Filomat 33:12 (2019), 3913–3924
https://doi.org/10.2298/FIL1912913A

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Radii of Starlikeness and Convexity for Harmonic Functions Defined by Shear Construction

Rasoul Aghalary[a], **Mehdi Jahani Rad**[a], **Santosh Joshi**[b]

[a] *Department of Mathematics, Faculty of Science, Urmia University, Urmia, Iran*
[b] *Department of Mathematics, Walchand College, Sangli-416415 Maharashtra, India*

Abstract. Using the convolution of harmonic functions, we introduce a generalization for a previously defined class of right half-strip harmonic mappings and determine sharp radii of univalence, full convexity and starlikeness for such functions.

1. Introduction

Let $\mathcal{H}$ be the class of all complex-valued harmonic functions f in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ normalized by $f(0) = f_z(0) - 1 = 0$. It is well known [2] that each $f \in \mathcal{H}$ can be decomposed as $f = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$ such that

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad and \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \qquad (|b_1| < 1). \tag{1}$$

The Jacobian of $f = h + \overline{g}$ is given by $J_f(z) = |h'(z)|^2 - |g'(z)|^2$. According to *Lewy's* Theorem [11], f is locally univalent in $\mathbb{D}$ if and only if $J_f(z) \neq 0$ for any $z \in \mathbb{D}$.

It is also known (see [2]) that necessary and sufficient condition for the harmonic function $f = h + \overline{g}$ to be sense preserving and locally univalent in $\mathbb{D}$ is that the Jacobian J_f is positive in $\mathbb{D}$. Denote by $\mathcal{S}_H$ the class of univalent and orientation-preserving functions $f \in \mathcal{H}$. We note that if $f = h + \overline{g} \in \mathcal{S}_H$ and $g(z) \equiv 0$ in $\mathbb{D}$, then $f = h \in \mathcal{S}$, where $\mathcal{S}$ denotes the well-known class of normalized univalent analytic functions in $\mathbb{D}$. Also let $\mathcal{K}_H$, $(\mathcal{K})$, $\mathcal{S}^*_H$, $(\mathcal{S}^*)$ and $\mathcal{C}_H$, $(\mathcal{C})$ be the subclass of $\mathcal{S}_H$, $(\mathcal{S})$ consisting of mapping $\mathbb{D}$ onto convex, starlike and close-to-convex domains, respectively. Denoted by $\mathcal{K}^0_H, \mathcal{S}^{*0}_H, \mathcal{C}^0_H$ and $\mathcal{S}^0_H$ the class consisting of those functions f in $\mathcal{K}_H, \mathcal{S}^*_H, \mathcal{C}_H$ and $\mathcal{S}_H$ respectively, for which $f_{\overline{z}}(0) = b_1 = 0$.
A harmonic mapping f of $\mathbb{D}$ is said to be fully convex of order α, $0 \le \alpha < 1$, if it maps every circle $|z| = r < 1$ in a one-to-one manner onto a convex curve satisfying

$$\frac{\partial}{\partial\theta}\left(arg\left(\frac{\partial}{\partial\theta} f(re^{i\theta})\right)\right) > \alpha, \quad 0 \le \theta < 2\pi, \; 0 < r < 1.$$

2010 *Mathematics Subject Classification*. Primary 30C45; Secondary 30C80
Keywords. Harmonic starlike and convex functions, Shear construction, Univalent harmonic function, Radii problem
Received: 30 April 2018; Revised: 21 October 2018; Accepted: 13 November 2018
Communicated by Dragan S. Djordjević
Email addresses: raghalary@yahoo.com and r.aghalary@urmia.ac.ir (Rasoul Aghalary), m.jahanirad1365@gmail.com (Mehdi Jahani Rad), joshisb@hotmail.com (Santosh Joshi)

If $\alpha = 0$, then f is said to be fully convex.
Similarly, a harmonic mapping f of $\mathbb{D}$ with $f(0) = 0$ is said to be fully starlike of order α, $0 \le \alpha < 1$, if it maps every circle $|z| = r < 1$ in a one-to-one manner onto a curve that bounds a domain starlike with respect to the origin satisfying

$$\frac{\partial}{\partial\theta}(\arg(f(re^{i\theta}))) > \alpha, 0 \le \theta < 2\pi, 0 < r < 1.$$

If $\alpha = 0$, then f is said to be fully starlike.
Let $\mathcal{FK}_H(\alpha)$ and $\mathcal{FS}^*{}_H(\alpha)$ denote the subclass of $\mathcal{K}_H$ consisting of fully convex functions of order α and the subclass of $\mathcal{S}_H^*$ consisting of fully starlike functions of order α, respectively.

For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$, de Branges [1] obtained the sharp coefficient bound that $|a_n| \le n$, $n \ge 2$. But this coefficient bound is not sufficient for f to be univalent. For example, $f(z) = z + 2z^2$ is clearly not a member of $\mathcal{S}$.
We remark that several subclasses of $\mathcal{S}$ possess a similar coefficient bound. For instance, the nth coefficients of starlike analytic functions, convex analytic functions in the direction of imaginary axis, and close-to-convex functions satisfy $|a_n| \le n\,(n \ge 2)$ (see [14-15]).
Other examples include functions which are convex, starlike of order $\alpha = 1/2$, and starlike with respect to symmetric points. The nth coefficients of these analytic functions satisfy $|a_n| \le 1\,(n \ge 2)$, see [16]. Also we note that a normalized analytic function f with $Ref'(z) > 0$ satisfies $|a_n| \le \frac{2}{n}$ for $n \ge 2$.
Simple examples show that these bounds are not sufficient to characterize the geometric properties of the classes of functions. The problem of determining sharp radius of univalence, or starlikeness of subclasses of analytic or harmonic functions have been investigated by many authors (see [6-8-9-10-16-17]) .
Gavrilov [6] showed that the radius of univalence of normalized analytic functions f satisfying $|a_n| \le n\,(n \ge 2)$ is the real root $r_0 \simeq 0.164$ of the equation $2(1-r)^3 - (1+r) = 0$, and the result is sharp for $f(z) = 2z - \frac{z}{(1-z)^2}$. Gavrilov also proved that the radius of univalence of functions f satisfying the coefficient bound $|a_n| \le M\,(n \ge 2)$ is $1 - \sqrt{\frac{M}{1+M}}$. The condition $|a_n| \le M$ clearly holds for functions $f \in \mathcal{A}$ satisfying $|f(z)| \le M$, and for these functions, Landau [12] proved that the radius of univalence is $M - \sqrt{M^2-1}$. In fact, Yamashita [17] showed that the radius of of univalence obtained by Gavrilov [6] is also the radius of of starlikeness for functions f satisfying $|a_n| \le n$ or $|a_n| \le M$. Additionally, Yamashita [17] determined that the radius of convexity for functions $f \in \mathcal{A}$ satisfying $|a_n| \le n$ is the real root $r_0 \simeq 0.090$ of the equation $2(1-r)^4 - (1 + 4r + r^2) = 0$, while the radius of convexity for functions $f \in \mathcal{A}$ satisfying $|a_n| \le M$ is the real root $(M+1)(1-r)^3 - M(1+r) = 0$.
Recently, Kalaj et al. [9] obtained the radii of univalence, starlikeness, and convexity for harmonic mappings satisfying certain coefficient inequalities. Also Long and Huang [10] obtained the radii of univalence, starlikeness, and convexity for the convolution and convex combination harmonic mappings satisfying certain coefficient inequalities.
The following lammas give sufficient conditions for functions $f \in \mathcal{H}$ to be in the classes $\mathcal{FK}_H(\alpha)$ and $\mathcal{FS}^*{}_H(\alpha)$, respectively.

Lemma 1.1. *([7]). Let $f = h + \overline{g}$, where h and g are given by (1). Furthermore, let*

$$\sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha}|a_n| + \sum_{n=1}^{\infty} \frac{n+\alpha}{1-\alpha}|b_n| \le 1$$

and $0 \le \alpha < 1$. Then f is harmonic univalent in $\mathbb{D}$, and $f \in \mathcal{FS}_H^*(\alpha)$.

Lemma 1.2. *([8]). Let $f = h + \overline{g}$, where h and g are given by (1). Furthermore, let*

$$\sum_{n=2}^{\infty} \frac{n(n-\alpha)}{1-\alpha}|a_n| + \sum_{n=1}^{\infty} \frac{n(n+\alpha)}{1-\alpha}|b_n| \le 1$$

and $0 \le \alpha < 1$. Then f is harmonic univalent in $\mathbb{D}$, and $f \in \mathcal{FK}_H(\alpha)$.

The convolution of two harmonic functions

$$f(z) = h(z) + \overline{g(z)} == z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n},$$

and

$$F(z) = H(z) + \overline{G(z)} == z + \sum_{n=2}^{\infty} A_n z^n + \overline{\sum_{n=1}^{\infty} B_n z^n}.$$

is defined as

$$(f * F)(z) = (h * H)(z) + \overline{(g * G)(z)} == z + \sum_{n=2}^{\infty} a_n A_n z^n + \overline{\sum_{n=1}^{\infty} b_n B_n z^n}.$$

There have been some results about harmonic convolution, (see [2-3-5]). The harmonic convolution $f * F$ of two harmonic functions f and F may not preserve the properties of f or F, such as convexity or even univalence.

In 1984, Clunie and Sheil-Small [2] introduced what is now the well-known shear construction for producing a planar harmonic mapping on $\mathbb{D}$. One interesting example is the harmonic right half-plane mapping $L_0 : \mathbb{D} \mapsto \mathbb{C}$ defined as

$$L_0(z) = \frac{I(z) + zI'(z)}{2} + \overline{\frac{I(z) - zI'(z)}{2}},$$

where $I(z) = z/(1-z)$. Note that L_0 is often considered the harmonic counterpart to the normalized analytic half-plane mapping I. Recently Muir [13] introduced a family of right half- strip harmonic mapping and investigate convolution preserving properties for this family. Motivated by her work we consider the generalized half-plane mappings $L_c : \mathbb{D} \mapsto \mathbb{C}$ defined as

$$L_c(z) = \frac{(1+c)zI'(z) + (1-c)I(z)}{2} + \overline{\frac{(1+c)zI'(z) - (1-c)I(z)}{2}}, \tag{2}$$

where $z \in \mathbb{D}$ and $0 \le c < 1$. For

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=2}^{\infty} b_n z^n$$

define

$$L_c[f, g](z) = L_c(z) * h(z), \tag{3}$$

where $h(z) = f(z) + \overline{g(z)}$. We note that

$$L_c[f, g](z) = z + \sum_{n=2}^{\infty} \left(\frac{(1+c)n + (1-c)}{2} \right) a_n z^n + \overline{\sum_{n=2}^{\infty} \left(\frac{(1+c)n - (1-c)}{2} \right) b_n z^n}.$$

For simplification we set $L_c[f, g](z) = z + \sum_{n=2}^{\infty} A_n z^n + \overline{\sum_{n=2}^{\infty} B_n z^n} = H(z) + \overline{G(z)}$, where

$$A_n = \left(\frac{(1+c)n + (1-c)}{2} \right) a_n \quad \textit{and} \quad B_n = \left(\frac{(1+c)n - (1-c)}{2} \right) b_n.$$

In this paper analogous to the works of Kalaj and et al. [9] and Long and et al. [10] we obtain the radii of univalence, full convexity, and starlikeness of order α, for the $L_c[f,g]$.

To prove our theorems in the next few sections, we shall need the following identities.

$$\sum_{n=1}^{\infty} r^{n-1} = \frac{1}{1-r}, \ \sum_{n=1}^{\infty} nr^{n-1} = \frac{1}{(1-r)^2}, \ \sum_{n=1}^{\infty} n^2r^{n-1} = \frac{1+r}{(1-r)^3},$$
$$\sum_{n=1}^{\infty} n^3r^{n-1} = \frac{1+4r+r^2}{(1-r)^4}, \ \sum_{n=1}^{\infty} n^4r^{n-1} = \frac{(1+r)(1+10r+r^2)}{(1-r)^5}. \tag{4}$$

2. Radius constants concerning $|a_n| \le n, \quad |b_n| \le n$

Theorem 2.1. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $g(z) = \sum_{n=2}^{\infty} b_n z^n$ *with*

$$|a_n| \le n, \quad |b_n| \le n \tag{5}$$

for $n \ge 2$. *Then for* $L_c[f,g]$, $\alpha \in [0,1)$, $c \in [0,1)$,

(1) the radius of full starlikeness of order α *is* r_s, *where* $r_s = r_s(\alpha, c)$ *is the unique real root of the equation*

$$(1+c)(1+4r+r^2) - (1-c)\alpha(1-r)^2 = [2(1-\alpha) + c(1+\alpha)](1-r)^4 \tag{6}$$

in the interval $(0,1)$.
(2) The radius of univalence is r_u, *where* r_u *is the unique real root of the equation*

$$(1+c)(1+4r+r^2) = (2+c)(1-r)^4 \tag{7}$$

in the interval $(0,1)$.
Furthermore, all the results are sharp.

Proof. For $r \in (0,1)$ with $r \le r_s$, it is sufficient to show that $L_c[f_r, g_r] \in \mathcal{FS}_H^*(\alpha)$ in $\mathbb{D}$, where

$$L_c[f_r, g_r](z) = \frac{L_c[f(rz), g(rz)]}{r}$$
$$= z + \sum_{n=2}^{\infty}\left(\frac{(1+c)n + (1-c)}{2}\right) a_n r^{n-1} z^n + \overline{\sum_{n=2}^{\infty}\left(\frac{(1+c)n - (1-c)}{2}\right) b_n r^{n-1} z^n}.$$

Consider the sum

$$S = \sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha}|A_n| r^{n-1} + \sum_{n=2}^{\infty} \frac{n+\alpha}{1-\alpha}|B_n| r^{n-1}.$$

According to Lemma 1.1, it is enough to show that $S \le 1$. By putting the values of $|A_n|$ and $|B_n|$ in the last equation we show that

$$\sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha}\left(\frac{(1+c)n + (1-c)}{2}\right) nr^{n-1} + \sum_{n=2}^{\infty} \frac{n+\alpha}{1-\alpha}\left(\frac{(1+c)n - (1-c)}{2}\right) nr^{n-1} \le 1.$$

Using the identities (4), the last inequality reduces to

$$P(r,c,\alpha) = (1+c)(1+4r+r^2) - (1-c)\alpha(1-r)^2 - [2(1-\alpha) + c(1+\alpha)](1-r)^4 \le 0.$$

We note that $P(0,c,\alpha) = -(1-\alpha) < 0$ and $P(1,c,\alpha) = 6(1+c) > 0$, and so intermediate value theorem shows that the equation (6) has a root in the interval (0, 1). It is easy to check that $P(r,c,\alpha)$ is increasing as a function of r. Thus, $L_c[f_r, g_r] \in \mathcal{FS}^*_H(\alpha)$ for $r \le r_s$, where r_s is the unique real root of (6). Also, taking $\alpha = 0$, equation (6) reduces to (7). Thus by Lemma 1.1, we obtain that $L_c[f,g]$ is harmonic univalent in $|z| \le r_u$, where $r_u = r_s(0,c)$.
To prove sharpness, we take

$$f_0(z) = 2z - \frac{z}{(1-z)^2} \quad \textit{and} \quad g_0(z) = z - \frac{z}{(1-z)^2}.$$

Then $L_c[f_0, g_0](z) = H_0(z) + \overline{G_0(z)}$, where

$$H_0(z) = \left(\frac{1+c}{2}\right)\left(2z - \frac{z(1+z)}{(1-z)^3}\right) + \left(\frac{1-c}{2}\right)\left(2z - \frac{z}{(1-z)^2}\right),$$

$$G_0(z) = \left(\frac{1+c}{2}\right)\left(z - \frac{z(1+z)}{(1-z)^3}\right) - \left(\frac{1-c}{2}\right)\left(z - \frac{z}{(1-z)^2}\right).$$

Direct computation gives us

$$H_0'(r) = \frac{2r^4 - 8r^3 + (12-c)r^2 - 2(5+c)r + 1}{(1-r)^4} \quad \textit{and} \quad G_0'(r) = \frac{(cr^3 - 4cr^2 + (6c-1)r - 6c - 2)r}{(1-r)^4}$$

Considering equation (7), for $r = r_u$, we have

$$H_0'(r_u) + G_0'(r_u) = 0.$$

Hence,

$$J_{L_c[f_0,g_0]}(r_u) = [H_0'(r_u) + G_0'(r_u)][H_0'(r_u) - G_0'(r_u)] = 0.$$

Therefore, in view of *Lewy's* Theorem, the function $L_c[f_0, g_0]$ is not univalent in $|z| < r$ if $r > r_u$. This shows that r_u is sharp.
Furthermore,

$$\begin{aligned}\frac{\partial}{\partial\theta}(arg(L_c[f_0, -g_0](re^{i\theta}))) &= \frac{rH_0'(r) + rG_0'(r)}{H_0(r) - G_0(r)} \\ &= \frac{(c+2)r^4 - 4(c+2)r^3 + (5c+11)r^2 - 4(2c+3)r + 1}{(1-r)^2((2-c)r^2 - 2(2-c)r + 1)}.\end{aligned} \tag{8}$$

At the same time, from equation (6), we have

$$\alpha = \frac{(c+2)r^4 - 4(c+2)r^3 + (5c+11)r^2 - 4(2c+3)r + 1}{(1-r)^2((2-c)r^2 - 2(2-c)r + 1)}. \tag{9}$$

Thus it follows from (8), (9) and for $r = r_s(\alpha, c)$

$$\frac{\partial}{\partial\theta}(arg(L_c[f_0, -g_0](re^{i\theta}))) = \alpha.$$

This shows that bound r_s is the best possible. □

Theorem 2.2. *Under the hypothesis of Theorem* 2.1, $L_c[f,g]$ *is fully convex of order* α *in* $|z| \le r_c$, *where* r_c *is the unique root of the equation*

$$(1+c)(1+r)(1+10r+r^2) - (1-c)\alpha(1+r)(1-r)^2 = [2(1-\alpha) + c(1+\alpha)](1-r)^5 \tag{10}$$

in the interval $(0,1)$. *Moreover, the result is sharp.*

Proof. For $r \in (0,1)$ with $r \leq r_u$, it is sufficient to show that $L_c[f_r, g_r](z) \in \mathcal{FK}_H^*(\alpha)$ in $\mathbb{D}$. The proof of this part of theorem is similar to the argument of the proof of Theorem 2.1 and so we omit details.
To prove sharpness, we take

$$f_0(z) = 2z - \frac{z}{(1-z)^2} \quad and \quad g_0(z) = z - \frac{z}{(1-z)^2}.$$

Then $L_c[f_0, g_0](z) = H_0(z) + \overline{G_0(z)}$, where

$$H_0(z) = \left(\frac{1+c}{2}\right)\left(2z - \frac{z(1+z)}{(1-z)^3}\right) + \left(\frac{1-c}{2}\right)\left(2z - \frac{z}{(1-z)^2}\right),$$

$$G_0(z) = \left(\frac{1+c}{2}\right)\left(z - \frac{z(1+z)}{(1-z)^3}\right) - \left(\frac{1-c}{2}\right)\left(z - \frac{z}{(1-z)^2}\right).$$

Direct computation, gives us

$$\frac{\partial}{\partial\theta}\left(arg\left(\frac{\partial}{\partial\theta}L_c[f_0, g_0](re^{i\theta})\right)\right) = \frac{H_0'(r) + G_0'(r) + r(H_0''(r) + G_0''(r))}{H_0'(r) - G_0'(r)} \tag{11}$$
$$= \frac{-(c+2)r^5 + 5(c+2)r^4 - (11c+21)r^3 + (9-c)r^2 - (16c+21)r + 1}{(1-r)^2((c-2)r^3 + (-3c+6)r^2 + (4c-7)r + 1)}.$$

At the same time, from equation (10), we have

$$\alpha = \frac{-(c+2)r^5 + 5(c+2)r^4 - (11c+21)r^3 + (9-c)r^2 - (16c+21)r + 1}{(1-r)^2((c-2)r^3 + (-3c+6)r^2 + (4c-7)r + 1)}. \tag{12}$$

Thus, from (11) and (12), we have

$$\frac{\partial}{\partial\theta}\left(arg\left(\frac{\partial}{\partial\theta}L_c[f_0, g_0](r_c e^{i\theta})\right)\right) = \alpha.$$

This shows that the bound r_c given by equation (10) is sharp. □

By putting $c = 0$ in the Theorems 2.1 and 2.2 we have the following corollary.

Corollary 2.3. *Let $f_0 = g_0 + \overline{h_0} \in \mathcal{K}_H^0$ and $g, h \in \mathcal{S}^*$. Then for $F = g_0 * g + \overline{h_0 * h}$,*
(1) the radius of full starlikeness of order α is r_s, where $r_s = r_s(\alpha)$ is the unique real root of the equation

$$(1 + 4r + r^2) - \alpha(1-r)^2 = 2(1-\alpha)(1-r)^4$$

in the interval $(0, 1)$.
(2) The radius of univalence is r_u, where r_u is the unique real root of the equation

$$(1 + 4r + r^2) = 2(1-r)^4$$

in the interval $(0, 1)$.
(3) The radius of full convexity of order α is r_c, where $r_c = r_c(\alpha)$ is the unique root of the equation

$$(1+r)(1 + 10r + r^2) - \alpha(1+r)(1-r)^2 = 2(1-\alpha)(1-r)^5$$

in the interval $(0, 1)$.

3. Radius constants concerning $|a_n| \le M, \quad |b_n| \le M$

Theorem 3.1. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *and* $g(z) = \sum_{n=2}^{\infty} b_n z^n$, *with*

$$|a_n| \le M, \ \ |b_n| \le M \tag{13}$$

for $n \ge 2$. *Then for* $L_c[f, g]$, $\alpha \in [0, 1)$, $c \in [0, 1)$, $M > 0$,

(1) the radius of full starlikeness of order α *is* r_s, *where* $r_s = r_s(\alpha, c, M)$ *is the unique real root of the equation*

$$M(1 + c)(1 + r) - M(1 - c)\alpha(1 - r)^2 = [(M + 1)(1 - \alpha) + Mc(1 + \alpha)](1 - r)^3 \tag{14}$$

in the interval $(0, 1)$.
(2) The radius of univalence is r_u, *where* r_u *is the unique real root of the equation*

$$((1 + c)M + 1)r^3 - 3((c + 1)M + 1)r^2 + (4(c + 1)M + 3)r - 1 = 0 \tag{15}$$

in the interval $(0, 1)$.
Furthermore, all the results are sharp.

Proof. The first part of the proof is similar to Theorem 2.1 and so we omit the details. To prove sharpness, we take

$$f_0(z) = z - M\frac{z^2}{1 - z} \quad \text{and} \quad g_0(z) = -M\frac{z^2}{1 - z}.$$

Then $L_c[f_0, g_0](z) = H_0(z) + \overline{G_0(z)}$, where

$$H_0(z) = \left(\frac{1 + c}{2}\right)\left(z - M\frac{z^2(2 - z)}{(1 - z)^2}\right) + \left(\frac{1 - c}{2}\right)\left(z - M\frac{z^2}{1 - z}\right),$$

$$G_0(z) = -\left(\frac{1 + c}{2}\right)M\frac{z^2(2 - z)}{(1 - z)^2} + \left(\frac{1 - c}{2}\right)M\frac{z^2}{1 - z}.$$

Then with a direct computation we have

$$H_0'(r) = \frac{-(M + 1)r^3 + 3(M + 1)r^2 - (3 + (c + 3)M)r + 1}{(1 - r)^3} \quad \text{and} \quad G_0'(r) = \frac{-rM((r^2 - 3r + 3)c + 1)}{(1 - r)^3}.$$

Considering equation (15), for $r = r_u$, we have

$$H_0'(r_u) + G_0'(r_u) = 0.$$

Hence,

$$J_{L_c[f_0,g_0]}(r_u) = [H_0'(r_u) + G_0'(r_u)][H_0'(r_u) - G_0'(r_u)] = 0.$$

Therefore, in view of *Lewy's* Theorem, the function $L_c[f_0, g_0]$ is not univalent in $|z| < r$ if $r > r_u$. This shows that r_u is sharp.
Furthermore,

$$\frac{\partial}{\partial\theta}(arg(L_c[f_0, g_0](re^{i\theta}))) = \frac{rH_0'(r) - rG_0'(r)}{H_0(r) + G_0(r)}$$

$$= \frac{-(1 + (c + 1)M)r^3 + 3((c + 1)M + 1)r^2 - (3 + 4(c + 1)M)r + 1}{(1 - r)^2(1 - (1 + (1 - c)M)r)} \tag{16}$$

At the same time, we have

$$\alpha = \frac{-(1+(c+1)M)r^3 + 3((c+1)M+1)r^2 - (3+4(c+1)M)r + 1}{(1-r)^2(1-(1+(1-c)M)r)} \tag{17}$$

Thus it follows from (16) and (17) and for $r = r_s(\alpha, c, M)$

$$\frac{\partial}{\partial\theta}(arg(L_c[f_0, g_0](re^{i\theta}))) = \alpha.$$

This shows that bound r_s is the best possible. □

Theorem 3.2. *Under the hypothesis of Theorem* 3.1*,* $L_c[f, g]$ *is fully convex of order* α *in* $|z| \leq r_c$*, where* r_c *is the unique root of the equation*

$$M(1+c)(1+4r+r^2) - M(1-c)\alpha(1-r)^2 = [(M+1)(1-\alpha) + Mc(1+\alpha)](1-r)^4 \tag{18}$$

in the interval $(0, 1)$*. Moreover, the result is sharp.*

Proof. The proof of equation (18) is the same as proof of Theorem 3.1 and so we omit the details. Now to prove sharpness, we take

$$f_0(z) = z - M\frac{z^2}{1-z} \quad and \quad g_0(z) = -M\frac{z^2}{1-z}.$$

Then $L_c[f_0, g_0](z) = H_0(z) + \overline{G_0(z)}$, where

$$H_0(z) = \left(\frac{1+c}{2}\right)\left(z - M\frac{z^2(2-z)}{(1-z)^2}\right) + \left(\frac{1-c}{2}\right)\left(z - M\frac{z^2}{1-z}\right)$$

and

$$G_0(z) = -\left(\frac{1+c}{2}\right)M\frac{z^2(2-z)}{(1-z)^2} + \left(\frac{1-c}{2}\right)M\frac{z^2}{1-z}.$$

Direct computation, yields

$$\begin{aligned}\frac{\partial}{\partial\theta}\left(arg\left(\frac{\partial}{\partial\theta}L_c[f_0, g_0](re^{i\theta})\right)\right) &= \frac{H_0'(r) + G_0'(r) + r(H_0''(r) + G_0''(r))}{H_0'(r) - G_0'(r)} \\ &= \frac{1 + M(1+c) - M(1+c)\frac{1+4z+z^2}{(1-z)^4}}{1 + M(1-c)\frac{1}{(1-z)^2}}.\end{aligned} \tag{19}$$

At the same time from equation (18), we have

$$\alpha = \frac{[1+M(1+c)](1-z)^4 - M(1+c)(1+4z+z^2)}{[1+M(1-c)](1-z)^4 - M(1-c)(1-z)^2}. \tag{20}$$

Thus, from (19) and (20), we have

$$\frac{\partial}{\partial\theta}\left(arg\left(\frac{\partial}{\partial\theta}L_c[f_0, g_0](r_c e^{i\theta})\right)\right) = \alpha.$$

This shows that the bound r_c given by equation (18) is sharp. □

Corollary 3.3. *Let $f_0 = g_0 + \overline{h_0} \in \mathcal{K}_H^0$ and $g, h \in \mathcal{K}$. Then for $F = g_0 * g + \overline{h_0 * h}$,*
(1) the radius of full starlikeness of order α is r_s, where $r_s = r_s(\alpha)$ is the unique real root of the equation

$$(1 + r) - \alpha(1 - r)^2 = 2(1 - \alpha)(1 - r)^3$$

in the interval $(0, 1)$.
(2) The radius of univalence is r_u, where r_u is the unique real root of the equation

$$2r^3 - 6r^2 + 7r - 1 = 0$$

in the interval $(0, 1)$.
(3) The radius of full convexity of order α is r_c, where $r_c = r_c(\alpha)$ is the unique root of the equation

$$(1 + 4r + r^2) - \alpha(1 - r)^2 = 2(1 - \alpha)(1 - r)^4$$

in the interval $(0, 1)$.

4. Radius constants concerning $|a_n| \leq \frac{M}{n}$, $|b_n| \leq \frac{M}{n}$

Theorem 4.1. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $g(z) = \sum_{n=2}^{\infty} b_n z^n$, with*

$$|a_n| \leq \frac{M}{n}, \quad |b_n| \leq \frac{M}{n} \tag{21}$$

for $n \geq 2$. Then for $L_c[f, g]$, $\alpha \in [0, 1)$, $c \in [0, 1)$, $M > 0$,

(1) the radius of full starlikeness of order α is r_s, where $r_s = r_s(\alpha, c, M)$ is the unique real root of the equation

$$M(1 + c) + M(1 - c)\alpha \frac{\log(1 - r)}{r}(1 - r)^2 = [(M + 1)(1 - \alpha) + Mc(1 + \alpha)](1 - r)^2. \tag{22}$$

in the interval $(0, 1)$.
(2) The radius of univalence is r_u, where r_u is the unique real root of the equation

$$((c + 1)M + 1)r^2 - 2((c + 1)M + 1)r + 1 = 0 \tag{23}$$

in the interval $(0, 1)$.
Furthermore, all the results are sharp.

Proof. For $r \in (0, 1)$ with $r \leq r_s$, it is sufficient to show that $L_c[f_r, g_r] \in \mathcal{FS}_H^*(\alpha)$ in $\mathbb{D}$, where

$$L_c[f_r, g_r](z) = \frac{L_c[f(rz), g(rz)]}{r} = z + \sum_{n=2}^{\infty} \left(\frac{(1 + c)n + (1 - c)}{2} \right) a_n r^{n-1} z^n + \overline{\sum_{n=2}^{\infty} \left(\frac{(1 + c)n - (1 - c)}{2} \right) b_n r^{n-1} z^n}.$$

Consider the sum

$$S = \sum_{n=2}^{\infty} \frac{n - \alpha}{1 - \alpha} |A_n| r^{n-1} + \sum_{n=2}^{\infty} \frac{n + \alpha}{1 - \alpha} |B_n| r^{n-1}.$$

According to Lemma 1.1, it is enough to show that $S \leq 1$. Putting the coefficients $|A_n|$ and $|B_n|$ in the last equation, we have

$$\sum_{n=2}^{\infty} \frac{n - \alpha}{1 - \alpha} \left(\frac{(1 + c)n + (1 - c)}{2} \right) \frac{M}{n} r^{n-1} + \sum_{n=2}^{\infty} \frac{n + \alpha}{1 - \alpha} \left(\frac{(1 + c)n - (1 - c)}{2} \right) \frac{M}{n} r^{n-1} \leq 1.$$

Using the identities (4), the last inequality reduces to

$$M(1+c)+M(1-c)\alpha\frac{\log(1-r)}{r}(1-r)^2-[(M+1)(1-\alpha)+Mc(1+\alpha)](1-r)^2\leq 0.$$

Set

$$q(r)=M(1+c)+M(1-c)\alpha\frac{\log(1-r)}{r}(1-r)^2-[(M+1)(1-\alpha)+Mc(1+\alpha)](1-r)^2$$

We note that $q(0)=-(1-\alpha)<0$ and $q(1)=M(1+c)>0$, and so intermediate value theorem shows that the equation (22) has a root in the interval (0, 1). Also is easy to verify that $q(r)$ is increasing as a function of r. Hence the equation (22) has exactly one root in the (0, 1).
Thus, $L_c[f_r,g_r]\in\mathcal{FS}^*_H(\alpha)$ for $r\leq r_s$, where r_s is the unique real root of (22). Also, taking $\alpha=0$, equation (14) reduces to (23). Then by Lemma 1.1, we know that $L_c[f,g]$ is harmonic univalent in $|z|\leq r_u$, where $r_u=r_s(0,c,M)$. To prove sharpness, we take

$$f_0(z)=(1+M)z+Mlog(1-z)\quad and\quad g_0(z)=M(z+log(1-z))$$

Then $L_c[f_0,g_0](z)=H_0(z)+\overline{G_0(z)}$, where

$$H_0(z)=\left(\frac{1+c}{2}\right)\left((1+M)z-\frac{Mz}{1-z}\right)+\left(\frac{1-c}{2}\right)((1+M)z+Mlog(1-z)),$$

$$G_0(z)=\left(\frac{1+c}{2}\right)M\left(z-\frac{z}{1-z}\right)-\left(\frac{1-c}{2}\right)M(z+log(1-z))$$

Direct computation, implies

$$H_0'(r)=\frac{2(M+1)r^2-(4+(c+3)M)r+2}{2(1-r)^2}\quad and\quad G_0'(r)=\frac{Mr(2cr-3c-1)}{2(1-r)^2}.$$

According to equation (23), for $r=r_u$, we have

$$H_0'(r_u)+G_0'(r_u)=0.$$

Hence,

$$J_{L_c[f_0,g_0]}(r_u)=[H_0'(r_u)+G_0'(r_u)][H_0'(r_u)-G_0'(r_u)]=0.$$

Therefore, in view of *Lewy's* Theorem, the function $L_c[f_0,g_0]$ is not univalent in $|z|<r$ if $r>r_u$. This shows that r_u is sharp.
Furthermore,

$$\begin{aligned}\frac{\partial}{\partial\theta}(arg(L_c[f_0,g_0](re^{i\theta})))&=\frac{rH_0'(r)+rG_0'(r)}{H_0(r)-G_0(r)}\\&=\frac{r(1+(1+(1+c)M)r^2+(-2+(-2c-2)M)r)}{(1-r)^2(-M(-1+c)log(1-r)-r(-1+M(-1+c)))}.\end{aligned}\tag{24}$$

At the same time, we have

$$\alpha=\frac{r(1+(1+(1+c)M)r^2+(-2+(-2c-2)M)r)}{(1-r)^2(-M(-1+c)log(1-r)-r(-1+M(-1+c)))}.\tag{25}$$

Thus it follows from (24) and (25) and for $r=r_s(\alpha,c,M)$

$$\frac{\partial}{\partial\theta}(arg(L_c[f_0,g_0](re^{i\theta})))=\alpha.$$

This shows that bound r_s is the best possible. □

Theorem 4.2. *Under the hypothesis of Theorem* 4.1, $L_c[f,g]$ *is fully convex of order* α *in* $|z| \le r_c$, *where* r_c *is the unique root of the equation*

$$M(1+c)(1+r) - M(1-c)\alpha(1-r)^2 = [(M+1)(1-\alpha) + Mc(1+\alpha)](1-r)^3 \tag{26}$$

in the interval $(0,1)$. *Moreover, the result is sharp.*

Proof. The proof of (26) is the same as proof of Theorem 4.1 and so we omit details. To prove sharpness, we take

$$f_0(z) = (1+M)z + Mlog(1-z) \quad and \quad g_0(z) = M(z + log(1-z)).$$

Then $L_c[f_0, g_0](z) = H_0(z) + \overline{G_0(z)}$, where

$$H_0(z) = \left(\frac{1+c}{2}\right)\left((1+M)z - \frac{Mz}{1-z}\right) + \left(\frac{1-c}{2}\right)((1+M)z + Mlog(1-z)),$$

$$G_0(z) = \left(\frac{1+c}{2}\right)M\left(z - \frac{z}{1-z}\right) - \left(\frac{1-c}{2}\right)M(z + log(1-z))$$

By direct computation, we have

$$\frac{\partial}{\partial\theta}\left(arg\left(\frac{\partial}{\partial\theta}L_c[f_0,g_0](re^{i\theta})\right)\right) = \frac{H_0'(r) + G_0'(r) + r(H_0''(r) + G_0''(r))}{H_0'(r) - G_0'(r)}$$
$$= \frac{(-1 + (-1-c)M)r^3 + (3 + (3c+3)M)r^2 + (-3 + (-4c-4)M)r + 1}{(1-r)^2((c-1)M-1)r+1)}. \tag{27}$$

Also from equation (26), we obtain

$$\alpha = \frac{(-1 + (-1-c)M)r^3 + (3 + (3c+3)M)r^2 + (-3 + (-4c-4)M)r + 1}{(1-r)^2((c-1)M-1)r+1)}. \tag{28}$$

Thus, relations (27) and (28), yields

$$\frac{\partial}{\partial\theta}\left(arg\left(\frac{\partial}{\partial\theta}L_c[f_0,g_0](r_c e^{i\theta})\right)\right) = \alpha.$$

This shows that the bound r_c given by equation (26) is sharp. □

Corollary 4.3. *Let* $f_0 = g_0 + \overline{h_0} \in \mathcal{K}_H^0$ *and the normalized functions* g, h *satisfy the condition* $Reg'(z) > 0$, $Reh'(z) > 0$. *Then for* $F = g_0 * g + \overline{h_0 * h}$,
(1) the radius of full starlikeness of order α *is* r_s, *where* $r_s = r_s(\alpha)$ *is the unique real root of the equation*

$$2 + 2\alpha\frac{\log(1-r)}{r}(1-r)^2 = 3(1-\alpha)(1-r)^2.$$

in the interval $(0,1)$.
(2) The radius of univalence is r_u, *where* r_u *is the unique real root of the equation*

$$3r^2 - 6r + 1 = 0$$

in the interval $(0,1)$.
(3) The radius of full convexity of order α *is* r_c, *where* $r_c = r_c(\alpha)$ *is the unique root of the equation*

$$2(1+r) - 2\alpha(1-r)^2 = 3(1-\alpha)(1-r)^3$$

in the interval $(0,1)$.

Acknowledgments. The authors would like to express their thanks to the reviewers for many valuable advices regarding a previous version of this paper.

References

[1] L. de Branges, A proof of the bieberbach conjecture, Acta Mathematatica 154(1-2) (1985) 137–152.
[2] J. Clunie and T. Sheil-Small, Harmonic univalent functions, Annales Academia Scientiarum Fennica 9 (1984) 3–25.
[3] M. Droff, Convolutions of plannar harmonic convex mappings, Complex Var. Theory Appl 45 (2001) 263–271.
[4] A. W. Goodman, Univalent functions, Mariner Publishing, Tampa, FL, USA, 1 (1983).
[5] M. R. Goodloe, Hadamard products of convex harmonic mappings, Complex Var. Theory Appl 47 (2)(2002) 81–92.
[6] V. I. Gavrilov, Remarks on the radius of univalence of holomorphic functions, Matematicheskie Zametki 7 (1970) 295–298.
[7] J. Jahangiri, Harmonic functions starlike in the unit disk, J. Math. Anal. Appl 235 (1999) 470–477.
[8] J. Jahangiri, Coefficient bounds and univalence criteria for harmonic functions with negative coefficient, Ann. Univ. Mariae Curie-Sklodowska Sect A 2 (1998) 57–66.
[9] D. Kalaj, S. Ponnusamy and M. Vuorinen, Radius of close-to-convexity of harmonic functions, Complex Var. Elliptic Equ 59 (4) (2014) 539–552.
[10] B. Y. Long and H. Hang, Radii of harmonic mapping in the plane, J. Aust. Math. Soc (2016) 1-7.
[11] H. Lewy, On the non-vanishing of the Jacobian in certain one-to-one mappings, Bull. Amer. Math. Soc 42 (1936) 689–692.
[12] E. Landau, Der Picard-Schottysche Satz und die Blochsche Konstanten, Sitzungsber. Preuss. Akad. Wiss. Berlin Phys.-Math. Kl (1926) 467–474.
[13] S. Muir, Convex combinations of planar harmonic mappings realized through convolutions with half-strip mappings, Bull. Malays. Math. Sci. Soc 40 (2017) 857-880.
[14] R. Nevanlinna, Uber die konforme Abbildung Sterngebieten, Oeversikt av Finska-Vetenskaps Societeten Forthandlingar A 63 (6) (1921) 1–21.
[15] M. O. Reade, On close-to-convex univalent functions, The Michigan Math. J 3 (1955) 59–62.
[16] K. Sakaguchi, On a certain univalent mappings, J. Math. Soc. Japan 11 (1959) 72–75.
[17] S. Yamashita, Radii of univalence, starlikeness, and convexity, Bull. Austral. Math. Soc 25 (3)(1982) 453–457.