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The Polar-Like Decomposition and its Applications

Hongxing Wang[a], **Xiaoji Liu**[a]

[a] *School of Science, Guangxi Key Laboratory of Hybrid Computation and IC Design Analysis, Guangxi University for Nationalities, Nanning, 530006, P.R. China*

Abstract. In this paper, we present a unique polar-like decomposition theorem for rectangular complex matrices. Applying this decomposition, we define on the set of rectangular matrices a new partial ordering called WL(weak Löwner) partial order – an extension of the GL(generalized Löwner) partial order, and derive some basic properties of the new partial ordering.

1. Introduction

In this paper, we use the following notations. The symbol $\mathbb{C}_{m,n}$ denotes the set of $m \times n$ matrices with complex entries; $\mathbb{C}_n^{\mathrm{H}}$ and $\mathbb{C}_n^{\geq}$ denote the set of $n \times n$ Hermitian matrices and Hermitian nonnegative definite matrices, respectively. The symbols A^*, $\mathcal{R}(A)$ and $\mathrm{rk}\,(A)$ represent the conjugate transpose, range space (or column space) and rank of $A \in \mathbb{C}_{m,n}$. The symbol $|A|$ denotes the modulus of $A \in \mathbb{C}_{m,n}$, i.e., $|A| = (AA^*)^{\frac{1}{2}}$. The *Moore-Penrose inverse* of $A \in \mathbb{C}_{m,n}$ is defined as the unique matrix $X \in \mathbb{C}_{n,m}$ satisfying the equations

$$(1)\ AXA = A,\ \ (2)\ XAX = X,\ \ (3)\ (AX)^* = AX,\ \ (3)\ (XA)^* = XA,$$

and is usually denoted as $X = A^\dagger$ (see [18]). Some other generalized inverses have been studied, see for example, [3, 18]

A binary relation is called a *partial order* if it is reflexive, transitive and anti-symmetric on a non-empty set. For matrices A and B (see [6, 15]), we say

(i) A is below B with respect to the *star partial order*, i.e. $A \overset{*}{\leq} B$, if $A^*A = A^*B$ and $AA^* = BA^*$, in which A and $B \in \mathbb{C}_{m,n}$;

(ii) A is below B with respect to the *Löwner partial order*, i.e. $A \overset{\mathrm{L}}{\leq} B$, if exists K such that $B - A = KK^*$, in which A, B and $K \in \mathbb{C}_{m,m}$.

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Email addresses: winghongxing0902@163.com (Hongxing Wang), xiaojiliu72@126.com (Xiaoji Liu)

Matrix decomposition, an important tool in the study of partial order theory, is used to prove some characterizations and properties of partial orders, and, furthermore, establish some partial orders. For example, the core-nilpotent, core and core-EP partial orders are derived on the basis of the core-nilpotent, core and core-EP decompositions , respectively, [2, 15, 19, etc]. A particular concern is a generalized polar decomposition, [3, Chapter 6(Theorem 7)].

Theorem 1.1. *Let $A \in \mathbb{C}_{m,n}$. Then A can be written as*

$$A = G_A E_A = E_A H_A, \tag{1.1}$$

where $E_A \in \mathbb{C}_{m,n}$ is a partial isometry, i.e., $E_A^* = E_A^\dagger$, and $G_A \in \mathbb{C}_{m,m}$, $H_A \in \mathbb{C}_{n,n}$ are Hermitiam nonnegative definite matrices. The matrices E_A, G_A and H_A are uniquely determined by $\mathcal{R}(E_A) = \mathcal{R}(G_A)$ and $\mathcal{R}\left(E_A^*\right) = \mathcal{R}(H_A)$, in which case $G_A = |A|$, $H_A = |A^*|$ and $E_A = G_A^\dagger A = AH_A^\dagger$.

Applying the generalized polar decomposition, Hauke and Markiewicz characterize the notion of GL partial order in [9, Definition 1 and Theorem 5].

Theorem 1.2. [9, Theorem 5] *Let $A, B \in \mathbb{C}_{m,n}$, and $A = G_A E_A$ and $B = G_B E_B$ be their polar decompositions, where $\mathcal{R}(E_A) = \mathcal{R}(G_A)$ and $\mathcal{R}(E_B) = \mathcal{R}(G_B)$. Then*

$$A \overset{\mathrm{GL}}{\leq} B \Leftrightarrow E_A \overset{*}{\leq} E_B \text{ and } G_A \overset{\mathrm{L}}{\leq} G_B, \tag{1.2}$$

$$\Leftrightarrow E_A \overset{*}{\leq} E_B \text{ and } H_A \overset{\mathrm{L}}{\leq} H_B. \tag{1.3}$$

Furthermore, the simultaneous polar decomposability of a pair of rectangular matrices is derived in [14, Definition 1], and a new characterization of the GL partial order is given in [14, Proposition 3]. The unique weighted polar decomposition theorem is given in [21, Theorem 3.5] and the WGL partial order is derived in [21, Definition 4.2]. Note that, the generalized polar decompositions are important in the numerical calculation as well. For more results about the generalized polar decompositions and related problems, refer to [4, 5, 7, 10, 12, 13, 16, 17, etc]. In this paper, we give, on the basis of Theorem 1.1, the notion of WL partial order, a generalization of the GL partial order. We derive properties and characterizations of the WL partial order, and consider its differences from the GL partial order.

2. Main Results

Theorem 2.1. *Let $A \in \mathbb{C}_{m,n}$. Then A can be written as*

$$A = G_A^{\frac{1}{2}} E_A H_A^{\frac{1}{2}}, \tag{2.1}$$

where E_A , G_A and H_A are given in Theorem 1.1.

Proof. Let $A \in \mathbb{C}_{m,n}$, $\mathrm{rk}(A) = r$, and

$$A = U_A \Sigma_A V_A^*$$

be the SVD decomposition of A, where $U_A \in \mathbb{C}_{m,r}$ and $V_A \in \mathbb{C}_{n,r}$ are unitary matrices, $U_A^* U_A = I_r = V_A^* V_A$, Σ_A is a diagonal positive definite matrix. Then

$$G_A = U_A \Sigma_A U_A^*,\ E_A = U_A V_A^* \text{ and } H_A = V_A \Sigma_A V_A^*.$$

Therefore,

$$A = U_A \Sigma_A^{\frac{1}{2}} U_A^* U_A V_A^* V_A \Sigma_A^{\frac{1}{2}} V_A^* = |A|^{\frac{1}{2}}\, E_A\, |A^*|^{\frac{1}{2}} = G_A^{\frac{1}{2}} E_A H_A^{\frac{1}{2}}.$$

□

We call (2.1) as the polar-like decomposition of A. It is easy to check that

$$E_A^* G_A^{\frac{1}{2}} E_A = V_A U_A^* U_A \Sigma_A^{\frac{1}{2}} U_A^* U_A V_A^* = V_A \Sigma_A^{\frac{1}{2}} V_A^* = H_A^{\frac{1}{2}},$$
$$E_A H_A^{\frac{1}{2}} E_A^* = G_A^{\frac{1}{2}}.$$

Consider the binary operation:

$$A \overset{\mathtt{WL}}{\leq} B \;\Leftrightarrow\; G_A^{\frac{1}{2}} \overset{\mathtt{L}}{\leq} G_B^{\frac{1}{2}},\; E_A \overset{*}{\leq} E_B,\; H_A^{\frac{1}{2}} \overset{\mathtt{L}}{\leq} H_B^{\frac{1}{2}}, \tag{2.2}$$

in which $A = G_A^{\frac{1}{2}} E_A H_A^{\frac{1}{2}}$ and $B = G_B^{\frac{1}{2}} E_B H_B^{\frac{1}{2}}$ are the polar-like decompositions of A and B, respectively. Since the decomposition of a given matrix is unique, it is easy to check that the binary operation is a partial order. We call it the weak GL partial order (the WL partial order for short).

Theorem 2.2. *The binary operation (2.2) is a partial order.*

Theorem 2.3. *Let $A, B \in \mathbb{C}_{m,n}$. Then*

$$A \overset{\mathtt{WL}}{\leq} B \;\Leftrightarrow\; A^* \overset{\mathtt{WL}}{\leq} B^*. \tag{2.3}$$

Proof. Let $A = G_A^{\frac{1}{2}} E_A H_A^{\frac{1}{2}}$. Since $G_A^{\frac{1}{2}} = H_{A^*}^{\frac{1}{2}}$, $E_A = E_{A^*}$ and $H_A^{\frac{1}{2}} = G_{A^*}^{\frac{1}{2}}$, we derive (2.3). □

Theorem 2.4. *Let $A, B \in \mathbb{C}_{m,n}$, $A = G_A^{\frac{1}{2}} E_A H_A^{\frac{1}{2}}$ and $B = G_B^{\frac{1}{2}} E_B H_B^{\frac{1}{2}}$ be their polar-like decompositions, and $E_A \overset{*}{\leq} E_B$. Then*

$$G_A^{\frac{1}{2}} \overset{\mathtt{L}}{\leq} G_B^{\frac{1}{2}} \;\Leftrightarrow\; H_A^{\frac{1}{2}} \overset{\mathtt{L}}{\leq} H_B^{\frac{1}{2}}. \tag{2.4}$$

Proof. Let $G_A^{\frac{1}{2}} \overset{\mathtt{L}}{\leq} G_B^{\frac{1}{2}}$ and $E_A \overset{*}{\leq} E_B$. Then

$$U_A \Sigma_A^{\frac{1}{2}} U_A^* \overset{\mathtt{L}}{\leq} U_B \Sigma_B^{\frac{1}{2}} U_B^*, \tag{2.5}$$
$$U_A V_A^* \overset{*}{\leq} U_B V_B^*. \tag{2.6}$$

Applying (2.6), we have $V_A U_A^* U_A V_A^* = V_A U_A^* U_B V_B^*$. It follows from $U_A^* U_A = I = V_A^* V_A$ and

$$V_A^* \left(V_A U_A^* U_A V_A^* \right) V_B = V_A^* \left(V_A U_A^* U_B V_B^* \right) V_B$$

that

$$V_A^* V_B = U_A^* U_B. \tag{2.7}$$

Applying (2.5) and (2.7), we have $V_B^* V_A \Sigma_A^{\frac{1}{2}} V_A^* V_B \overset{\mathtt{L}}{\leq} \Sigma_B^{\frac{1}{2}}$. Therefore,

$$V_B V_B^* V_A \Sigma_A^{\frac{1}{2}} V_A^* V_B V_B^* \overset{\mathtt{L}}{\leq} V_B \Sigma_B^{\frac{1}{2}} V_B^*. \tag{2.8}$$

Applying (2.6), we have $\left(U_A V_A^* \right)^* U_A V_A^* \overset{*}{\leq} \left(U_B V_B^* \right)^* U_B V_B^*$, i.e.,

$$V_A V_A^* \overset{*}{\leq} V_B V_B^*.$$

It follows that $V_AV_A^*V_AV_A^* = V_AV_A^* = V_AV_A^*V_BV_B^*$. Therefore, $V_A^*V_AV_A^* = V_A^*V_AV_A^*V_BV_B^*$, that is, $V_A^* = V_A^*V_BV_B^*$. It follows from (2.8) that

$$V_A\Sigma_A^{\frac{1}{2}}V_A^* \overset{\mathtt{L}}{\leq} V_B\Sigma_B^{\frac{1}{2}}V_B^*,$$

i.e., $H_A^{\frac{1}{2}} \overset{\mathtt{L}}{\leq} H_B^{\frac{1}{2}}$.

On the contrary, applying $E_A \overset{*}{\leq} E_B$ and $H_A^{\frac{1}{2}} \overset{\mathtt{L}}{\leq} H_B^{\frac{1}{2}}$, we obtain $G_A^{\frac{1}{2}} \overset{\mathtt{L}}{\leq} G_B^{\frac{1}{2}}$. □

Theorem 2.5. *Let $A, B \in \mathbb{C}_{m,n}$. Then*

$$A \overset{\mathtt{WL}}{\leq} B \Leftrightarrow G_A^{\frac{1}{2}} \overset{\mathtt{L}}{\leq} G_B^{\frac{1}{2}}, E_A \overset{*}{\leq} E_B, \tag{2.9}$$

$$\Leftrightarrow E_A \overset{*}{\leq} E_B, H_A^{\frac{1}{2}} \overset{\mathtt{L}}{\leq} H_B^{\frac{1}{2}}. \tag{2.10}$$

It is well known that the star partial order is preserved for the Moore-Penrose inverse, that is,

$$A \overset{*}{\leq} B \;\Leftrightarrow\; A^\dagger \overset{*}{\leq} B^\dagger.$$

Theorem 2.6. *Let $A \in \mathbb{C}_{m,n}$. Then the polar-like decomposition of $A^\dagger$ is*

$$A^\dagger = \left(H_A^\dagger\right)^{\frac{1}{2}} E_A^* \left(G_A^\dagger\right)^{\frac{1}{2}}, \tag{2.11}$$

where E_A, G_A and H_A are given in Theorem 1.1

Proof. [1)] Let A be written as in (1.1), and let $X = E_A^*G_A^\dagger$. Now, since $G_A = G_A^*$, we have $\left(G_A^\dagger\right)^* = G_A^\dagger$, and thus

$$AX = G_AE_AE_A^*G_A^\dagger = G_AE_AE_A^\dagger G_A^\dagger = G_AG_AG_A^\dagger G_A^\dagger,$$

hence

$$(AX)^* = \left(G_A^\dagger\right)^* G_AG_A^\dagger G_A^* = G_A^\dagger G_AG_A^\dagger G_A = G_A^\dagger G_A,$$

which proves that AX is Hermitian and $AX = G_A^\dagger G_A$. Also,

$$XA = E_A^*G_A^\dagger A = E_A^*E_A \text{ is Hermitian,}$$
$$AXA = A(XA) = G_AE_AE_A^*E_A = G_AE_AE_A^\dagger E_A = G_AE_A = A,$$
$$XAX = (XA)X = E_A^*E_AE_A^*G_A^\dagger = E_A^\dagger E_AE_A^\dagger G_A^\dagger = E_A^\dagger G_A^\dagger = E_A^*G_A^\dagger = X.$$

This proves that $A^\dagger = E_A^*G_A^\dagger$. In the same way, we have $A^\dagger = H_A^\dagger E_A^*$. Therefore, applying Theorem 2.1, we have (2.11). □

Note that, the Löwner partial order may not be preserved for the Moore-Penrose inverse. Even when $A, B \in \mathbb{C}_n^{\geq}$,

$$A \overset{\mathtt{L}}{\leq} B \;\not\Leftrightarrow\; B^\dagger \overset{\mathtt{L}}{\leq} A^\dagger.$$

It follows from Theorem 2.6 that we drive the following Theorem 2.7.

[1)]This proof was provided by an anonymous reviewer.

Theorem 2.7. *Let $A, B \in \mathbb{C}_{m,n}$, $A = G_A^{\frac{1}{2}} E_A H_A^{\frac{1}{2}}$ and $B = G_B^{\frac{1}{2}} E_B H_B^{\frac{1}{2}}$ be their polar-like decompositions. Then*

$$A^{\dagger} \overset{\mathtt{WL}}{\leq} B^{\dagger} \Leftrightarrow G_A^{\dagger\frac{1}{2}} \overset{\mathtt{L}}{\leq} G_B^{\dagger\frac{1}{2}} \text{ and } E_A \overset{*}{\leq} E_B,$$
$$\Leftrightarrow E_A \overset{*}{\leq} E_B \text{ and } H_A^{\dagger\frac{1}{2}} \overset{\mathtt{L}}{\leq} H_B^{\dagger\frac{1}{2}}.$$

Lemma 2.8. [1] *For Hermitian nonnegative definite matrices A and B consider the following:*

$$(\mathrm{a}_1)\ A \overset{\mathtt{L}}{\leq} B,\ (\mathrm{a}_2)\ A \overset{*}{\leq} B,\ (\mathrm{b}_1)\ A^2 \overset{\mathtt{L}}{\leq} B^2,\ (\mathrm{b}_2)\ A^2 \overset{*}{\leq} B^2,\ (\mathrm{c})\ AB = BA.$$

Then

$$(\mathrm{a}_1), (\mathrm{c}) \Rightarrow (\mathrm{b}_1);\ (\mathrm{b}_1) \Rightarrow (\mathrm{a}_1);\ (\mathrm{a}_2) \Leftrightarrow (\mathrm{b}_2) \Rightarrow (\mathrm{c}).$$

Theorem 2.9. *Let $A, B \in \mathbb{C}_{m,n}$ and $A \overset{\mathtt{GL}}{\leq} B$. Then $A \overset{\mathtt{WL}}{\leq} B$.*

Proof. Let $A \overset{\mathtt{GL}}{\leq} B$, that is,

$$H_A \overset{\mathtt{L}}{\leq} H_B,\ E_A \overset{*}{\leq} E_B,\ G_A \overset{\mathtt{L}}{\leq} G_B.$$

Applying Lemma 2.8, we have

$$G_A \overset{\mathtt{L}}{\leq} G_B \Rightarrow G_A^{\frac{1}{2}} \overset{\mathtt{L}}{\leq} G_B^{\frac{1}{2}},$$
$$H_A \overset{\mathtt{L}}{\leq} H_B \Rightarrow H_A^{\frac{1}{2}} \overset{\mathtt{L}}{\leq} H_B^{\frac{1}{2}}.$$

Therefore, we derive $A \overset{\mathtt{WL}}{\leq} B$. □

The condition $A \overset{\mathtt{WL}}{\leq} B$ does not imply the condition $A \overset{\mathtt{GL}}{\leq} B$ as the following example shows.

Example 2.10 ([1]). *Let $A = \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix}, B = \begin{bmatrix} 9 & 0 \\ 0 & 36 \end{bmatrix}$. Then*

$$G_A = \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix},\ E_A = \begin{bmatrix} 0.2 & 0.4 \\ 0.4 & 0.8 \end{bmatrix},\ H_A = \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix},$$
$$G_B = \begin{bmatrix} 9 & 0 \\ 0 & 36 \end{bmatrix},\quad E_B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\quad H_B = \begin{bmatrix} 9 & 0 \\ 0 & 36 \end{bmatrix}.$$

Since

$$G_A^{\frac{1}{2}} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix},\ H_A^{\frac{1}{2}} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix},\ G_B^{\frac{1}{2}} = \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix},\ H_B^{\frac{1}{2}} = \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix},$$

we derive

$$G_A^{\frac{1}{2}} \overset{\mathtt{L}}{\leq} G_B^{\frac{1}{2}},\ E_A \overset{*}{\leq} E_B,\ H_A^{\frac{1}{2}} \overset{\mathtt{L}}{\leq} H_B^{\frac{1}{2}},$$

that is, $A \overset{\mathtt{WL}}{\leq} B$. Furthermore,

$$B - A = H_B - H_A = G_B - G_A = \begin{bmatrix} 4 & -10 \\ -10 & 16 \end{bmatrix},$$
$$\mathrm{rk}(B) = 2,\ \mathrm{rk}(A) = 1,\ \mathrm{rk}(B - A) = 2,$$

$\mathcal{V}(A) = 0$, $\mathcal{V}(B) = 0$ and $\mathcal{V}(B - A) = 1$ are 0, 0 and 1, respectively, where $\mathcal{V}(A)$ denotes the numbers of negative eigenvalues of A. Then

(1.) *since* $\det(B-A) = -36$, *A is not below B with respect to the GL partial order;*

(2.) *since* $\mathrm{rk}(B-A) \neq \mathrm{rk}(B) - \mathrm{rk}(A)$, *A is not below B with respect to the minus partial order;*

(3.) *since* $\mathcal{V}(B-A) \neq \mathcal{V}(B) - \mathcal{V}(A)$, *A is not below B with respect to the "* $\overset{\circ}{\leq}$ *" partial order.*

The " $\overset{\circ}{\leq}$ *" partial order is given [15, Theorem 8.5.4]*

$$A \overset{\circ}{\leq} B \;\Leftrightarrow\; \mathcal{R}(A) \subseteq \mathcal{R}(B) \text{ and } \mathcal{V}(B-A) = \mathcal{V}(B) - \mathcal{V}(A),$$

where $A, B \in \mathbb{C}_n^{\mathrm{H}}$.

Note that, when $A, B \in \mathbb{C}_n^{\geq}$,

$$A \overset{\mathtt{GL}}{\leq} B \Leftrightarrow A \overset{\mathtt{L}}{\leq} B \text{ [9, Theorem 3]},$$
$$\Leftrightarrow A \overset{\mathtt{CL}}{\leq} B, \text{[20, Corollary 3.8, Corollary 3.9]}.$$

From the Example 2.10 above, we can see that such a property is not valid for the WL partial order, that is, $A \overset{\mathtt{WL}}{\leq} B \;\not\Leftrightarrow\; A \overset{\mathtt{L}}{\leq} B$, even when $A, B \in \mathbb{C}_n^{\geq}$.

Theorem 2.11. *Let* $A, B \in \mathbb{C}_n^{\geq}$, *and* $AB = BA$ *(or* $AB \in \mathbb{C}_n^{\geq}$*). Then*

$$A \overset{\mathtt{WL}}{\leq} B \;\Leftrightarrow\; A \overset{\mathtt{GL}}{\leq} B \;\Leftrightarrow\; A \overset{\mathtt{CL}}{\leq} B \;\Leftrightarrow\; A \overset{\mathtt{L}}{\leq} B. \tag{2.12}$$

Proof. Let $A, B \in \mathbb{C}_n^{\geq}$ and $A \overset{\mathtt{WL}}{\leq} B$. It is well known that, if A commutes with B, then AB is a Hermitian nonnegative definite matrix, and $A^{\frac{1}{2}}$ commutes with $B^{\frac{1}{2}}$, [11] . It follows from Lemma 2.8, $G_A^{\frac{1}{2}} = A^{\frac{1}{2}}$ and $G_B^{\frac{1}{2}} = B^{\frac{1}{2}}$ that $G_A \overset{\mathtt{L}}{\leq} G_B$. Therefore, applying Theorem 2.2 and Theorem 1.2, we derive $A \overset{\mathtt{GL}}{\leq} B$.

On the contrary, applying Theorem 2.9, we have $A \overset{\mathtt{GL}}{\leq} B \;\Rightarrow\; A \overset{\mathtt{WL}}{\leq} B$.

Furthermore, applying [9, Theorem 3] and [20, Corollary 3.8, Corollary 3.9] we obtain (2.12). □

A binary relation is called a *pre-order* if it is reflexive and transitive on a non-empty set. It is well known that the Drazin order,

$$A \overset{D}{\leq} B \Leftrightarrow AA^D = BA^D = A^D B, \tag{2.13}$$

is a pre-order. Especially, when $\mathrm{Ind}(A) = 1$, the Drazin order is reduced to the well-known partial order: the sharp order. In [8, Page 164], a pre-order is characterized by:

$$A \prec B : \;\; E_A \overset{*}{\leq} E_B, \tag{2.14}$$

in which $A = G_A^{\frac{1}{2}} E_A H_A^{\frac{1}{2}}$ and $B = G_B^{\frac{1}{2}} E_B H_B^{\frac{1}{2}}$ are the polar-like decompositions of A and B, respectively.

From Theorem 2.1, we know that the polar-like decomposition of a given A is unique. In Theorem 2.5, we reduce the number of the conditions to two. But we cannot derive $A \overset{\mathtt{WL}}{\leq} B$ by applying $H_A^{\frac{1}{2}} \overset{\mathtt{L}}{\leq} H_B^{\frac{1}{2}}$ and $G_A^{\frac{1}{2}} \overset{\mathtt{L}}{\leq} G_B^{\frac{1}{2}}$. For example, let

$$A = I_n \;\text{ and }\; B = -I_n, \tag{2.15}$$

then $G_A = G_B = H_A = H_B = I_n$, $E_A = I_n$ and $E_B = -I_n$. It is obvious that $A \neq B$, although $H_A \overset{\text{L}}{\leq} H_B$, $H_B \overset{\text{L}}{\leq} H_A$, $G_A \overset{\text{L}}{\leq} G_B$ and $G_B \overset{\text{L}}{\leq} G_A$.

Consider the binary operation

$$A \overset{\text{P}}{<} B: \;\; H_A^{\frac{1}{2}} \overset{\text{L}}{\leq} H_B^{\frac{1}{2}} \;\text{ and }\; G_A^{\frac{1}{2}} \overset{\text{L}}{\leq} G_B^{\frac{1}{2}}, \tag{2.16}$$

in which $A = G_A^{\frac{1}{2}} E_A H_A^{\frac{1}{2}}$ and $B = G_B^{\frac{1}{2}} E_B H_B^{\frac{1}{2}}$ are as in (2.2). It is easy to check that the binary operation (2.16) is reflexive and transitive. From (2.15), we see that the binary operation is not antisymmetric. Therefore, it is a pre-order.

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