\sigma_2^2}{8}\Big(1+\frac{a_2x}{1+b_2x}\Big)^2(-x^2+2x)\sqrt{y}\\
& -\frac{\sigma_2^2}{4}d_1\Big(1+\frac{a_2x}{1+b_2x}\Big)x\sqrt{y}-\frac{\sigma_2^2}{2}\Big(1+\frac{a_2x}{1+b_2x}\Big)^2x+\frac{\sigma_2^2}{2}d_2\Big(1+\frac{a_2x}{1+b_2x}\Big)x\\
& +\frac{\sigma_2^2}{8}(2-\sqrt{y})\Big[d_2^2+\Big(1+\frac{a_2x}{1+b_2x}\Big)^2(1-2y)-2d_2\Big(1+\frac{a_2x}{1+b_2x}\Big)(1-y)\Big]-\frac{\sigma_2^2}{4}\Big(1+\frac{a_2x}{1+b_2x}\Big)x\sqrt{y}\\
\le\ & -\frac{\sigma_1^2}{8}\Big(1+\frac{a_1y}{1+b_1y}\Big)^2x^{\frac{5}{2}}-\frac{r_1}{2}\Big(1+\frac{a_1y}{1+b_1y}\Big)x^{\frac{3}{2}}-\frac{\sigma_2^2}{2}\Big(1+\frac{a_2x}{1+b_2x}\Big)^2x+\frac{r_1}{2}\Big[-d_1+\Big(1+\frac{a_1y}{1+b_1y}\Big)\Big]\sqrt{x}\\
& -\frac{\sigma_2^2}{8}\Big(1+\frac{a_2x}{1+b_2x}\Big)^2y^{\frac{5}{2}}-\frac{r_2}{2}\Big(1+\frac{a_2x}{1+b_2x}\Big)y^{\frac{3}{2}}-\frac{\sigma_1^2}{2}\Big(1+\frac{a_1y}{1+b_1y}\Big)^2y+\frac{r_2}{2}\Big[-d_2+\Big(1+\frac{a_2y}{1+b_2y}\Big)\Big]\sqrt{y}\\
& +\frac{\sigma_2^2}{2}d_2\Big(1+\frac{a_2x}{1+b_2x}\Big)x+\frac{\sigma_2^2}{4}\Big(1+\frac{a_2x}{1+b_2x}\Big)^2M_1x+\frac{\sigma_1^2}{8}\Big(1+\frac{a_1y}{1+b_1y}\Big)^2M_2\sqrt{x}\\
& +\frac{\sigma_1^2}{2}d_1\Big(1+\frac{a_1y}{1+b_1y}\Big)y+\frac{\sigma_1^2}{4}\Big(1+\frac{a_1y}{1+b_1y}\Big)^2M_3y+\frac{\sigma_2^2}{8}\Big(1+\frac{a_2x}{1+b_2x}\Big)^2M_4\sqrt{y}\\
& +\frac{\sigma_1^2}{8}(2-\sqrt{x})\Big[d_1^2+\Big(1+\frac{a_1y}{1+b_1y}\Big)^2(1-2x)-2d_1\Big(1+\frac{a_1y}{1+b_1y}\Big)(1-x)\Big]\\
& +\frac{\sigma_2^2}{8}(2-\sqrt{y})\Big[d_2^2+\Big(1+\frac{a_2x}{1+b_2x}\Big)^2(1-2y)-2d_2\Big(1+\frac{a_2x}{1+b_2x}\Big)(1-y)\Big]\\
& -\frac{r_1}{2}\Big[-d_1+\Big(1+\frac{a_1y}{1+b_1y}\Big)(1-x-y)\Big]-\frac{r_2}{2}\Big[-d_2+\Big(1+\frac{a_2y}{1+b_2y}\Big)(1-x-y)\Big]\\
\le\ & K.
\end{aligned}
$$

Since all coefficients of system (SMM) are positive constants, we deduce that the function $LV(x,y)$ is upper bounded, say by K. By the similar proof of Theorem 2.1 of [18], we can obtain the desired assertion. □

Theorem 2.1 shows that (SMM) has a globally positive solution which is essential for a population system. This result provides us with a great opportunity to construct some Lyapunov functions to study the stability of the positive equilibrium of model (SMM).

Let (x^*,y^*) be a positive equilibrium point of system (SMM) which is the solution of the following algebraic equations:

$$-d_1+\Big(1+\frac{a_1y}{1+b_1y}\Big)(1-x-y)=0, \tag{5}$$

$$-d_2+\Big(1+\frac{a_2x}{1+b_2x}\Big)(1-x-y)=0. \tag{6}$$

Then system (SMM) can be rewritten as :

$$
\begin{aligned}
dx\ =\ & -x\Big[(x-x^*)+(y-y^*)+\frac{-a_1+a_1x^*+2a_1y^*+a_1b_1(y^*)^2}{(1+b_1y^*)(1+b_1y)}(y-y^*)\\
& +\frac{a_1y^*}{1+b_1y}(x-x^*)+\frac{a_1}{1+b_1y}(x-x^*)(y-y^*)+\frac{a_1}{1+b_1y}(y-y^*)^2\Big]\cdot[r_1dt+\sigma_1dB_1(t)],
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
dy &= -y\Big[(x-x^*)+(y-y^*)+\frac{-a_2+a_2y^*+2a_2x^*+a_2b_2(x^*)^2}{(1+b_2x^*)(1+b_2x)}(x-x^*)\\
&\quad+\frac{a_2x^*}{1+b_2x}(y-y^*)+\frac{a_2}{1+b_2x}(x-x^*)(y-y^*)+\frac{a_2}{1+b_2x}(x-x^*)^2\Big]\cdot[r_2dt+\sigma_2dB_2(t)]. \qquad (8)
\end{aligned}
$$

We let

$$
\begin{aligned}
A_1 &= -r_1+\frac{\sigma_1^2}{2}x^*\Big(1+\frac{a_1y}{1+b_1y}\Big)^2+\frac{\sigma_2^2}{2}y^*\Big(1+\frac{a_2x}{1+b_2x}-a_2\frac{1-x^*-y^*}{(1+b_2x^*)(1+b_2x)}\Big)^2,\\
B_1 &= -r_2+\frac{\sigma_2^2}{2}y^*\Big(1+\frac{a_2x}{1+b_2x}\Big)^2+\frac{\sigma_1^2}{2}x^*\Big(1+\frac{a_1y}{1+b_1y}-a_1\frac{1-x^*-y^*}{(1+b_1y^*)(1+b_1y)}\Big)^2,\\
C_1 &= \Big[-r_1+\sigma_1^2x^*\Big(1+\frac{a_1y}{1+b_1y}\Big)\Big]\Big[1+\frac{a_1y}{1+b_1y}-a_1\frac{1-x^*-y^*}{(1+b_1y^*)(1+b_1y)}\Big]\\
&\quad+\Big[-r_2+\sigma_2^2y^*\Big(1+\frac{a_2x}{1+b_2x}\Big)\Big]\Big[1+\frac{a_2x}{1+b_2x}-a_2\frac{1-x^*-y^*}{(1+b_2x^*)(1+b_2x)}\Big],\\
A_2 &= -r_1+\frac{\sigma_1^2}{2}x^*\Big(1+\frac{a_1}{1+b_1}\Big)^2+\frac{\sigma_2^2}{2}y^*\Big(1+\frac{a_2}{1+b_2}\Big)^2,\\
B_2 &= -r_2+\frac{\sigma_2^2}{2}y^*\Big(1+\frac{a_2}{1+b_2}\Big)^2+\frac{\sigma_1^2}{2}x^*\Big(1+\frac{a_1}{1+b_1}\Big)^2,\\
C_2 &= (r_1-\sigma_1^2x^*)\Big(1-a_1\frac{1-x^*-y^*}{1+b_1y^*}\Big)+\Big[r_2-\sigma_2^2y^*\Big(1+\frac{a_2x^*}{1+b_2x^*}\Big)\Big]\Big[1+\frac{a_2x^*}{1+b_2x^*}-a_2\frac{1-x^*-y^*}{(1+b_2x^*)^2}\Big],\\
C_3 &= (r_2-\sigma_2^2y^*)\Big(1-a_2\frac{1-x^*-y^*}{1+a_1x^*}\Big)+\Big[r_1-\sigma_1^2x^*\Big(1+\frac{a_1y^*}{1+b_1y^*}\Big)\Big]\Big[1+\frac{a_1y^*}{1+b_1y^*}-a_1\frac{1-x^*-y^*}{(1+b_1y^*)^2}\Big].
\end{aligned}
$$

Now, we are in the position to analyze the globally asymptotic stability of the stochastic model (SMM).

Theorem 2.2. *If*

$$
A_2<0 \quad and \quad 4A_2B_2-C_2^2>0 \quad and \quad 4A_2B_2-C_3^2>0. \qquad (9)
$$

Then the equilibrium position (x^*, y^*) of model (SMM) is stochastically asymptotically stable in the large, i.e., for any initial data (x_0, y_0), the solution of model (SMM) has the property that

$$
\lim_{t\to\infty}x(t)=x^*,\ \lim_{t\to\infty}y(t)=y^*, \qquad (10)
$$

almost surely.

Proof. From the theory of stability of stochastic differential equations, we only need to find a Lyapunov function $V(z)$ satisfying $LV(z)\le 0$ and the identity holds if and only if $z=z^*$ (see e.g. [25]), where $z=z(t)$ is the solution of the n-dimensional stochastic differential equation

$$
dz(t)=f(z(t),t)dt+g(z(t),t)dB(t). \qquad (11)
$$

and

$$
LV(z)=V_t(z)+V_z(z)f(t,z)+\frac{1}{2}trace[g^T(t,z)V_{zz}(z)g(t,z)].
$$

Now define Lyapunov functions

$$
V_1(x)=x-x^*-x^*\ln(\frac{x}{x^*}),\quad V_2(y)=y-y^*-y^*\ln(\frac{y}{y^*}).
$$

The non-negativity of this function can be observed from $u-1-\ln u \geq 0$ on $u>0$. If $(x(t), y(t)) \in R_{+}^{2}$, applying *Itô's* formula leads to

$$
\begin{aligned}
LV_1(x) &= r_1(x-x^*)\Big[-(x-x^*)-(y-y^*)-\frac{-a_1+a_1x^*+2a_1y^*+a_1b_1(y^*)^2}{(1+b_1y^*)(1+b_1y)}(y-y^*)\\
&\quad -\frac{a_1y^*(x-x^*)+a_1(x-x^*)(y-y^*)+a_1(y-y^*)^2}{1+b_1y}\Big]\\
&\quad +\frac{\sigma_1^2}{2}x^*\Big[-(x-x^*)-(y-y^*)-\frac{-a_1+a_1x^*+2a_1y^*+a_1b_1(y^*)^2}{(1+b_1y^*)(1+b_1y)}(y-y^*)\\
&\quad -\frac{a_1y^*(x-x^*)+a_1(x-x^*)(y-y^*)+a_1(y-y^*)^2}{1+b_1y}\Big]^2\\
&\leq -r_1(x-x^*)^2-r_1(x-x^*)(y-y^*)-r_1\frac{-a_1+a_1x^*+2a_1y^*+a_1b_1(y^*)^2}{(1+b_1y^*)(1+b_1y)}(x-x^*)(y-y^*)\\
&\quad -a_1r_1\frac{y-y^*}{1+b_1y}(x-x^*)(y-y^*)+\frac{\sigma_1^2}{2}x^*\Big(1+\frac{a_1y^*}{1+b_1y}\Big)^2(x-x^*)^2\\
&\quad +\sigma_1^2x^*\Big(1+\frac{a_1y^*}{1+b_1y}\Big)\Big(1+\frac{-a_1+a_1x^*+2a_1y^*+a_1b_1(y^*)^2}{(1+b_1y^*)(1+b_1y)}\Big)(x-x^*)(y-y^*)\\
&\quad +\sigma_1^2a_1x^*\Big(1+\frac{a_1y^*}{1+b_1y}\Big)\frac{y-y^*}{1+b_1y}(x-x^*)^2+\sigma_1^2a_1x^*\Big(1+\frac{a_1y^*}{1+b_1y}\Big)\frac{y-y^*}{1+b_1y}(x-x^*)(y-y^*)\\
&\quad +\frac{\sigma_1^2}{2}x^*\Big[1+\frac{-a_1+a_1x^*+2a_1y^*}{(1+b_1y^*)(1+b_1y)}+\frac{a_1b_1(y^*)^2}{(1+b_1y^*)(1+b_1y)}\Big]^2(y-y^*)^2\\
&\quad +\sigma_1^2a_1x^*\Big[1+\frac{-a_1+a_1x^*+2a_1y^*}{(1+b_1y^*)(1+b_1y)}+\frac{a_1b_1(y^*)^2}{(1+b_1y^*)(1+b_1y)}\Big]\frac{y-y^*}{1+b_1y}(x-x^*)(y-y^*)\\
&\quad +\sigma_1^2a_1x^*\frac{y-y^*}{1+b_1y}\Big[1+\frac{-a_1+a_1x^*+2a_1y^*+a_1b_1(y^*)^2}{(1+b_1y^*)(1+b_1y)}\Big](y-y^*)^2+\frac{\sigma_1^2}{2}a_1^2x^*\Big(\frac{y-y^*}{1+b_1y}\Big)^2(x-x^*)^2\\
&\quad +\sigma_1^2a_1^2x^*\Big(\frac{y-y^*}{1+b_1y}\Big)^2(x-x^*)(y-y^*)+\frac{\sigma_1^2}{2}a_1^2x^*\Big(\frac{y-y^*}{1+b_1y}\Big)^2(y-y^*)^2\\
&= \Big[-r_1+\sigma_1^2x^*\Big(1+\frac{a_1y}{1+b_1y}\Big)\Big]\Big[1+\frac{a_1y}{1+b_1y}-a_1\frac{1-x^*-y^*}{(1+b_1y^*)(1+b_1y)}\Big](x-x^*)(y-y^*)\\
&+ \Big[-r_1+\frac{\sigma_1^2}{2}x^*\Big(1+\frac{a_1y}{1+b_1y}\Big)^2\Big](x-x^*)^2+\frac{\sigma_1^2}{2}x^*\Big[1+\frac{a_1y}{1+b_1y}-a_1\frac{1-x^*-y^*}{(1+b_1y^*)(1+b_1y)}\Big]^2(y-y^*)^2.
\end{aligned}
$$

and

$$
\begin{aligned}
LV_2(x) &= r_2(y-y^*)\Big[-(x-x^*)-(y-y^*)-\frac{-a_2+a_2y^*+2a_2x^*+a_2b_2(x^*)^2}{(1+b_2x^*)(1+b_2x)}(x-x^*)\\
&\quad -\frac{a_2x^*(y-y^*)+a_2(x-x^*)(y-y^*)+a_2(x-x^*)^2}{1+b_2x}\Big]\\
&\quad +\frac{\sigma_2^2}{2}y^*\Big[-(x-x^*)-(y-y^*)-\frac{-a_2+a_2y^*+2a_2x^*+a_2b_2(x^*)^2}{(1+b_2x^*)(1+b_2x)}(x-x^*)\\
&\quad -\frac{a_2x^*(y-y^*)+a_2(x-x^*)(y-y^*)+a_2(x-x^*)^2}{1+b_2x}\Big]^2\\
&\leq -r_2(y-y^*)^2-r_2(x-x^*)(y-y^*)-r_2\frac{-a_2+a_2y^*+2a_2x^*+a_2b_2(x^*)^2}{(1+b_2x^*)(1+b_2x)}(x-x^*)(y-y^*)
\end{aligned}
$$

$$
\begin{aligned}
&-a_2r_2\frac{x-x^*}{1+b_2x}(x-x^*)(y-y^*)+\frac{\sigma_2^2}{2}y^*\Big(1+\frac{a_2x^*}{1+b_2x}\Big)^2(y-y^*)^2\\
&+\sigma_2^2y^*\Big(1+\frac{a_2x^*}{1+b_2x}\Big)\Big(1+\frac{-a_2+a_2y^*+2a_2x^*+a_2b_2(x^*)^2}{(1+b_2x^*)(1+b_2x)}\Big)(x-x^*)(y-y^*)\\
&+\sigma_2^2a_2y^*\Big(1+\frac{a_2x^*}{1+b_2x}\Big)\frac{x-x^*}{1+b_2x^*}(y-y^*)^2+\sigma_2^2a_2y^*\Big(1+\frac{a_2x^*}{1+b_2x}\Big)\frac{x-x^*}{1+b_2x}(x-x^*)(y-y^*)\\
&+\frac{\sigma_2^2}{2}y^*\Big[1+\frac{-a_2+a_2y^*+2a_2x^*}{(1+b_2x^*)(1+b_2x)}+\frac{a_2b_2(x^*)^2}{(1+b_2x^*)(1+b_2x)}\Big]^2(x-x^*)^2\\
&+\sigma_2^2a_2y^*\Big[1+\frac{-a_2+a_2y^*+2a_2x^*}{(1+b_2x^*)(1+b_2x)}+\frac{a_2b_2(x^*)^2}{(1+b_2x^*)(1+b_2x)}\Big]\frac{x-x^*}{1+b_2x}(x-x^*)(y-y^*)\\
&+\sigma_2^2a_2y^*\frac{x-x^*}{1+b_2x}\Big[1+\frac{-a_2+a_2y^*+2a_2x^*+a_2b_2(x^*)^2}{(1+b_2x^*)(1+b_2x)}\Big](x-x^*)^2+\frac{\sigma_2^2}{2}a_2^2y^*\Big(\frac{x-x^*}{1+b_2x}\Big)^2(x-x^*)^2\\
&+\sigma_2^2a_2^2y^*\Big(\frac{x-x^*}{1+b_2x}\Big)^2(x-x^*)(y-y^*)+\frac{\sigma_2^2}{2}a_2^2y^*\Big(\frac{x-x^*}{1+b_2x}\Big)^2(y-y^*)^2\\
=\;&\frac{\sigma_2^2}{2}y^*\Big[1+\frac{a_2x}{1+b_2x}-a_2\frac{1-x^*-y^*}{(1+b_2x^*)(1+b_2x)}\Big]^2(x-x^*)^2+\Big[-r_2+\frac{\sigma_2^2}{2}y^*(1+\frac{a_2x}{1+b_2x})^2\Big](y-y^*)^2\\
+\;&\Big[-r_2+\sigma_2^2y^*(1+\frac{a_2x}{1+b_2x})\Big]\Big[1+\frac{a_2x}{1+b_2x}-a_2\frac{1-x^*-y^*}{(1+b_2x^*)(1+b_2x)}\Big](x-x^*)(y-y^*).
\end{aligned}
$$

Define $V(x,y) = V_1(x,y) + V_2(x,y)$, we compute

$$
\begin{aligned}
LV(x,y) &= LV_1(x,y) + LV_2(x,y)\\
&= A_1(x-x^*)^2 + B_1(y-y^*)^2 + C_1(x-x^*)(y-y^*).
\end{aligned}
$$

When $x-x^* > 0$ *and* $y-y^* > 0$ or $x-x^* < 0$ *and* $y-y^* < 0$, we can easily get $LV(x,y) < 0$.
When $x-x^* > 0$ *and* $y-y^* < 0$, we know that

$$LV(x,y) \le A_2(x-x^*)^2 + B_2(y-y^*)^2 + C_2(x-x^*)(y-y^*).$$

Let $(Z-Z^*) = (x-x^*, y-y^*)^T$. Then

$$LV(x,y) \le \frac{1}{2}(Z-Z^*)\begin{pmatrix} 2A_2 & C_2 \\ C_2 & 2B_2 \end{pmatrix}(Z-Z^*). \tag{12}$$

Therefore $LV(x,y) < 0$.

When $x-x^* < 0$ *and* $y-y^* > 0$, we can also get $LV(x,y) < 0$. Obviously $LV(x,y) < 0$ along all trajectories in R_+^2 except (x^*,y^*). Then we can get the desired assertion immediately. □

3. Numerical simulations

In this section we will use the Milstein method mentioned in Higham [26] to substantiate the analytical findings. Consider the discretization equations:

$$
\begin{aligned}
x_{k+1} &= x_k + r_1x_k\Big[-d_1+\Big(1+\frac{a_1y_k}{1+b_1y_k}\Big)(1-x_k-y_k)\Big]\triangle t\\
&+ \sigma_1x_k\Big[-d_1+\Big(1+\frac{a_1y_k}{1+b_1y_k}\Big)(1-x_k-y_k)\Big]\sqrt{\triangle t}\xi_k\\
&+ \frac{\sigma_1^2}{2}x_k^2\Big[-d_1+\Big(1+\frac{a_1y_k}{1+b_1y_k}\Big)(1-x_k-y_k)\Big]^2(\xi_k^2-1)\triangle t,
\end{aligned}
$$

$$
\begin{aligned}
y_{k+1} &= y_k + r_2 y_k\Big[-d_2 + \Big(1+\frac{a_2x_k}{1+b_2x_k}\Big)(1-x_k-y_k)\Big]\triangle t\\
&+ \sigma_2 y_k\Big[-d_2 + \Big(1+\frac{a_2x_k}{1+b_2x_k}\Big)(1-x_k-y_k)\Big]\sqrt{\triangle t}\eta_k\\
&+ \frac{\sigma_2^2}{2}y_k^2\Big[-d_2 + \Big(1+\frac{a_2x_k}{1+b_2x_k}\Big)(1-x_k-y_k)\Big]^2(\eta_k^2-1)\triangle t,
\end{aligned}
$$

where ξ_k and η_k, $k = 1, 2, ..., n$, are the Gaussian random variables that follow $N(0, 1)$.

As pointed out in Theorem 2.2, if $A_2 < 0$, $4A_2B_2 - C_2^2 > 0$ and $4A_2B_2 - C_3^2 > 0$, then the positive equilibrium position (x^*, y^*) is stochastically asymptotically stable in the large. In all figures, we choose $a_1 = 0.4$, $a_2 = 0.6$, $b_1 = 0.5$, $b_2 = 0.7$ and $x_0 = 0.1$, $y_0 = 0.2$, $\sigma_1 = \sigma_2 = 0.15$, $r_1 = 0.513$, $r_2 = 0.601$, $d_1 = \frac{0.4}{0.513} = 0.78$, $d_2 = \frac{0.5}{0.601} = 0.832$, then $x^* = 0.1736$, $y^* = 0.0653$, $A_2 = -0.509 < 0$, $B_2 = -0.597 < 0$, $C_2 = 0.789$, $C_3 = 0.72$. So we have $A_2 = -0.509 < 0$, $4A_2B_2 - C_2^2 = 0.593 > 0$, $4A_2B_2 - C_3^2 = 0.697 > 0$, i.e. the conditions of Theorem 2.2 are satisfied, thus the stochastic model is stochastically asymptotically stable in the large. Figure 1, Figure 2 and Figure 3 can confirm the conclusion.

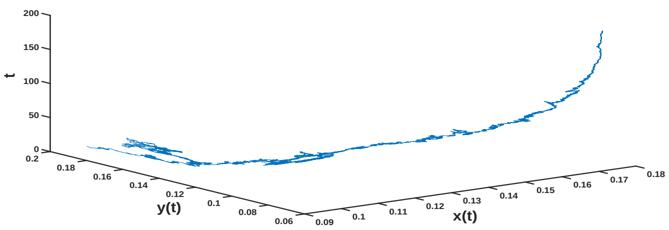

Figure 1: Its globally asymptotic stability in three dimensional space.

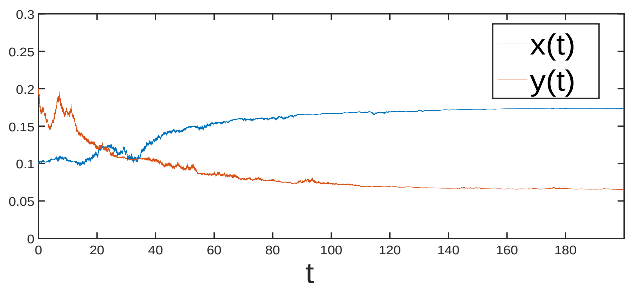

Figure 2: The horizontal axis represents the time t, it reflects the sample path is globally asymptotically stable.

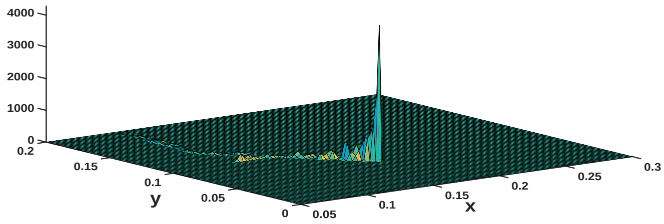

Figure 3: The joint distribution of the system in the three dimensional space.

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