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Pseudospectra in a Non-Archimedean Banach Space and Essential Pseudospectra in $\mathbb{E}_\omega$

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Abstract. In this work, we introduce and study the pseudospectra and the essential pseudospectra of linear operators in a non-Archimedean Banach space and in the non-Archimedean Hilbert space $\mathbb{E}_\omega$, respectively. In particular, we characterize these pseudospectra. Furthermore, inspired by T. Diagana and F. Ramaroson [12], we establish a relationship between the essential pseudospectrum of a closed linear operator and the essential pseudospectrum of this closed linear operator perturbed by completely continuous operator in the non-Archimedean Hilbert space $\mathbb{E}_\omega$.

1. Introduction

Non-Archimedean functional analysis has long whetted the interest and drew the attention of various researchers. Indeed, it was introduced independently by F. Q. Gouva [13], A. F. Monna [16, 17], P. Schneider [18] and A. C. M. van Rooij [22]. One of the main objectives of this theory is to study the operator theory. The theory of a non-Archimedean operators, from which many valued results were obtained, has been extensively studied (see [5–7, 9]). In recent years, a number of papers presented by diverse authors about the spectral theory of linear operators in a non-Archimedean Banach and Hilbert space have appeared such as, e.g., [10–12].

The principal aim of this paper is to extend the concept of pseudospectra (resp. essential pseudospectra) to in a non-Archimedean Banach space (resp. Hilbert space $\mathbb{E}_\omega$). This work is devoted to the investigation of some properties as well as the characterization of these pseudospectra.

There are many interesting works on pseudospectra in the classical Banach space (see [8, 19–21, 23]). In [23], J. M. Varah was the pioneer consider this notion. In [19, 20], L. N. Trefethen developed the pseudospectrum for matrices and operators, and applied it to such multiple fields of science as mathematics and physics. This concept of the closed linear operator A is usually defined by

$$\sigma_\varepsilon(A) := \sigma(A) \cup \Big\{\lambda \in \mathbb{C} : \|(\lambda - A)^{-1}\| > \frac{1}{\varepsilon}\Big\},$$

where $\varepsilon > 0$ and $\sigma(A)$ is the spectrum of the operator A. By convention, $\lambda \in \sigma(A)$ if, and only if, $\|(\lambda - A)^{-1}\| = \infty$. In [8], E. B. Davies found a characterization of the pseudospectrum of the closed linear operator A, for

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every $\varepsilon > 0$, defined by

$$\sigma_\varepsilon(A) = \bigcup_{\|B\|<\varepsilon} \sigma(A + B).$$

This is equivalent to saying that a number λ belongs to the spectrum of some perturbed $A + B$ with $\|B\| < \varepsilon$ if, and only if, it belongs to the pseudospectrum of A.
In the literature, there are several definitions of the essential spectrum of a closed densely defined linear operator in a Banach space, which are not equivalent. In particular, the Weyl essential spectrum of the closed densely defined linear operator A in a Banach space X is defined by

$$\sigma_w(A) = \bigcap_{K\in\mathcal{K}(X)} \sigma(A + K),$$

where $\mathcal{K}(X)$ designates the subspace of all compact operator on X. Inspired by the notions of pseudospectra and essential spectrum, the new concept of the essential pseudospectra was declared (see [1–4]). In [2], A. Ammar and A. Jeribi introduced the notion of Weyl pseudospectra of densely closed, linear operator A in a Banach space X which is defined by

$$\sigma_{w,\varepsilon}(A) = \bigcap_{K\in\mathcal{K}(X)} \sigma_\varepsilon(A + K),$$

and characterized by

$$\sigma_{w,\varepsilon}(A) = \bigcup_{\|B\|<\varepsilon} \sigma_w(A + B), \text{(see [2, Remark 2. 4]).}$$

Our paper is organized in the following way: In Section 2, some results are recalled from the operator theory and the spectral theory in a non-Archimedean Banach space. Such results are used to prove the main theorems. In Section 3, some properties of the pseudospectra (resp. the essential pseudospectra) of a linear operator in the non-Archimedean Banach space (resp. Hilbert space $\mathbb{E}_\omega$) are introduced and studied. The main focus of this section is Theorems 3.4 and 3.11 in where we investigate a characterization of these pseudospectra.

2. Preliminary and auxiliary results

The goal of this section consists in recalling the basic concepts of non-Archimedean functional analysis, including valuation, norm and Banach space, and some results of the theory of linear operator in a non-Archimedean Banach space needed in the sequel.

Definition 2.1. *Let $\mathbb{K}$ be a field. A valuation on $\mathbb{K}$ is a map $|\cdot| : \mathbb{K} \longrightarrow \mathbb{R}$ satisfying*
(*i*) $|x| \geq 0$ *for any $x \in \mathbb{K}$ with equality only for $x = 0$.*
(*ii*) $|xy| = |x| \cdot |y|$ *for any $x, y \in \mathbb{K}$.*
(*iii*) *For some real number $c \geq 1$ and any $x \in \mathbb{K}$, if $|x| \leq 1$, then $|x + 1| \leq c$.* ◊

Definition 2.2. (*i*) *A valuation $|\cdot|$ on $\mathbb{K}$ satisfies the triangle inequality if for any $x, y \in \mathbb{K}$*

$$|x + y| \leq |x| + |y|.$$

(*ii*) *A valuation $|\cdot|$ on $\mathbb{K}$ satisfies the ultrametric inequality if for any $x, y \in \mathbb{K}$*

$$|x + y| \leq \max\{|x|, |y|\}. \tag{1}$$

(*iii*) *A valuation on $\mathbb{K}$ is called non-Archimedean if it satisfies* (1). ◊

Remark 2.3. (*i*) [12, Proposition 1.6] *Let* $|\cdot|$ *be a valuation on* $\mathbb{K}$*, then it satisfies the triangle inequality if, and only if, one can take* $c = 2$.

(*ii*) [12, Proposition 1.13] *Let* $|\cdot|$ *be a valuation on* $\mathbb{K}$*, then it satisfies the ultrametric inequality if, and only if, one can take* $c = 1$.

(*iii*) *It is known that there are two kinds of valuation, one is the Archimedean valuation, as in the cases of* $\mathbb{C}$ *and* $\mathbb{R}$*, and the other is the non-Archimedean valuation.* ◊

In the sequel of the paper, let $(\mathbb{K}, |\cdot|)$ be a complete non-Archimedean filed. Now, we shall recall a some basics properties of non-Archimedean norm.

Definition 2.4. *Let* X *be a vector space over* $\mathbb{K}$*. A non-Archimedean norm on* X *is a map* $\|\cdot\| : X \to \mathbb{R}_+^*$ *satisfying*

(*i*) $\|x\| = 0$ *if, and only if,* $x = 0$,

(*ii*) $\|\lambda x\| = |\lambda|\,\|x\|$ *for any* $x \in X$ *and any* $\lambda \in \mathbb{K}$*, and*

(*iii*) $\|x + y\| \leq \max\Big\{\|x\|, \|y\|\Big\}$ *for any* $x, y \in X$. ◊

Definition 2.5. *A non-Archimedean Banach space is a non-Archimedean normed vector space, which is complete.* ◊

Now, let us recall the space $\mathbb{E}_\omega$ which plays a very important role in the sequel. The reader interested in this space may also refer to T. Diagana and F. Ramaroson [12].

Definition 2.6. *Let* $\omega = (\omega_i)_i$ *be a sequence of non-zero elements in* $\mathbb{K}$*. We define* $\mathbb{E}_\omega$ *by*

$$\mathbb{E}_\omega = \Big\{x = (x_i)_i : x_i \in \mathbb{K}, \forall\, i \in \mathbb{N} \text{ and } \lim_{i \to +\infty} |\omega_i|^{1/2}|x_i| = 0\Big\},$$

and it is equipped with the norm

$$x = (x_i)_i \subset \mathbb{E}_\omega,\ \|x\| = \sup_{i \in \mathbb{N}} \Big(|\omega_i|^{1/2}|x_i|\Big). \quad \diamond$$

Remark 2.7. (*i*) [12, Example 2.21] *The space* $\Big(\mathbb{E}_\omega, \|\cdot\|\Big)$ *is a non-Archimedean Banach space.*

(*ii*) [12, Section 2.4] *For* $x = (x_i)_i$ *and* $y = (y_i)_i$*, the inner product is defined by*

$$\begin{array}{rcl} \langle\cdot,\cdot\rangle : \mathbb{E}_\omega \times \mathbb{E}_\omega & \longrightarrow & \mathbb{K} \\ (x, y) & \longmapsto & \displaystyle\sum_{i=0}^{+\infty} x_i y_i w_i. \end{array}$$

Hence, the space $\Big(\mathbb{E}_\omega, \|\cdot\|, \langle\cdot,\cdot\rangle\Big)$ *is called a p-adic (or non-Archimedean) Hilbert space.*

(*iii*) [12, Remark 2.44] *The orthogonal basis* $\{e_i : i = 0, 1, 2, \dots\}$ *is called the canonical basis of* $\mathbb{E}_\omega$. ◊

Let $(X, \|\cdot\|)$ be a non-Archimedean Banach space. The operator A acting on X is called linear if $\mathcal{D}(A)$, which designate its domain, is a linear subspace of X, and if $A(\alpha x + \beta y) = \alpha Ax + \beta Ay$, for all $\alpha, \beta \in \mathbb{K}$ and x, $y \in \mathcal{D}(A)$.

The symbols $R(A)$, $N(A)$ and $G(A)$ stand respectively for the range, the null space and the graph of the operator A, which are defined by

$$\begin{array}{rcl} R(A) & = & \Big\{Ax : x \in \mathcal{D}(A)\Big\}, \\ N(A) & = & \Big\{x \in \mathcal{D}(A) : Ax = 0\Big\}, \text{ and} \\ G(A) & = & \Big\{(x, Ax) : x \in \mathcal{D}(A)\Big\}. \end{array}$$

Definition 2.8. *Let $(X, \|\cdot\|)$ be a non-Archimedean Banach space. The linear operator $A : X \longrightarrow X$ is called bounded, if there exists $M \geq 0$ such that*

$$\|Ax\| \leq M \, \|x\|, \text{ for all } x \in X. \qquad \diamond$$

The collection of all bounded linear operators on X is denoted by $\mathcal{L}(X)$.

Remark 2.9. *Let $(X, \|\cdot\|)$ a non-Archimedean Banach space.*

(i) If $A \in \mathcal{L}(X)$, then $\|A\| := \sup\limits_{x \in X \setminus \{0\}} \dfrac{\|Ax\|}{\|x\|}$ is finite.

(ii) If $A \in \mathcal{L}(X)$, then $\|Ax\| \leq \|A\| \, \|x\|$, for all $x \in X$. $\diamond$

Definition 2.10. *Let X be a non-Archimedean Banach space. If $A \in \mathcal{L}(X)$, then*

(i) A is said to be one-to-one if $N(A) = \{0\}$.

(ii) A is said to be onto if $R(A) = X$.

(iii) A is said to be invertible if it is both one-to-one and onto. $\diamond$

Remark 2.11. *Let X be a non-Archimedean Banach space and let $A \in \mathcal{L}(X)$. If A is invertible, then there exists a unique bounded linear operator denoted $A^{-1} : X \longrightarrow X$ called the inverse of A such that $A^{-1}A = AA^{-1} = I$, where $I : X \to X$ is the identity operator.* $\diamond$

The following theorem is developed by T. Diagana and F. Ramaroson in [12].

Theorem 2.12. *Let X be a non-Archimedean Banach space.*

(i) If $A, B \in \mathcal{L}(X)$ and $\lambda \in \mathbb{K}$, then $A + B$, λA, AB and BA belong to $\mathcal{L}(X)$.

(ii) The space $(\mathcal{L}(X), \|\cdot\|)$ of bounded linear operators on X, is a non-Archimedean Banach space.

(iii) Let $A \in \mathcal{L}(X)$. If $\|A\| < 1$, then $I - A$ is invertible and $(I - A)^{-1} = \sum\limits_{n \geq 0} A^n$. $\diamond$

Remark 2.13. *Let X be a non-Archimedean Banach space. Then, $\mathcal{L}(X, \mathbb{K})$ which is called the dual of X and denoted X', is a non-Archimedean Banach space.* $\diamond$

Theorem 2.14. [14] *Let X be a non-Archimedean Banach space. For any non zero $x \in X$, there exists $x^* \in X'$ such that $x^*(x) = 1$ and $\|x^*\| = \|x\|^{-1}$.* $\diamond$

Definition 2.15. *Let $A \in \mathcal{L}(\mathbb{E}_\omega)$. The linear operator B is called the adjoint of A if $\langle Ax, y \rangle = \langle x, By \rangle$, for all $x, y \in \mathbb{E}_\omega$, where $\langle ., . \rangle$ is the inner product of $\mathbb{E}_\omega$.* $\diamond$

Remark 2.16. *Let $A \in \mathcal{L}(\mathbb{E}_\omega)$.*

(i) In the classical Banach space, any bounded linear operator admit an adjoint, but in the non-Archimedean Banach space, it is not true (see [9, Examples 17 and 18]).

(ii) The adjoint of an operator A is denoted by A^*. If A^* exists, then it is unique.

(iii) A^* is an adjoint for A if, and only if, $\langle Ae_i, e_j \rangle = \langle e_i, A^* e_j \rangle$, for all $i, j \in \mathbb{N}$.

(iv) If M is a subspace of $\mathbb{E}_\omega$, then $M^\perp = \{x \in \mathbb{E}_\omega : \langle x, y \rangle = 0, \text{ for all } y \in \mathbb{E}_\omega\}$. $\diamond$

The collection of all bounded linear operators on $\mathbb{E}_\omega$ whose adjoint operators do exist is denoted by $\tilde{\mathcal{L}}(\mathbb{E}_\omega)$.

Proposition 2.17. [12, Proposition 3.20] *If $A \in \tilde{\mathcal{L}}(\mathbb{E}_\omega)$ and for $\lambda \in \mathbb{K}$, then*

(i) $(\lambda + A)^* = \lambda + A^*$.

(ii) $\|A\| = \|A^*\|$. $\diamond$

Lemma 2.18. *If $A \in \tilde{\mathcal{L}}(\mathbb{E}_\omega)$, then $N(A^*) = R(A)^\perp$.* ◊

Proof. Let $x \in N(A^*)$, then $A^*x = 0$. This implies that $\langle y, A^*x\rangle = 0$, for all $y \in \mathbb{E}_\omega$. Hence, we have $\langle Ay, x\rangle = 0$, for all $y \in \mathbb{E}_\omega$, which yields $x \in R(A)^\perp$. We conclude that $N(A^*) \subset R(A)^\perp$. Conversely, let $x \in R(A)^\perp$, then for all $y \in \mathbb{E}_\omega$, we have $\langle Ay, x\rangle = 0$. This implies that $\langle y, A^*x\rangle = 0$, for all $y \in \mathbb{E}_\omega$. Consequently, we obtain $x \in N(A^*)$. □

Definition 2.19. *Let X be a non-Archimedean Banach space and let $A \in \mathcal{L}(X)$. A is called an operator of finite rank if $R(A)$ is a finite dimensional subspace of X.* ◊

The collection of all finite rank operators on X is denoted by $\mathcal{F}_0(X)$.

Definition 2.20. *Let X be a non-Archimedean Banach space and let $A \in \mathcal{L}(X)$. A is said to be completely continuous if, there exists a sequence $(A_n)_n$ in $\mathcal{F}_0(X)$ such that $\|A - A_n\|$ converge to 0 as $n \longrightarrow +\infty$.* ◊

The collection of completely continuous linear operators on X is denote by $C_c(X)$.

Remark 2.21. (*i*) *Every finite rank operators on non-Archimedean Banach space is completely continuous.*
(*ii*) *Let X be a non-Archimedean Banach space. If $A, B \in C_c(X)$ and $C \in \mathcal{L}(X)$, then $A + B$, AC and CA all belong to $C_c(X)$* (*see* [12, Theorem 3.32]). ◊

Definition 2.22. *Let X be a non-Archimedean Banach space. An unbounded linear operator $A : \mathcal{D}(A) \subset X \longrightarrow X$ is said to be closed if, its graph as a subset of $X \times X$ is closed.* ◊

The collection of closed linear operator on X is denote by $C(X)$.

Remark 2.23. *Every bounded linear operator on non-Archimedean Banach space is closed* (*see* [12, Remark 6.5]).◊

Definition 2.24. *Let X be a non-Archimedean Banach space and let $A \in \mathcal{L}(X)$. A is said to be a Fredholm operator if, it satisfies the following conditions:*
(*i*) $\alpha(A) := \dim N(A)$ *is finite,*
(*ii*) $R(A)$ *is closed, and*
(*iii*) $\beta(A) := \dim (X/R(A))$ *is finite.* ◊

The collection of all bounded Fredholm operators on X is denote by $\Phi^b(X)$. The set of unbounded Fredholm operators on a non-Archimedean Banach space X is defined by

$$\Phi(X) := \Big\{A \in C(X) : \ \alpha(A) < \infty \text{ and } \ \beta(A) < \infty\Big\}.$$

If A is a Fredholm operator, then its index is defined by $i(A) := \alpha(A) - \beta(A)$.
The following theorem give a result of Fredholm operators in a non-Archimedean Hilbert $\mathbb{E}_\omega$.

Theorem 2.25. [12, Theorem 6.18] *If $A \in \Phi(\mathbb{E}_\omega)$, then for all $C \in C_c(\mathbb{E}_\omega)$, we have*

$$A + C \in \Phi(\mathbb{E}_\omega) \text{ and } i(A + C) = i(A). \quad ◊$$

At the end of this section, we shall recall the definition of the spectrum and essential spectrum of linear operators on a non-Archimedean Banach space.

Definition 2.26. *Let X be a non-Archimedean Banach space.*
(*i*) *The resolvent set of a linear operator A is defined by*

$$\rho(A) := \{\lambda \in \mathbb{K} : (\lambda - A) \text{ is invertible in } \mathcal{L}(X)\}.$$

(*ii*) *The spectrum of a linear operator A is defined by $\sigma(A) = \mathbb{K} \backslash \rho(A)$.*

(*iii*) *The essential spectrum of a linear operator A is defined by*

$$\sigma_e(A) = \Big\{\lambda \in \mathbb{K} : \lambda - A \text{ is not a Fredholm operator of index } 0\Big\}. \quad ◊$$

Remark 2.27. *Let X be a non-Archimedean Banach space and let A be a linear operator on X. Then, we have $\sigma_e(A) \subset \sigma(A)$ (see [12]).* $\diamond$

The following theorem gives a characterization of the essential spectrum of bounded linear operators on $\mathbb{E}_\omega$.

Theorem 2.28. *Let $A \in \tilde{\mathcal{L}}(\mathbb{E}_\omega)$. Then,*

$$\sigma_e(A) = \bigcap_{C \in C_c(\mathbb{E}_\omega)} \sigma(A + C). \qquad \diamond$$

Proof. Let us assume that $\lambda \notin \bigcap_{C \in C_c(\mathbb{E}_\omega)} \sigma(A + C)$, then there exists $C \in C_c(\mathbb{E}_\omega)$ such that $\lambda \notin \sigma(A + C)$. This equivalent to say that $\lambda \in \rho(A + C)$. Therefore, $\lambda - A - C \in \Phi^b(\mathbb{E}_\omega)$ and $i(\lambda - A - C) = 0$ and, by using Theorem 2.25, we infer that

$$\lambda - A \in \Phi^b(\mathbb{E}_\omega) \text{ and } i(\lambda - A) = 0.$$

Hence, $\lambda \notin \sigma_e(A)$. As a result, $\sigma_e(A) \subset \bigcap_{C \in C_c(\mathbb{E}_\omega)} \sigma(A + C)$, as desired. Conversely, we suppose that $\lambda \notin \sigma_e(A)$, then

$$\lambda - A \in \Phi^b(\mathbb{E}_\omega) \text{ and } i(\lambda - A) = 0. \tag{2}$$

Let $n = \alpha(\lambda - A) = \beta(\lambda - A)$. It follows from Proposition 2.17 (i) and Lemma 2.18 that $\dim N(\lambda - A^*) = n$. Let $\{x_1, \ldots, x_n\}$ be a basis for $N(\lambda - A)$ and $\{y_1^*, \ldots, y_n^*\}$ be a basis for $N(\lambda - A^*)$. By [15, Lemma 2.1.1], there are functionals $x_1^*, \ldots, x_n^*$ in $\mathbb{E}'_\omega$ and elements $y_1, \ldots, y_n$ in $\mathbb{E}_\omega$ such that $x_k^*(x_j) = \delta_{jk}$ and $y_k^*(y_j) = \delta_{jk}$, for all $1 \leq j, k \leq n$ with

$$\delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise.} \end{cases}$$

Let us consider the following operator

$$\begin{array}{rcl} C : \mathbb{E}_\omega & \longrightarrow & \mathbb{E}_\omega \\ x & \longmapsto & \displaystyle\sum_{k=1}^{n} x_k^*(x) y_k. \end{array}$$

Since $\mathcal{D}(C) = \mathbb{E}_\omega$ and

$$\begin{aligned} \|Cx\| &= \left\| \sum_{k=1}^{n} x_k^*(x) y_k \right\| \\ &\leq \max_{1 \leq k \leq n} \left\| x_k^*(x) y_k \right\| \\ &\leq \max_{1 \leq k \leq n} \left\{ \|x_k^*\| \, \|y_k\| \right\} \|x\|, \end{aligned}$$

then C is bounded. Moreover, $R(C)$ is contained in a finite dimensional subspace of $\mathbb{E}_\omega$. So, C is a finite rank operator. It follows from Remark 2.21 (i) that $C \in C_c(\mathbb{E}_\omega)$. Now, we show that

$$N(\lambda - A) \cap N(C) = \{0\} \text{ and} \tag{3}$$

$$R(\lambda - A) \cap R(C) = \{0\}. \tag{4}$$

Let $x \in N(\lambda - A) \cap N(C)$. On the one hand, if $x \in N(\lambda - A)$, then

$$x = \sum_{k=1}^{n} \alpha_k x_k, \text{ where } \alpha_1, \ldots, \alpha_n \text{ are scalars in } \mathbb{K}.$$

This implies that $x_j^*(x) = \sum_{k=1}^{n} \alpha_k \delta_{k,j} = \alpha_j$, for all $1 \leq j \leq n$. On the other hand, if $x \in N(C)$, then $Cx = 0$. This implies that

$$\sum_{k=1}^{n} x_k^*(x) y_k = 0.$$

Therefore, we obtain $x_j^*(x) = 0$, for all $1 \leq j \leq n$. Hence, we conclude $x = 0$. Thus,

$$N(\lambda - A) \cap N(C) = \{0\}.$$

Now, we show that (4) holds. Let $y \in R(\lambda - A) \cap R(C)$. On the one hand, if $y \in R(C)$, then

$$y = \sum_{k=1}^{n} \alpha_k y_k, \text{ where } \alpha_1, \dots, \alpha_n \text{ are scalars in } \mathbb{K}.$$

Hence, we get $y_j^*(y) = \alpha_j$, for all $1 \leq j \leq n$. On the other hand, if $y \in R(\lambda - A)$, then $y_j^*(y) = 0$, for all $1 \leq j \leq n$. Therefore, $y = 0$. This leads to

$$R(\lambda - A) \cap R(C) = \{0\}.$$

It follows from Theorem 2.25 and (2) that $\lambda - A - C \in \Phi^b(\mathbb{E}_\omega)$ and $i(\lambda - A - C) = 0$. Consequently, we have

$$\alpha(\lambda - A - C) = \beta(\lambda - A - C). \tag{5}$$

Let us assume that $x \in N(\lambda - A - C)$, then $(\lambda - A)x = Cx \in R(\lambda - A) \cap R(C)$. It follows from (4) that $(\lambda - A)x = Cx = 0$, which yields $x \in N(\lambda - A) \cap N(C)$, and by using (3), we infer that $x = 0$. This implies that $\alpha(\lambda - A - C) = 0$, and from (5), we deduce that $R(\lambda - A - C) = X$. Consequently, $\lambda - A - C$ is invertible and we can conclude that $\lambda \notin \bigcap_{C \in \mathcal{C}_c(\mathbb{E}_\omega)} \sigma(A + C)$. □

3. Main results

In this section, we study the pseudospectra and the essential pseudospectra of linear operator in a non-Archimedean Banach space.

Definition 3.1. *Let X be a non-Archimedean Banach space and $\varepsilon > 0$. The pseudospectrum of a linear operator A on X is defined by*

$$\sigma_\varepsilon(A) := \sigma(A) \bigcup \Big\{\lambda \in \mathbb{K} : \|(\lambda - A)^{-1}\| > \frac{1}{\varepsilon}\Big\},$$

by convention $\|(\lambda - A)^{-1}\| = +\infty$ if, and only if, $\lambda \in \sigma(A)$. ◊

Proposition 3.2. *Let X be a non-Archimedean Banach space and $\varepsilon > 0$. Let A be a linear operator on X.*
(i) If $0 < \varepsilon_1 < \varepsilon_2$, then $\sigma(A) \subset \sigma_{\varepsilon_1}(A) \subset \sigma_{\varepsilon_2}(A)$.
(ii) $\sigma(A) = \bigcap_{\varepsilon > 0} \sigma_\varepsilon(A)$. ◊

Proof. (*i*) It is clear that

$$\sigma(A) \subset \sigma_\varepsilon(A), \text{ for all } \varepsilon > 0. \tag{6}$$

Let $0 < \varepsilon_1 < \varepsilon_2$, we show that

$$\sigma_{\varepsilon_1}(A) \subset \sigma_{\varepsilon_2}(A).$$

Let us assume that $\lambda \in \sigma_{\varepsilon_1}(A)\backslash\sigma(A)$, then $\|(\lambda - A)^{-1}\| > \dfrac{1}{\varepsilon_1}$. By the assumption $0 < \varepsilon_1 < \varepsilon_2$, we obtain $\|(\lambda - A)^{-1}\| > \dfrac{1}{\varepsilon_2}$. This leads to $\Big\{\lambda \in \mathbb{K} : \|(\lambda - A)^{-1}\| > \dfrac{1}{\varepsilon_1}\Big\} \subset \sigma_{\varepsilon_2}(A)$. It follows from (6) that $\sigma_{\varepsilon_1}(A) \subset \sigma_{\varepsilon_2}(A)$, for all $\varepsilon_1 < \varepsilon_2$. Finally, the use of (6) gives the wanted inclusion and achieves the proof of (i).
(ii) Let $\varepsilon > 0$, we have

$$\begin{aligned}\bigcap_{\varepsilon>0}\sigma_\varepsilon(A) &= \bigcap_{\varepsilon>0}\Big(\sigma(A)\bigcup\Big\{\lambda \in \mathbb{K} : \|(\lambda - A)^{-1}\| > \frac{1}{\varepsilon}\Big\}\Big)\\ &= \sigma(A)\bigcup\Big(\bigcap_{\varepsilon>0}\Big\{\lambda \in \mathbb{K} : \|(\lambda - A)^{-1}\| > \frac{1}{\varepsilon}\Big\}\Big).\end{aligned}$$

It suffices to prove that

$$\bigcap_{\varepsilon>0}\Big\{\lambda \in \mathbb{K} : \|(\lambda - A)^{-1}\| > \frac{1}{\varepsilon}\Big\} \subset \sigma(A).$$

If $\lambda \in \bigcap_{\varepsilon>0}\Big\{\lambda \in \mathbb{K} : \|(\lambda - A)^{-1}\| > \dfrac{1}{\varepsilon}\Big\}$, then $\|(\lambda - A)^{-1}\| > \dfrac{1}{\varepsilon}$, for all $\varepsilon > 0$. Hence, $\|(\lambda - A)^{-1}\| = +\infty$ as $\varepsilon \to 0^+$. This implies that $\lambda \in \sigma(A)$. This enables us to conclude that $\bigcap_{\varepsilon>0}\sigma_\varepsilon(A) \subset \sigma(A)$. □

Proposition 3.3. *Let $\mathbb{E}_\omega$ be a p-adic Hilbert space over $\mathbb{K}$. Let $(\lambda_i)_i$ be a sequence of element in $\mathbb{K}$ such that $\lim\limits_{i\to+\infty}|\lambda_i| = +\infty$. Let A be an unbounded diagonal operators on $\mathbb{E}_\omega$ defined by*

$$\mathcal{D}(A) = \Big\{x = (x_i)_i \subset \mathbb{E}_\omega : \lim_{i\to+\infty}|x_i|\cdot|\lambda_i|\cdot\|e_i\| = 0\Big\},$$

and

$$Ax = \sum_{i\in\mathbb{N}}\lambda_i\, x_i\, e_i, \text{ for all } x \in \mathcal{D}(A).$$

Let B be an unbounded diagonal operators on $\mathbb{E}_\omega$ defined by

$$\mathcal{D}(B) = \Big\{x = (x_i)_i \subset \mathbb{E}_\omega : \lim_{i\to+\infty}|x_i|\cdot|\mu_i|\cdot\|e_i\| = 0\Big\},$$

and

$$Bx = \sum_{i\in\mathbb{N}}\mu_i\, x_i\, e_i, \text{ for all } x \in \mathcal{D}(B).$$

(i) *The pseudospectrum of the operator A is*

$$\sigma_\varepsilon(A) = \Big\{\lambda_i : i \in \mathbb{N}\Big\}\bigcup\Big\{\lambda \in \mathbb{K} : \inf_{i\in\mathbb{N}}|\lambda - \lambda_i| < \varepsilon\Big\}.$$

(ii) *Suppose that $|\mu_i| < |\lambda_i|$ for each $i \in \mathbb{N}$, then the pseudospectrum of the operator $A + B$ is*

$$\sigma_\varepsilon(A+B) = \Big\{\delta_i, i \in \mathbb{N}\Big\}\bigcup\Big\{\lambda \in \mathbb{K} : \inf_{i\in\mathbb{N}}|\lambda - \delta_i| < \varepsilon\Big\},$$

where $\delta_i = \lambda_i + \mu_i$, for all $i \in \mathbb{N}$.
(iii) *Suppose that $\lim\limits_{i\to+\infty}|\mu_i\lambda_i| = +\infty$, then the pseudospectrum of the operator AB is*

$$\sigma_\varepsilon(AB) = \Big\{\delta_i, i \in \mathbb{N}\Big\}\bigcup\Big\{\lambda \in \mathbb{K} : \inf_{i\in\mathbb{N}}|\lambda - \delta_i| < \varepsilon\Big\},$$

where $\delta_i = \lambda_i\mu_i$, for all $i \in \mathbb{N}$. ◊

Proof. (i) From [9, Proposition 28], we have $\|(\lambda - A)^{-1}\| = \sup_{i\in\mathbb{N}} \frac{1}{|\lambda - \lambda_i|}$ and $\sigma(A) = \Big\{\lambda_i : i \in \mathbb{N}\Big\}$. Then, we can write

$$\sigma_\varepsilon(A) = \Big\{\lambda_i : i \in \mathbb{N}\Big\} \bigcup \left\{\lambda \in \mathbb{K} : \sup_{i\in\mathbb{N}} \frac{1}{|\lambda - \lambda_i|} > \frac{1}{\varepsilon}\right\}.$$

Since $\left\{\lambda \in \mathbb{K} : \sup_{i\in\mathbb{N}} \frac{1}{|\lambda - \lambda_i|} > \frac{1}{\varepsilon}\right\} = \left\{\lambda \in \mathbb{K} : \inf_{i\in\mathbb{N}} |\lambda - \lambda_i| < \varepsilon\right\}$, then we deduce that

$$\sigma_\varepsilon(A) = \Big\{\lambda_i : i \in \mathbb{N}\Big\} \bigcup \left\{\lambda \in \mathbb{K} : \inf_{i\in\mathbb{N}} |\lambda - \lambda_i| < \varepsilon\right\}.$$

(ii) The operator $A + B$ is defined by $(A + B)x = \sum_{i\in\mathbb{N}} \delta_i\ x_i\ e_i$, for all $x \in \mathcal{D}(A + B)$, with $\mathcal{D}(A + B) = \Big\{x = (x_i)_i \subset \mathbb{E}_\omega : \lim_{i\to+\infty} |x_i| \cdot |\delta_i| \cdot \|e_i\| = 0\Big\}$, where $\delta_i = \lambda_i + \mu_i$, for all $i \in \mathbb{N}$. It follows from [9, Corollary 7], that $\sigma_\varepsilon(A + B)$ can be expressed in the form

$$\sigma_\varepsilon(A + B) = \Big\{\delta_i : i \in \mathbb{N}\Big\} \bigcup \left\{\lambda \in \mathbb{K} : \sup_{i\in\mathbb{N}} \frac{1}{|\lambda - \delta_i|} > \frac{1}{\varepsilon}\right\}. \tag{7}$$

Using the fact that $\left\{\lambda \in \mathbb{K} : \sup_{i\in\mathbb{N}} \frac{1}{|\lambda - \delta_i|} > \frac{1}{\varepsilon}\right\} = \left\{\lambda \in \mathbb{K} : \inf_{i\in\mathbb{N}} |\lambda - \delta_i| < \varepsilon\right\}$ and by (7), we conclude that

$$\sigma_\varepsilon(A + B) = \Big\{\delta_i, i \in \mathbb{N}\Big\} \bigcup \left\{\lambda \in \mathbb{K} : \inf_{i\in\mathbb{N}} |\lambda - \delta_i| < \varepsilon\right\}.$$

(iii) The operator AB is defined by, $(AB)x = \sum_{i\in\mathbb{N}} \delta_i\ x_i\ e_i$, for all $x \in \mathcal{D}(AB)$, with $\mathcal{D}(AB) = \Big\{x = (x_i)_i \subset \mathbb{E}_\omega : \lim_{i\to+\infty} |\lambda_i| \cdot |\mu_i| \cdot |x_i| \cdot \|e_i\| = 0\Big\}$, where $\delta_i = \lambda_i\mu_i$, for all $i \in \mathbb{N}$. From [9, Corollary 8], we can write

$$\sigma_\varepsilon(AB) = \Big\{\delta_i : i \in \mathbb{N}\Big\} \bigcup \left\{\lambda \in \mathbb{K} : \sup_{i\in\mathbb{N}} \frac{1}{|\lambda - \delta_i|} > \frac{1}{\varepsilon}\right\}.$$

Since $\left\{\lambda \in \mathbb{K} : \sup_{i\in\mathbb{N}} \frac{1}{|\lambda - \delta_i|} > \frac{1}{\varepsilon}\right\} = \left\{\lambda \in \mathbb{K} : \inf_{i\in\mathbb{N}} |\lambda - \delta_i| < \varepsilon\right\}$, we conclude that

$$\sigma_\varepsilon(AB) = \Big\{\delta_i, i \in \mathbb{N}\Big\} \cup \left\{\lambda \in \mathbb{K} : \inf_{i\in\mathbb{N}} |\lambda - \delta_i| < \varepsilon\right\}.$$ □

Theorem 3.4. *Let X be a non-Archimedean Banach space such that $\|X\| \subseteq |\mathbb{K}|$. Let A be a linear operator on X and $\varepsilon > 0$. Then,*

$$\sigma_\varepsilon(A) = \bigcup_{\|B\|<\varepsilon} \sigma(A + B). \qquad \diamond$$

Proof. Let us assume that $\lambda \in \bigcup_{\|B\|<\varepsilon} \sigma(A + B)$. We argue by contradiction. Suppose that $\lambda \in \rho(A)$ and $\|(\lambda - A)^{-1}\| \leq \frac{1}{\varepsilon}$. let us consider the bounded linear operator C defined on X by

$$C := \sum_{n=0}^{+\infty} (\lambda - A)^{-1}\Big(B(\lambda - A)^{-1}\Big)^n.$$

By Theorem 2.12 (*iii*), we can write $C = (\lambda - A)^{-1}\Big(I - B(\lambda - A)^{-1}\Big)^{-1}$. This implies that C is invertible with $R(C) = \mathcal{D}(A)$. For all $x \in X$, we have $C\Big(I - B(\lambda - A)^{-1}\Big)x = (\lambda - A)^{-1}x$. Putting $y := (\lambda - A)^{-1}x$, we deduce that $C(\lambda - A - B)y = y$, for all $y \in \mathcal{D}(A)$. Moreover, we have $(\lambda - A - B)C\, x = x$, for all $x \in X$. Hence, we conclude that $(\lambda - A - B)$ is invertible with $C = (\lambda - A - B)^{-1}$. Conversely, suppose that $\lambda \in \sigma_\varepsilon(A)$. We discuss two cases.

First case. If $\lambda \in \sigma(A)$, then we may put $B = 0$.

Second case. Assume that $\lambda \in \sigma_\varepsilon(A)$ and $\lambda \notin \sigma(A)$. Then, there exists $z_0 \in X\backslash\{0\}$ such that

$$\frac{\|(\lambda - A)^{-1}z_0\|}{\|z_0\|} > \frac{1}{\varepsilon}. \tag{8}$$

Since $\|X\| \subseteq |\mathbb{K}|$, then we infer that there exists $c_0 \in \mathbb{K}\backslash\{0\}$ such that $|c_0| = \|z_0\|$. Then, setting $y_0 = c_0^{-1}z_0$, one indeed has $\|y_0\| = 1$. Hence, we obtain

$$\|(\lambda - A)^{-1}y_0\| = \|(\lambda - A)^{-1}c_0^{-1}z_0\| = \frac{\|(\lambda - A)^{-1}z_0\|}{|c_0|} = \frac{\|(\lambda - A)^{-1}z_0\|}{\|z_0\|}.$$

This implies from (8) that

$$\|(\lambda - A)^{-1}y_0\| > \frac{1}{\varepsilon}. \tag{9}$$

By the same reasoning above, we infer that there exists $c_1 \in \mathbb{K}\backslash\{0\}$ such that $|c_1| = \|(\lambda - A)^{-1}y_0\|$. Then, setting $x_0 = c_1^{-1}(\lambda - A)^{-1}y_0$, which yields $x_0 \in \mathcal{D}(A)$ and $\|x_0\| = 1$. Consequently, we get that

$$\|(\lambda - A)x_0\| = \left\|(\lambda - A)\frac{(\lambda - A)^{-1}y_0}{c_1}\right\| = \frac{\|y_0\|}{|c_1|}.$$

Using the fact that $\|y_0\| = 1$, we deduce from (9) that

$$\begin{aligned}\|(\lambda - A)x_0\| &= \|(\lambda - A)^{-1}y_0\|^{-1}\\ &< \varepsilon.\end{aligned}$$

At this level, by using Theorem 2.14, there exists a linear function ψ defined on X satisfy $\psi(x_0) = 1$ and $\|\psi\| = \|x_0\|^{-1} = 1$. We consider the following linear operator

$$By := \psi(y)(\lambda - A)x_0.$$

Let us observe that $\mathcal{D}(B) = X$ and for all $y \in X$, we have

$$\begin{aligned}\|By\| &\leq \|\psi(y)\|\|(\lambda - A)x\|\\ &\leq \varepsilon\|y\|,\end{aligned}$$

then we infer that $\|B\| \leq \varepsilon$ and $\mathcal{D}(B) = X$. This implies that B is bounded. Moreover, we have $(\lambda - A - B)x_0 = 0$. So, $\lambda - A - B$ is not invertible. This enables us to conclude that,

$$\lambda \in \bigcup_{\|B\|<\varepsilon} \sigma(A + B). \qquad \square$$

Lemma 3.5. *Let X be a non-Archimedean Banach space such that $\|X\| \subseteq |\mathbb{K}|$. Let A be a linear operator on X and $\varepsilon, \delta > 0$. Then, we have*

(i) $\sigma(A) + \Big\{\lambda \in \mathbb{K} : |\lambda| < \varepsilon\Big\} \subseteq \sigma_\varepsilon(A)$.

(ii) $\sigma_\delta(A) + \Big\{\lambda \in \mathbb{K} : |\lambda| < \varepsilon\Big\} \subseteq \sigma_{\varepsilon+\delta}(A)$. ◊

Proof. First, we have prove that

$$\lambda + \mu \in \sigma(A + \mu), \text{ for all } \lambda \in \sigma(A) \text{ and } \mu \in \mathbb{K}. \tag{10}$$

Let $\lambda \in \sigma(A)$ and $\mu \in \mathbb{K}$. We assume that $\lambda+\mu \in \rho(A+\mu)$, then $\lambda+\mu-(A+\mu)$ is invertible and $\Big(\lambda+\mu-(A+\mu)\Big)^{-1} \in \mathcal{L}(X)$. Hence, we infer that $\lambda \in \rho(A)$. This contradiction implies that $\lambda + \mu \in \sigma(A + \mu)$.

(i) Suppose that $\lambda \in \sigma(A) + \Big\{\lambda \in \mathbb{C} : |\lambda| < \varepsilon\Big\}$. Then, there exists $\lambda_1 \in \sigma(A)$ and $|\lambda_2| < \varepsilon$ such that $\lambda = \lambda_1 + \lambda_2$. By using (10), we infer that

$$\lambda = \lambda_1 + \lambda_2 \in \sigma(A + \lambda_2).$$

By the assumption $|\lambda_2| < \varepsilon$ and by Theorem 3.4, we conclude that $\lambda \in \sigma_\varepsilon(A)$.

(ii) Let us assume that $\lambda \in \sigma_\varepsilon(A) + \Big\{\lambda \in \mathbb{C} : |\lambda| < \delta\Big\}$. Then, there exists $\lambda_1 \in \sigma_\varepsilon(A)$ and $|\lambda_2| < \delta$ such that $\lambda = \lambda_1 + \lambda_2$. From the assumption $\lambda_1 \in \sigma_\varepsilon(A)$ and Theorem 3.4, we infer that there exists $B \in \mathcal{L}(X)$ such that $\|B\| < \varepsilon$ and $\lambda_1 \in \sigma(A + B)$. From (10), allows us to conclude that

$$\lambda = \lambda_1 + \lambda_2 \in \sigma(A + B + \lambda_2).$$

Furthermore, $B + \lambda_2 \in \mathcal{L}(X)$ and

$$\begin{aligned} \|B + \lambda_2\| &\leq \varepsilon + |\lambda_2| \\ &< \varepsilon + \delta. \end{aligned}$$

Hence, by using Theorem 3.4, we deduce that $\lambda \in \sigma_{\delta+\varepsilon}(A)$. □

Lemma 3.6. *Let X be a non-Archimedean Banach space such that $\|X\| \subseteq |\mathbb{K}|$, $\varepsilon > 0$ and let A be a linear operator on X. The following statements are equivalent*

(i) $\sigma(A) \bigcup \Big\{\lambda \in \mathbb{K} : \exists\, x \in \mathcal{D}(A) \text{ and } \|(\lambda - A)x\| < \varepsilon\|x\|\Big\}$.

(ii) $\sigma(A) \bigcup \Big\{\lambda \in \mathbb{K} : \exists\, x_n \in \mathcal{D}(A), \|x_n\| = 1 \text{ and } \lim\limits_{n\to+\infty} \|(\lambda - A)x_n\| < \varepsilon\Big\}$. ◊

Proof. This proof is analogous to proof in the classical Banach space. □

Now, we characterize the essential pseudospectra of linear operator in non-Archimedean Hilbert space $\mathbb{E}_\omega$.

Definition 3.7. *Let $A \in \mathcal{C}(\mathbb{E}_\omega)$ and let $\varepsilon > 0$. The essential pseudospectrum of the linear operator A is defined by*

$$\sigma_{e,\varepsilon}(A) = \mathbb{K}\backslash\Big\{\lambda \in \mathbb{K} : \lambda - A - B \in \Phi_0(\mathbb{E}_\omega), \text{ for all } B \in \mathcal{L}(\mathbb{E}_\omega), \|B\| < \varepsilon\Big\},$$

where $\Phi_0(X)$ designates the set of all unbounded Fredholm operators on X of index 0.

Theorem 3.8. *Let $A \in \mathcal{C}(\mathbb{E}_\omega)$ and $\varepsilon > 0$. Then,*

$$\sigma_{e,\varepsilon}(A) = \bigcup_{\|B\|<\varepsilon} \sigma_e(A + B).$$ ◊

Proof. Let $\lambda \notin \sigma_{e,\varepsilon}(A)$. Then, for all $B \in \mathcal{L}(\mathbb{E}_\omega)$ such that $\|B\| < \varepsilon$,

$$\lambda - (A + B) \in \Phi(\mathbb{E}_\omega) \text{ and } i(\lambda - (A + B)) = 0.$$

Hence, $\lambda \notin \sigma_e(A + B)$, for all $B \in \mathcal{L}(\mathbb{E}_\omega)$ such that $\|B\| < \varepsilon$. This is equivalent to say that

$$\lambda \notin \bigcup_{\|B\|<\varepsilon} \sigma_e(A + B).$$

This shows that $\bigcup\limits_{\|B\|<\varepsilon} \sigma_e(A+B) \subset \sigma_{e,\varepsilon}(A)$. Conversely, let $\lambda \notin \bigcup\limits_{\|B\|<\varepsilon} \sigma_e(A+B)$. Then, for all $B \in \mathcal{L}(\mathbb{E}_\omega)$ such that $\|B\| < \varepsilon$, we have $\lambda \notin \sigma_e(A+B)$. This implies that $\lambda - A - B \in \Phi(\mathbb{E}_\omega)$ and $i(\lambda - A - B) = 0$, for all $B \in \mathcal{L}(\mathbb{E}_\omega)$ such that $\|B\| < \varepsilon$. Hence, $\lambda \notin \sigma_{e,\varepsilon}(A)$. This prove that $\sigma_{e,\varepsilon}(A) \subset \bigcup\limits_{\|B\|<\varepsilon} \sigma_e(A+B)$. □

The purpose of this result is to give a characterization of the essential pseudospectra of a closed linear operator on $\mathbb{E}_\omega$.

Theorem 3.9. *Let $A \in \mathcal{C}(\mathbb{E}_\omega)$ and $\varepsilon > 0$. Then,*

$$\sigma_{e,\varepsilon}(A) = \sigma_{e,\varepsilon}(A+C), \text{ for all } C \in \mathcal{C}_c(\mathbb{E}_\omega). \qquad \diamond$$

Proof. Let us assume that $\lambda \notin \sigma_{e,\varepsilon}(A)$. Then, for all $B \in \mathcal{L}(\mathbb{E}_\omega)$ such that $\|B\| < \varepsilon$, we have $\lambda - A - B \in \Phi(\mathbb{E}_\omega)$ and $i(\lambda - A - B) = 0$. Hence, by using Theorem 2.25, for all $C \in \mathcal{C}_c(\mathbb{E}_\omega)$ and $B \in \mathcal{L}(\mathbb{E}_\omega)$ such that $\|B\| < \varepsilon$, we infer that

$$\lambda - (A+C) - B \in \Phi(\mathbb{E}_\omega) \text{ and } i(\lambda - (A+C) - B) = 0. \tag{11}$$

Moreover, it follows from [12, Reamrk 6.5] that $A + C$ is a closed operator. Thus, by using (11), we deduce that $\lambda \notin \sigma_{e,\varepsilon}(A+C)$. As a result, $\sigma_{e,\varepsilon}(A+C) \subset \sigma_{e,\varepsilon}(A)$, as desired. The opposite inclusion follows from symmetry and we obtain

$$\sigma_{e,\varepsilon}(A) = \sigma_{e,\varepsilon}(A+C). \qquad \square$$

Remark 3.10. *From Theorem 3.9, it follows that the pseudospectrum of closed operator is invariant under perturbations of completely continuous operators in $\mathbb{E}_\omega$.* ◇

In the sequel, we assume that $\mathbb{K}$ is quadratically closed. The following result gives a characterization of the essential pseudospectrum of a closed linear operator by means of the spectra of all perturbed completely continuous operators.

Theorem 3.11. *Let $A \in \tilde{\mathcal{L}}(\mathbb{E}_\omega)$ and $\varepsilon > 0$. Then,*

$$\sigma_{e,\varepsilon}(A) = \bigcap_{C \in \mathcal{C}_c(\mathbb{E}_\omega)} \sigma_\varepsilon(A+C). \qquad \diamond$$

Proof. Let us assume that $\lambda \notin \bigcap\limits_{C \in \mathcal{C}_c(\mathbb{E}_\omega)} \sigma_\varepsilon(A+C)$, then there exists $C \in \mathcal{C}_c(\mathbb{E}_\omega)$ such that $\lambda \notin \sigma_\varepsilon(A+C)$. Since $\mathbb{K}$ is quadratically closed, then we have $\|\mathbb{E}_\omega\| \subseteq |\mathbb{K}|$. Now, by referring to Theorem 3.4, we obtain $\lambda \in \rho(A+B+C)$, for all $B \in \mathcal{L}(\mathbb{E}_\omega)$ such that $\|B\| < \varepsilon$. Therefore,

$$\lambda - (A+B+C) \in \Phi^b(\mathbb{E}_\omega) \text{ and } i(\lambda - (A+B+C)) = 0.$$

By using Theorem 2.25, we infer that $\lambda - A - B \in \Phi^b(\mathbb{E}_\omega)$ and $i(\lambda - A - B) = 0$. Thus, $\lambda \notin \sigma_{e,\varepsilon}(A)$. This proves that

$$\sigma_{e,\varepsilon}(A) \subset \bigcap_{C \in \mathcal{C}_c(\mathbb{E}_\omega)} \sigma_\varepsilon(A+C).$$

Conversely, we assume that $\lambda \notin \sigma_{e,\varepsilon}(A)$. By using Theorem 3.8, for all $B \in \mathcal{L}(\mathbb{E}_\omega)$ such that $\|B\| < \varepsilon$, we infer that $\lambda \notin \sigma_e(A+B)$. Hence, by applying Theorem 2.28, there exists $C \in \mathcal{C}_c(\mathbb{E}_\omega)$ such that $\lambda \notin \sigma(A+B+C)$. This implies that for all $B \in \mathcal{L}(\mathbb{E}_\omega)$ such that $\|B\| < \varepsilon$, we have $\lambda \in \rho(A+B+C)$. Hence,

$$\lambda \in \bigcap_{\|B\|<\varepsilon} \rho(A+B+C).$$

It follows from Theorem 3.4 that $\lambda \notin \sigma_\varepsilon(A+C)$. Hence, we deduce that

$$\lambda \notin \bigcup_{C\in\mathcal{C}_c(\mathbb{E}_\omega)} \sigma_\varepsilon(A+C).$$

As a result, $\bigcap_{C\in\mathcal{C}_c(\mathbb{E}_\omega)} \sigma_\varepsilon(A+C) = \sigma_{e,\varepsilon}(A)$, as desired. □

The following result gives a characterization of the essential pseudospectrum of a closed linear operator by means of the spectra of all perturbed rank finite operators.

Theorem 3.12. *Let $A \in \tilde{\mathcal{L}}(\mathbb{E}_\omega)$ and $\varepsilon > 0$. Then,*

$$\sigma_{e,\varepsilon}(A) = \bigcap_{F\in\mathcal{F}_0(\mathbb{E}_\omega)} \sigma_\varepsilon(A+F). \qquad \diamond$$

Proof. Let $O = \bigcap_{F\in\mathcal{F}_0(\mathbb{E}_\omega)} \sigma_\varepsilon(A+F)$. By using Remark 3.13 and Theorem 3.11, we infer that $\sigma_{e,\varepsilon}(A) \subset O$. Conversely, let $\lambda \notin O$. Then, there exists $F \in \mathcal{F}_0(\mathbb{E}_\omega)$ such that

$$\lambda \notin \sigma_\varepsilon(A+F).$$

Since $\mathbb{K}$ is quadratically closed, then we have $\|\mathbb{E}_\omega\| \subseteq |\mathbb{K}|$. Hence, by using Theorem 3.4, for all $B \in \mathcal{L}(\mathbb{E}_\omega)$ such that $\|B\| < \varepsilon$, we deduce that $\lambda \in \rho(A+B+F)$. Therefore, $\lambda - A - B - F$ is invertible. This implies that

$$\lambda - A - B - F \in \Phi^b(\mathbb{E}_\omega) \text{ and } i(\lambda - A - B - F) = 0.$$

It follows from Remark 2.21 (i) and Theorem 2.25 that for all $B \in \mathcal{L}(\mathbb{E}_\omega)$ such that $\|B\| < \varepsilon$, $\lambda - A - B \in \Phi^b(\mathbb{E}_\omega)$ and $i(\lambda - A - B) = 0$. This is equivalent to say that $\lambda \notin \sigma_{e,\varepsilon}(A)$. □

The following proposition gives some properties of the essential pseudospectra in $\mathbb{E}_\omega$.

Proposition 3.13. *Let $A \in \mathcal{C}(\mathbb{E}_\omega)$ and $\varepsilon > 0$.*
(i) $\sigma_{e,\varepsilon}(A) \subset \sigma_\varepsilon(A)$.
(ii) *Let* $0 < \varepsilon_1 < \varepsilon_2$, $\sigma_e(A) \subset \sigma_{e,\varepsilon_1}(A) \subset \sigma_{e,\varepsilon_2}(A)$. ◇

Proof. (i) Let $\lambda \in \sigma_{e,\varepsilon}(A)$. Then, by using Theorem 3.8, we have $\lambda \in \bigcup_{\|B\|<\varepsilon} \sigma_e(A+B)$. The fact that $\sigma_e(A+B) \subset \sigma(A+B)$ allows us to deduce that $\lambda \in \bigcup_{\|B\|<\varepsilon} \sigma(A+B)$. Since $\mathbb{K}$ is quadratically closed, then we have $\|\mathbb{E}_\omega\| \subseteq |\mathbb{K}|$. Finally, the use of Theorem 3.4 shows that $\lambda \in \sigma_\varepsilon(A)$. As a result, $\sigma_{e,\varepsilon}(A) \subset \sigma_\varepsilon(A)$, as desired.
(ii) Let $0 < \varepsilon_1 < \varepsilon_2$. We have to prove that

$$\sigma_e(A) \subset \sigma_{e,\varepsilon_1}(A). \tag{12}$$

Let us assume that $\lambda \notin \sigma_{e,\varepsilon_1}(A)$, then for all $B \in \mathcal{L}(\mathbb{E}_\omega)$ such that $\|B\| < \varepsilon_1$, we have $\lambda - A - B \in \Phi(\mathbb{E}_\omega)$ and $i(\lambda - A - B) = 0$. As $\varepsilon_1 \to 0$, then $\lambda - A \in \Phi(\mathbb{E}_\omega)$ and $i(\lambda - A) = 0$. Hence, we deduce that $\lambda \notin \sigma_e(A)$. Now, we propose to show that

$$\sigma_{e,\varepsilon_1}(A) \subset \sigma_{e,\varepsilon_2}(A).$$

Let us suppose that $\lambda \in \sigma_{e,\varepsilon_1}(A)$. Then, by using Theorem 3.8, we infer that $\lambda \in \bigcup_{\|B\|<\varepsilon_1} \sigma_e(A+B)$. Hence, by the assumption $\varepsilon_1 < \varepsilon_2$, we deduce that $\lambda \in \bigcup_{\|B\|<\varepsilon_2} \sigma_e(A+B)$. It follows from Theorem 3.8 that $\lambda \in \sigma_{e,\varepsilon_2}(A)$. This shows that $\sigma_{e,\varepsilon_1}(A) \subset \sigma_{e,\varepsilon_2}(A)$, for all $0 < \varepsilon_1 < \varepsilon_2$. □

Proposition 3.14. *Let $A \in \tilde{\mathcal{L}}(\mathbb{E}_\omega)$ and $\varepsilon > 0$. Then,*

$$\sigma_e(A) = \bigcap_{\varepsilon>0} \sigma_{e,\varepsilon}(A). \qquad \diamond$$

Proof. Assume that $\lambda \in \bigcap_{\varepsilon>0} \sigma_{e,\varepsilon}(A)$. Since $\mathbb{K}$ is quadratically closed, then we have $\|\mathbb{E}_\omega\| \subseteq |\mathbb{K}|$. It follows from Theorem 3.11 that

$$\lambda \in \bigcap_{\varepsilon>0} \bigcap_{C\in C_c(\mathbb{E}_\omega)} \sigma_\varepsilon(A+C) = \bigcap_{C\in C_c(\mathbb{E}_\omega)} \bigcap_{\varepsilon>0} \sigma_\varepsilon(A+C).$$

This implies from Proposition 3.2 (*ii*) that $\lambda \in \bigcap_{C\in C_c(\mathbb{E}_\omega)} \sigma(A+C)$. Hence, it follows from Theorem 2.28 that $\lambda \in \sigma_e(A)$. Conversely , let $\lambda \in \sigma_e(A)$. By referring to Theorem 2.28, we infer that $\lambda \in \bigcap_{C\in C_c(\mathbb{E}_\omega)} \sigma(A+C)$. In view of Proposition 3.2 (*ii*) implies that $\lambda \in \bigcap_{\varepsilon>0} \bigcap_{C\in C_c(\mathbb{E}_\omega)} \sigma_\varepsilon(A+C)$. Since $\mathbb{K}$ is quadratically closed, then we have $\|\mathbb{E}_\omega\| \subseteq |\mathbb{K}|$. Hence, it follows from Theorem 3.4 that

$$\lambda \in \bigcap_{\varepsilon>0} \bigcup_{\|B\|<\varepsilon} \bigcap_{C\in C_c(\mathbb{E}_\omega)} \sigma(A+B+C).$$

By using Theorem 2.28, we infer that $\lambda \in \bigcap_{\varepsilon>0} \bigcup_{\|B\|<\varepsilon} \sigma_e(A+B)$. Finally, the use of Theorem 3.8 allows us to conclude that $\lambda \in \bigcap_{\varepsilon>0} \sigma_{e,\varepsilon}(A)$. □

Example 3.15. *Let $p \geq 2$ be a prime. Let $\mathbb{E}_\omega$ be a p-adic Hilbert space over $\mathbb{Q}_p$ which is equipped with the usual p-adic absolute value $|\cdot|_p$. Let $\zeta \in \{-1, 1\}$. We shall recall the spaces $\mathbb{Z}$ and $\mathbb{Z}_P$, which are defined respectively by*

$$\mathbb{Z} := \left\{ \sum_{i=0}^{N} \zeta a_k p^k : 0 \leq a_k \leq p-1, N \in \mathbb{N} \right\},$$

and

$$\mathbb{Z}_P := \left\{ \sum_{k=0}^{+\infty} a_k p^k : 0 \leq a_k \leq p-1 \right\}.$$

Moreover, the closure of $\mathbb{Z}$ in $\mathbb{Q}_p$ is $\mathbb{Z}_P$ which called the ring of p-adic integers.
Let $(\lambda_i)_i$ be a sequence in $\mathbb{Q}_p$. Let us consider the bounded diagonal linear operator D on $\mathbb{E}_\omega$ defined by,

$$De_i = \lambda_i e_i, \text{ for all } i \in \mathbb{N}.$$

Then,

$$\sigma_\varepsilon(D) = \mathbb{Z}_P \bigcup \left\{ \lambda \in \mathbb{Q}_p : \inf_{i\in\mathbb{N}} |\lambda - \lambda_i|_p < \varepsilon \right\}. \qquad \diamond$$

Proof. Let $\lambda \neq \lambda_i$, for all $i \in \mathbb{N}$. Then, by [12, Proposition 3.55], we infer that $\lambda \in \rho(D)$. This implies that $(\lambda - D)$ is invertible in $\mathcal{L}(\mathbb{E}_\omega)$. Moreover, for all $i \in \mathbb{N}$, the bounded diagonal operator $(\lambda - D)^{-1}$ is expressed in the form $(\lambda - D)^{-1} e_i = \dfrac{1}{\lambda - \lambda_i} e_i$, and its norm is defined by

$$\begin{aligned} \|(\lambda - D)^{-1}\| &= \sup_{i\in\mathbb{N}} \frac{\|(\lambda - D)^{-1} e_i\|}{\|e_i\|} \\ &= \sup_{i\in\mathbb{N}} \frac{\|e_i\|}{|\lambda - \lambda_i|_p \|e_i\|} \\ &= \sup_{i\in\mathbb{N}} \frac{1}{|\lambda - \lambda_i|_p}. \end{aligned}$$

Hence,

$$\left\{\lambda \in \mathbb{Q}_p : \|(\lambda - D)^{-1}\| > \frac{1}{\varepsilon}\right\} = \left\{\lambda \in \mathbb{Q}_p : \inf_{i\in\mathbb{N}} |\lambda - \lambda_i|_p < \varepsilon\right\}. \tag{13}$$

Since $\sigma(D) = \mathbb{Z}_P$ (see [12]), then by (13), we have

$$\sigma_\varepsilon(D) = \mathbb{Z}_P \bigcup \left\{\lambda \in \mathbb{Q}_p : \inf_{i\in\mathbb{N}} |\lambda - \lambda_i|_p < \varepsilon\right\}. \qquad \square$$

References

[1] A. Ammar, A. Jeribi, A characterization of the essential pseudospectra and application to a transport equation, Extracta Math. 28 (2013), no. 1, 95–112.

[2] A. Ammar, A. Jeribi, A characterization of the essential pseudospectra on a Banach space, Arab. J. Math. (Springer) 2 (2013), no. 2, 139–145.

[3] A. Ammar, A. Jeribi, Measures of noncompactness and essential pseudospectra on Banach space, Math. Methods Appl. Sci. 37 (2014), no. 3, 447–452.

[4] A. Ammar, B. Boukettaya, A. Jeribi, A note on the essential pseudospectra and application, Linear Multilinear Algebra 64 (2016), no. 8, 1474–1483.

[5] S. Basu, T. Diagana, F. Ramaroson, A p-adic version of Hilbert-Schmidt operators and applications, J. Anal. Appl. 2 (2004), no. 3, 173–188.

[6] B. Diarra, An operator on some ultrametric Hilbert spaces, J. Anal. 6 (1998), 55–74.

[7] T. Diagana, Towards a theory of some unbounded linear operators on p-adic Hilbert spaces and applications. Ann. Math. Blaise Pascal 12 (2005), no. 1, 205–222.

[8] E. B. Davies, Linear operators and their spectra. Cambridge Studies in Advanced Mathematics, 106. Cambridge University Press, Cambridge, xii+451 pp. ISBN: 978-0-521-86629-3; 0-521-86629-4, 2007.

[9] T. Diagana, Non-Archimedean linear operators and applications, Nova Science Publishers, Inc., Huntington, NY, xiv+92 pp. ISBN: 978-1-60021-405-9; 1-60021-405-3, 2007.

[10] T. Diagana , R. Kerby, T. H. Miabey, F. Ramaroson, Spectral analysis for finite rank perturbations of diagonal operators in non-archimedean Hilbert space. p-Adic Numbers Ultrametric Anal. Appl. 6, no. 3, (2014) 171–187 .

[11] T. Diagana, F. Ramaroson, Spectral theory for finite rank perturbations of unbounded diagonal operators in non-Archimedean Hilbert space. Advances in non-Archimedean analysis, Contemp. Math., 665, Amer. Math. Soc., Providence, RI, (2016) 29–40.

[12] T. Diagana, F. Ramaroson, Non-Archimedean operator theory, SpringerBriefs in Mathematics. Springer, Cham, xiii+156 pp. ISBN: 978-3-319-27322-8; 978-3-319-27323-5, 2016.

[13] F. Q. Gouva, p-adic numbers. An introduction. Second edition. Universitext. Springer-Verlag, Berlin vi+298 pp. ISBN: 3-540-62911-4, 1997.

[14] A. W. Ingleton, The Hahn-Banach theorem for non-Archimedean valued fields, Proc. Cambridge Philos. Soc. 48 (1952) 41–45.

[15] A. Jeribi, Spectral theory and applications of linear operators and block operator matrices. Springer, Cham, xvi+599 pp. ISBN: 978-3-319-17565-2; 978-3-319-17566-9, 2015.

[16] A. F. Monna, On non-Archimedean linear spaces. (Dutch) Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde 52, (1943) 308–321.

[17] A. F. Monna, Analyse non-archimdienne. (French) Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 56. Springer-Verlag, Berlin-New York, vii+119 pp, 1970.

[18] P. Schneider, Nonarchimedean functional analysis. Springer Monographs in Mathematics. Springer-Verlag, Berlin, vi+156 pp. ISBN: 3-540-42533-0,2002.

[19] L. N. Trefethen, Pseudospectra of matrices, Numerical analysis 1991 (Dundee, 1991), , Pitman Res. Notes Math. Ser., 260, Longman Sci. Tech., Harlow (1992) 234–266.

[20] L. N. Trefethen, Pseudospectra of linear operators. SIAM Rev. 39 (1997), no. 3, 383–406.

[21] L. N. Trefethen, M. Embree, Spectra and pseudospectra. The behavior of nonnormal matrices and operators. Princeton University Press, Princeton, NJ, xviii+606 pp. ISBN: 978-0-691-11946-5; 0-691-11946-5, 2005.

[22] A. C. M. van Rooij, Non-Archimedean functional analysis. Monographs and Textbooks in Pure and Applied Math., 51. Marcel Dekker, Inc., New York, x+404 pp. ISBN: 0-8247-6556-7, 1978.

[23] J. M. Varah, The computation of bounds for the invariant subspaces of a general matrixoperator. Thesis (Ph.D.)Stanford University. ProQuest LLC, Ann Arbor, MI, 244 pp (1967).