



The Study of Global Stability of a Diffusive Beddington-Deangelis and Tanner Predator-Prey Model

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Abstract. In this article, a diffusive Beddington-DeAngelis and Tanner predator-prey model with no-flux boundary condition is investigated, and it is proved that the unique constant equilibrium is globally asymptotically stable under a new simpler parameter condition.

1. Introduction

In this paper, we consider a reaction-diffusion Beddington-DeAngelis and Tanner predator-prey model in the form given in [1]:

$$\begin{cases} u_t = d_1 \Delta u + u - u^2 - \frac{uv}{a + u + v}, & (x, t) \in \Omega \times (0, \infty), \\ v_t = d_2 \Delta v + v \left(\delta - \beta \frac{v}{u} \right), & (x, t) \in \Omega \times (0, \infty), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & (x, t) \in \partial \Omega \times (0, \infty), \\ u(x, 0) = u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0, \quad x \in \bar{\Omega}, \end{cases} \quad (1)$$

where $u(x, t)$ and $v(x, t)$ are the densities of prey and predator, respectively, Ω is a bounded domain with smooth boundary $\partial \Omega$, a , δ and β are positive constants. In this paper we assume that the two diffusion coefficients d_1 and d_2 are the diffusion coefficients corresponding to u and v , respectively, and are positive, but not necessary constants. The admissible initial data $u_0(x)$ and $v_0(x)$ are continuous functions on $\bar{\Omega}$. The homogeneous Neumann boundary condition means that (1) is self-contained and has no population flux across the boundary $\partial \Omega$.

In order to precisely describe the real ecological interactions between species such as mite and spider mite, lynx and hare, sparrow and sparrow hawk, etc. described by Tanner [3], Bianca et al [28, 29] and Wollkind et al. [4], Robert May developed a model, also known as the Holling-Tanner predator-prey model [5], in which he incorporated Holling's rate [7, 8]. Taking into account the inhomogeneous distribution of the species in different spatial locations within a fixed bounded domain $\Omega \subset \mathbb{R}^n$, after a scaling as in [1], this

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system reads as above. The functional response $\frac{uv}{a+u+v}$ was introduced by Beddington [9] and DeAngelis [10]. They proposed the following predator-prey model with Beddington-DeAngelis functional response

$$\begin{cases} \dot{x} = x(r - \theta x) - \frac{Exy}{a + bx + cy}, \\ \dot{y} = -dy + \frac{\beta xy}{a + bx + cy}. \end{cases} \tag{2}$$

Huang et al. [11, 12] proposed a class of virus dynamics model with Beddington-DeAngelis functional response. Liu and Kong [13] studied the dynamics of a predator-prey system with Beddington-DeAngelis functional response and delays.

Besides the Beddington-DeAngelis functional responses mentioned above, there are several other well-known functional responses, such as Holling type (I, II, III, IV), Monod-Haldane type and Hassel-Verley type functional responses etc. Some authors studied and raised some open questions for structured predator-prey models with different types of functional responses. Especially, in [14], Peng and Wang considered the steady states of a diffusive Holling-Tanner predator-prey model

$$\begin{cases} u_t = d_1 \Delta u + au - u^2 - \frac{uv}{m + u}, & (x, t) \in \Omega \times (0, \infty), \\ v_t = d_2 \Delta v + bv - \frac{v^2}{\gamma u}, & (x, t) \in \Omega \times (0, \infty), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & (x, t) \in \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x) > 0, \quad v(x, 0) = v_0(x) \geq 0, \quad x \in \bar{\Omega}. \end{cases} \tag{3}$$

They discussed the existence and non-existence of positive non-constant steady solutions for (3), and proved that (3) has no positive non-constant steady solution under a certain condition. In the another paper [15], by the construction of a Lyapunov function and a standard linearization procedure, they studied the stability of diffusive predator-prey system of Holling-Tanner type (3). Chen and Shi [16] concentrated on the steady states of (3). They used the comparison principle and defined iteration sequences to prove the global stability for the constant positive equilibrium. Their result improves the earlier one given in [15] which was established by Lyapunov method. We also note here that the (non-spatial) kinetic equation of system (3) was first proposed by Tanner [3] and May [17], see also [18, 19].

Recently, Qi and Zhu [20] studied the global stability of diffusive predator-prey system (3). Indeed, in [20], they established improved global asymptotic stability of the unique positive equilibrium solution. For more detailed biological implications of the model, besides the references mentioned above, one can see [21–27].

Motivated by the previous works [1], in this paper by incorporating the diffusion and ratio-dependent Beddington-DeAngelis functional response into system (3), we study the stability of the positive equilibrium solution of (1).

It is effortless to obtain that system (1) possesses a unique positive equilibrium (u^*, v^*) , where

$$u^* = \frac{\beta \left(1 - a + \sqrt{(1 - a)^2 + 4a \left(1 + \frac{\delta}{\beta} \right)} \right)}{2(\beta + \delta)}, \quad v^* = \frac{\delta}{\beta} u^*$$

where $a, \delta, \beta > 0$.

The main goal of our work is to study the local and global stability of the unique positive equilibrium (u^*, v^*) . As a result, we improve some results of [6, 14, 15].

2. Stability of (u^*, v^*) for system (1)

In this section, we study the local and global stability of (u^*, v^*) for system (1). Through this section, let $(u(x, t), v(x, t))$ be the unique solution of (1). It is easily seen that $(u(x, t), v(x, t))$ possesses globally and is

positive, namely, $u(x, t), v(x, t) > 0$ for all $x \in \overline{\Omega}$ and $t > 0$.

2.1. Lyapunov functional

In the theory of ordinary differential equations (ODEs), Lyapunov functions are scalar functions that may be used to prove the stability of an equilibrium of an ODE. Named after the Russian mathematician Aleksandr Mikhailovich Lyapunov, Lyapunov functions (also called the Lyapunovs second method for stability) are important to stability theory of dynamical systems and control theory. A similar concept appears in the theory of general state space Markov chains, usually under the name Foster-Lyapunov functions.

For certain classes of ODEs, the existence of Lyapunov functions is a necessary and sufficient condition for stability. Whereas there is no general technique for constructing Lyapunov functions for ODEs, in many specific cases the construction of Lyapunov functions is known. For instance, quadratic functions suffice for systems with one state; the solution of a particular linear matrix inequality provides Lyapunov functions for linear systems; and conservation laws can often be used to construct Lyapunov functions for physical systems.

Lemma 2.1 (Globally asymptotically stable equilibrium). *If the Lyapunov-candidate-function V is globally positive definite, radially unbounded and the time derivative of the Lyapunov-candidate-function is globally negative definite:*

$$\dot{V}(x) < 0 \quad \forall x \in \mathbb{R}^n \setminus \{0\},$$

then the equilibrium is proven to be globally asymptotically stable.

The Lyapunov-candidate-function $V(x)$ is radially unbounded if

$$\|x\| \rightarrow \infty \Rightarrow V(x) \rightarrow \infty.$$

(This is also referred to as norm-coercivity.)

2.2. Local stability of (u^*, v^*) for system (1)

In this subsection, we will devote consideration to the local stability of (u^*, v^*) for (1). For this purpose, we need to introduce some notation.

Let $0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots$ be the eigenvalues of the operator $-\Delta$ in Ω with the homogeneous Neumann boundary condition, and $E(\lambda_i)$ be the eigenspace corresponding to λ_i in $C^2(\overline{\Omega})$. Let $\mathbf{X} = \{(u, v) \in [C^2(\overline{\Omega})]^2 \mid \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega, \{\phi_{ij} : j = 1, 2, \dots, \dim E(\lambda_i)\} \text{ be an orthonormal basis of } E(\lambda_i), \text{ and } \mathbf{X}_{ij} = \{c\phi_{ij} \mid c \in \mathbb{R}^2\}$. Then,

$$\mathbf{X} = \bigoplus_{i=0}^{\infty} \mathbf{X}_i \quad \text{and} \quad \mathbf{X}_i = \bigoplus_{j=1}^{\dim E(\lambda_i)} \mathbf{X}_{ij}.$$

Before discussing the local stability of (u^*, v^*) , using the definition of (u^*, v^*) , we prove the following theorem.

Theorem 2.2. *Suppose that the parameters a, β, δ are all positive. Then the following holds:*

$$1 - 2u^* - \frac{(a + v^*)v^*}{(a + u^* + v^*)^2} = \frac{u^*}{a + u^* + v^*} \left(1 - a - \left(2 + \frac{\delta}{\beta} \right) u^* \right).$$

Proof. From the definitions of u^* and v^* , we easily see that

$$\left(1 + \frac{\delta}{\beta}\right)(u^*)^2 + (a-1)u^* - a = 0,$$

$$\frac{\delta}{\beta}u^* = (1-u^*)\left(a + u^* + \frac{\delta}{\beta}u^*\right)$$

and

$$v^* = \frac{\delta}{\beta}u^*.$$

Then direct computations deduce that

$$\begin{aligned} & 1 - 2u^* - \frac{(a+v^*)v^*}{(a+u^*+v^*)^2} \\ &= -u^* + \frac{v^*}{a+u^*+v^*} - \frac{(a+v^*)v^*}{(a+u^*+v^*)^2} \\ &= \frac{u^*}{(a+u^*+v^*)^2} \left(-(a+u^*+v^*)^2 + \frac{\delta}{\beta}(a+u^*+v^*) - \frac{\delta}{\beta}(a+v^*) \right) \\ &= \frac{u^*}{a+u^*+v^*} \left(1 - a - \left(2 + \frac{\delta}{\beta}\right)u^* \right). \end{aligned}$$

This finishes the proof. \square

With the above lemma, the linearized problem of (1) at (u^*, v^*) is

$$\begin{cases} u_t - d_1 \Delta u = \frac{u^* \left(1 - a - \left(2 + \frac{\delta}{\beta}\right)u^*\right)}{a + u^* + v^*} (u - u^*) - \frac{u^*}{a + u^* + v^*} (v - v^*) \\ (x, t) \in \Omega \times (0, \infty), \\ v_t - d_2 \Delta v = \frac{\delta^2}{\beta} (u - u^*) - \delta (v - v^*) \\ (x, t) \in \Omega \times (0, \infty). \end{cases}$$

That is,

$$\begin{cases} u_t - d_1 \Delta u = \frac{u^* \left(1 - a - \left(2 + \frac{\delta}{\beta}\right)u^*\right)}{a + u^* + v^*} u - \frac{u^*}{a + u^* + v^*} v & (x, t) \in \Omega \times (0, \infty), \\ v_t - d_2 \Delta v = \frac{\delta^2}{\beta} u - \delta v & (x, t) \in \Omega \times (0, \infty). \end{cases}$$

Theorem 2.3. Suppose $d = d(x, t)$ is strictly positive, bounded and continuous in $\Omega \times [0, +\infty)$, a , δ and β are positive constants, $\delta < 1$, then (u^*, v^*) is uniformly asymptotically stable for (1) in the sense of [2].

Proof. Define

$$\mathcal{L} = \begin{pmatrix} d_1 \Delta + \frac{u^* \left(1 - a - \left(2 + \frac{\delta}{\beta}\right)u^*\right)}{\frac{\delta^2}{\beta} a + u^* + v^*} & -\frac{u^*}{a + u^* + v^*} \\ & d_2 \Delta - \delta \end{pmatrix}$$

For each $i, i = 0, 1, 2, \dots$, X_i is invariant under the operator \mathcal{L} , and ξ is an eigenvalue of \mathcal{L} on X_i if and only if ξ is an eigenvalue of the matrix

$$\mathcal{A}_i = \begin{pmatrix} -d_1\lambda_i + \frac{u^*(1-a-(2+\frac{\delta}{\beta})u^*)}{\frac{\delta^2}{\beta} a+u^*+v^*} & -\frac{u^*}{a+u^*+v^*} \\ \frac{\delta^2}{\beta} & -d_2\lambda_i - \delta \end{pmatrix}.$$

Since

$$\det \mathcal{A}_i = d_1 d_2 \lambda_i^2 + \left[d_1 \delta - d_2 \frac{u^*(1-a-(2+\frac{\delta}{\beta})u^*)}{a+u^*+v^*} \right] \lambda_i - \frac{\delta u^*(1-a-(2+\frac{\delta}{\beta})u^*)}{a+u^*+v^*} + \frac{\frac{\delta^2}{\beta} u^*}{a+u^*+v^*},$$

and

$$\text{Tr} \mathcal{A}_i = -(d_1 + d_2)\lambda_i + \frac{u^*(1-a-(2+\frac{\delta}{\beta})u^*)}{a+u^*+v^*} - \delta,$$

where $\det \mathcal{A}_i$ and $\text{Tr} \mathcal{A}_i$ are respectively the determinant and trace of \mathcal{A}_i , by Theorem 2.2 it is easy to check that $\det \mathcal{A}_i > 0$, and $\text{Tr} \mathcal{A}_i < 0$ for all $i \geq 0$. Therefore, two eigenvalues ξ_i^+ and ξ_i^- of \mathcal{A}_i have negative real parts. Since λ_i is increasing with i and $\lambda_i \rightarrow \infty$ as $i \rightarrow \infty$, it follows that

$$\text{Re } \xi_0^\pm = \frac{1}{2} \left(\frac{u^*(1-a-(2+\frac{\delta}{\beta})u^*)}{a+u^*+v^*} - \delta \right) < 0,$$

and for any $i \geq 1$ the following hold:

(i) If $(\text{Tr} \mathcal{A}_i)^2 - 4 \det \mathcal{A}_i \leq 0$ then

$$\begin{aligned} \text{Re } \xi_i^\pm &= \frac{1}{2} \text{Tr} \mathcal{A}_i = \frac{1}{2} \left(-(d_1 + d_2)\lambda_i + \frac{u^*(1-a-(2+\frac{\delta}{\beta})u^*)}{a+u^*+v^*} - \delta \right) \\ &\leq \frac{1}{2} \left(-(d_1 + d_2)\lambda_1 + \frac{u^*(1-a-(2+\frac{\delta}{\beta})u^*)}{a+u^*+v^*} - \delta \right) < 0. \end{aligned}$$

(ii) If $(\text{Tr} \mathcal{A}_i)^2 - 4 \det \mathcal{A}_i > 0$ then (since $\det \mathcal{A}_i > 0$ and $\text{Tr} \mathcal{A}_i < 0$)

$$\text{Re } \xi_i^- = \frac{1}{2} \left(\text{Tr} \mathcal{A}_i - \sqrt{(\text{Tr} \mathcal{A}_i)^2 - 4 \det \mathcal{A}_i} \right) \leq \frac{1}{2} \text{Tr} \mathcal{A}_i \leq \frac{1}{2} \left(-(d_1 + d_2)\lambda_1 + \frac{u^*(1-a-(2+\frac{\delta}{\beta})u^*)}{a+u^*+v^*} - \delta \right) < 0,$$

$$\text{Re } \xi_i^+ = \frac{1}{2} \left(\text{Tr} \mathcal{A}_i + \sqrt{(\text{Tr} \mathcal{A}_i)^2 - 4 \det \mathcal{A}_i} \right) = \frac{2 \det \mathcal{A}_i}{\text{Tr} \mathcal{A}_i - \sqrt{(\text{Tr} \mathcal{A}_i)^2 - 4 \det \mathcal{A}_i}} \leq \frac{\det \mathcal{A}_i}{\text{Tr} \mathcal{A}_i} < -\epsilon$$

for some $\epsilon > 0$ which does not depend on i . The above arguments show that there exists a positive constant ϵ , which does not depend on i , such that $\text{Re } \xi_i^\pm < -\epsilon$ for all i . And consequently, the spectrum of \mathcal{L} lies in $\{\text{Re } \xi < -\epsilon\}$ (since the spectrum of \mathcal{L} consists of eigenvalues). By Theorem 5.1.1 of [2] we know that our result holds. The proof is complete. \square

2.3. Global stability of (u^*, v^*) for system (1)

First, we state a persistence property $(u(x, t), v(x, t))$, which implies that the prey and predator will always coexist at any time and any location of the habitat domain, no matter what their diffusion coefficients are.

Theorem 2.4. For $0 < \varepsilon \ll 1$, there exists $t_0 \gg 1$ such that the solution $(u(x, t), v(x, t))$ of (1) satisfies

$$K - \varepsilon < u(x, t) < 1 + \varepsilon, \quad \frac{\delta}{\beta}K - \varepsilon < v(x, t) < \frac{\delta}{\beta} + \varepsilon,$$

for all $x \in \overline{\Omega}$ and $t \geq t_0$, where K is defined by

$$K = \frac{1}{2} \left\{ \frac{1 - a + \sqrt{(1 - a)^2 + 4\left(a - \frac{\delta}{\beta}\right)}}{1 + \frac{\delta}{\beta}} \right\}.$$

Proof. For $0 < \varepsilon \ll 1$, from the first equation in (1) it is clear there exists $t_0 \gg 1$ such that $u(x, t) < 1 + \varepsilon$ for all $x \in \overline{\Omega}$ and $t \geq t_0$ by the comparison principle for the parabolic equation. Hence, $v(x, t)$ is a lower solution of the following problem:

$$\begin{cases} \frac{\partial z}{\partial t} - d_2 \Delta z = \frac{\frac{\delta}{\beta}(1 + \varepsilon) - z}{\frac{1}{\beta}(1 + \varepsilon)} z & \text{in } \Omega \times (t_0, \infty), \\ \frac{\partial z}{\partial \nu} = 0 & \text{on } \partial\Omega \times (t_0, \infty), \\ z(x, t_0) = v(x, t_0) > 0 & \text{on } \overline{\Omega}. \end{cases} \tag{4}$$

Let $v(t)$ be the unique positive solution of the problem

$$\begin{cases} w_t = \frac{\frac{\delta}{\beta}(1 + \varepsilon) - w}{\frac{1}{\beta}(1 + \varepsilon)} w & \text{in } (t_0, \infty), \\ w(t_0) = \max_{\overline{\Omega}} v(x, t_0) > 0. \end{cases}$$

Then $v(t)$ is an upper solution of (4). As $\lim_{t \rightarrow \infty} v(t) = \frac{\delta}{\beta}(1 + \varepsilon)$, taking larger t_0 if necessary, we get from the comparison principle that $v(x, t) < v(t) + \varepsilon < \frac{\delta}{\beta} + (\frac{\delta}{\beta} + 1)\varepsilon$ for all $x \in \overline{\Omega}$ and $t \geq t_0$. Thus, we can think that

$$v(x, t) < v(t) + \varepsilon < \frac{\delta}{\beta} + \varepsilon, \quad \forall x \in \overline{\Omega} \text{ and } t \geq t_0$$

In the following proof, without special statement, a disposal similar to the above will be made.

As a result, by the first equation in (1) again, we have that $u(x, t)$ is an upper solution of

$$\begin{cases} \frac{\partial z}{\partial t} - d_2 \Delta z = \frac{-z^2 + (1 - a)z + \left(a - \frac{\delta}{\beta}\right) - \varepsilon}{a + \left(1 + \frac{\delta}{\beta}\right)z} z & \text{in } \Omega \times (t_0, \infty), \\ \frac{\partial z}{\partial \nu} = 0 & \text{on } \partial\Omega \times (t_0, \infty), \\ z(x, t_0) = u(x, t_0) > 0 & \text{on } \overline{\Omega}. \end{cases} \tag{5}$$

Let $u(t)$ be the solution of the problem

$$\begin{cases} w_t = \frac{-w^2 + (1-a)w + (a - \frac{\delta}{\beta}) - \varepsilon}{a + (1 + \frac{\delta}{\beta})w} w & \text{in } (t_0, \infty), \\ w(t_0) = \min_{\bar{\Omega}} u(x, t_0) > 0. \end{cases}$$

Then $u(t)$ is a lower solution of (5), simple analysis shows that

$$\lim_{t \rightarrow \infty} u(t) = \frac{1}{2} \left\{ \frac{1 - a + \sqrt{(1-a)^2 + 4 \left[\left(a - \frac{\delta}{\beta} \right) - \varepsilon \right]}}{1 + \frac{\delta}{\beta}} \right\}$$

Therefore, we deduce that for $0 < \varepsilon \ll 1$, $x \in \bar{\Omega}$ and $t \geq t_0$.

$$u(x, t) \geq \frac{1}{2} \left\{ \frac{1 - a + \sqrt{(1-a)^2 + 4 \left(a - \frac{\delta}{\beta} \right)}}{1 + \frac{\delta}{\beta}} \right\} - \varepsilon = K - \varepsilon.$$

Using the above result, we can similarly verify that $v(x, t)$ satisfies the positive lower bounds in Theorem 2.4. The proof is complete. \square

Now, we give the result of global stability of (u^*, v^*) for (1). Biologically, our statement of the global stability of (\bar{u}, \bar{v}) means that, however quickly or slowly the two species diffuse, they will be spatially homogeneously distributed as time converges to infinity.

Theorem 2.5. *Suppose $d = d(x, t)$ is strictly positive, bounded and continuous in $\Omega \times [0, +\infty)$, a , δ and β are positive constants, $\delta < 1$, then the positive equilibrium solution (u^*, v^*) is globally asymptotically stable in the sense that every solution $u(x, t)$ of (1) satisfies*

$$\lim_{t \rightarrow \infty} (u(x, t), v(x, t)) = (u^*, v^*)$$

uniformly in $x \in \Omega$.

Proof. Let $(u(x, t), v(x, t))$ be the solution of (1). To prove our statement, we need to construct a Lyapunov functional. To this end, adapting the Lyapunov functional in [6] (see 2.1), we define

$$W(u, v) = \int \frac{u - u^*}{u^2} du + \alpha \int \frac{v - v^*}{v} dv,$$

$$E(t) = \int_{\Omega} W(u(x, t), v(x, t)) dx,$$

where α is a positive constant to be determined later.

By simple computations, it follows that

$$\begin{aligned} \frac{dE(t)}{dt} &= \int_{\Omega} \{W_u(u(x, t), v(x, t))u_t + W_v(u(x, t), v(x, t))v_t\} dx \\ &= \int_{\Omega} \left\{ \frac{u - u^*}{u^2} d_1 \Delta u + \alpha \frac{v - v^*}{v} d_2 \Delta v + \frac{u - u^*}{u} \left(1 - u - \frac{v}{a + u + v} \right) + \alpha(v - v^*) \left(\delta - \beta \frac{v}{u} \right) \right\} dx \\ &= - \int_{\Omega} \left\{ d_1 \frac{2u^* - u}{u^3} |\nabla u|^2 + d_2 \alpha \frac{v^*}{v^2} |\nabla v|^2 \right\} dx + \int_{\Omega} \left\{ \frac{u - u^*}{u} \left(u^* + \frac{v^*}{a + u^* + v^*} - u - \frac{v}{a + u + v} \right) \right. \\ &\quad \left. + \alpha(v - v^*) \left(\beta \frac{v^*}{u^*} - \beta \frac{v}{u} \right) \right\} dx \\ &= - \int_{\Omega} \left\{ d_1 \frac{2u^* - u}{u^3} |\nabla u|^2 + d_2 \alpha \frac{v^*}{v^2} |\nabla v|^2 \right\} dx + \int_{\Omega} \left\{ (u - u^*)^2 \left(-\frac{1}{u} + \frac{a - u^*}{u(a + u^* + v^*)} \right) \right. \\ &\quad \left. + (u - u^*)(v - v^*) \left(\frac{\delta \alpha}{u} - \frac{1}{u(a + u + v)} \right) - (v - v^*)^2 \frac{\alpha \beta}{u} \right\}, \end{aligned}$$

We note that $2u^* - u > 0$ for $x \in \bar{\Omega}$ and $t \gg 1$. Therefore, suppose that there exist positive constants α and t_0 such that for all $x \in \bar{\Omega}$ and $t \geq t_0$; the solution $(u(x, t), v(x, t))$ of (1) satisfies

$$(u - u^*)^2 \left(-\frac{1}{u} + \frac{1 - u^*}{u(a + u^* + v^*)} \right) + (u - u^*)(v - v^*) \left(\frac{\delta \alpha}{u} - \frac{1}{u(a + u + v)} \right) - (v - v^*)^2 \frac{\alpha \beta}{u} \leq 0 \tag{6}$$

and then $E'(t) \leq 0$ for $t \geq t_0$. Applying some standard argument, together with Theorem 2.4 again, it follows that

$$(u(x, t), v(x, t)) \rightarrow (u^*, v^*) \quad \text{in } [L^\infty(\Omega)]^2.$$

This shows that (u^*, v^*) attracts all solutions of (1). On the other hand, by Theorem 2.2, we see that (u^*, v^*) is global asymptotically stable. Thus, the whole proof is complete. \square

Remark 2.6. We note that

$$\lim_{a \rightarrow \infty} u^* = \lim_{a \rightarrow \infty} K = 1, \quad \lim_{\frac{\delta}{\beta} \rightarrow 0} u^* = \lim_{\frac{\delta}{\beta} \rightarrow 0} K = 1.$$

We observe that, as $\frac{\delta}{\beta} \rightarrow a$, $u^* \rightarrow \frac{1}{2} \{1 - 2a + \sqrt{1 + 4a^2}\}$, $K \rightarrow 0$ if $1 \leq a$ and $K \rightarrow 1 - a$ if $1 > a$. Hence, simple analysis shows that (u^*, v^*) is globally asymptotically stable for (1) if one of the following cases holds:

- (1) a is large enough;
- (2) $\frac{\delta}{\beta}$ is small enough;
- (3) $\frac{\delta}{\beta}$ is close to but less than a and $1 \leq a$;
- (4) $\frac{\delta}{\beta}$ is close to but less than a , $a < 1$ and $1 - a + \sqrt{1 + 4a^2} > (1 + a)a$.

We also point out that our result of Theorem 2.4 improves (i) of Theorem 3.5 in [14]. Roughly speaking, if the intrinsic growth δ of the predator is small, or the saturation rate of the prey is large, both of the species will be in homogeneous distribution as time converges to infinity.

Remark 2.7. In the proof of Theorem 2.5, if we choose some other kinds of Lyapunov functionals, for example, if we take

$$W(u, v) = \int \frac{u - u^*}{u} du + \alpha \int \frac{v - v^*}{v} dv$$

or

$$W(u, v) = \int \frac{u^2 - (u^*)^2}{u^2} du + \alpha \int \frac{v - v^*}{v} dv$$

where α is a positive constant to be determined, similar analysis shows that (u^*, v^*) is globally asymptotically stable for (1).

Remark 2.8. Let us denote by $(u(t), v(t))$ the solution of the corresponding kinetic dynamics of (1):

$$\begin{cases} \dot{u} = u - u^2 - \frac{uv}{a + u + v}, & t > 0, u(0) > 0, \\ \dot{v} = v \left(\delta - \beta \frac{v}{u} \right), & t > 0, u(0) > 0. \end{cases} \quad (7)$$

If we replace $(u(x, t), v(x, t))$ by $(u(t), v(t))$ in Theorem 2.4 and Theorem 2.5, since $(u(t), v(t))$ is also the solution of (1), Theorem 2.4 is true for $(u(t), v(t))$ and (u^*, v^*) is also globally asymptotically stable for (7) under the same condition as for Theorem 2.5. Therefore, in comparison with [6], simple analysis shows that, in some ranges of parameter, our result obtained here for the global stability of (u^*, v^*) is not included by Theorem 3.2 in [6]. That is, in some cases, we also improve the result of the global stability of (u^*, v^*) for (7) in [6].

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