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Asymmetric Truncated Toeplitz Operators and Conjugations

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Abstract. Truncated Toeplitz operators in a model space are C–symmetric with respect to a natural conjugation in that space. We show that this and another conjugation associated to an orthogonal decomposition possess unique properties and we study their relations with asymmetric truncated Toeplitz operators in terms of C–symmetry. New connections with Hankel operators are established through this approach.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space, and denote by $L(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. A *conjugation* on $\mathcal{H}$ is an antilinear involution $C\colon \mathcal{H} \to \mathcal{H}$ such that $\langle Cf, Cg\rangle = \langle g, f\rangle$ for all $f, g \in \mathcal{H}$. Conjugations and their relations with various classes of operators have been studied in Hilbert spaces for many years. A new motivation to study them came from [7], and many interesting results have recently appeared on this topic [2, 8, 10–12, 16]. In particular, the study of *C–symmetric* operators, i.e., operators $A \in L(\mathcal{H})$ such that $CAC = A^*$, has attracted much attention, with particular emphasis on the case where the underlying Hilbert spaces are model spaces, defined as follows.

Let us denote by L^2 the space $L^2(\mathbb{T}, m)$, where $\mathbb{T}$ is the unit circle and m is the normalized Lebesgue measure on $\mathbb{T}$, and let $H^2 = H^2(\mathbb{D})$ be the Hardy space on the unit disc, identified as usual with a subspace of L^2. If θ is an inner function, i.e., $\theta \in H^\infty$ ($H^\infty = H^\infty(\mathbb{D})$ denotes the space of all bounded analytic functions in $\mathbb{D}$), $|\theta(t)| = 1$ a.e. on $\mathbb{T}$, the model space K_θ is defined by $K_\theta = H^2 \ominus \theta H^2$. It follows from Beurling's theorem that these are the invariant subspaces for the classical backward shift S^*. We denote by P_θ the orthogonal projection from L^2 onto K_θ, and by K_θ^∞ the dense subset of K_θ defined by $K_\theta^\infty = K_\theta \cap H^\infty$ ([15]).

One of the most important classes of operators on model spaces is that of truncated Toeplitz operators ([15]), which have been widely studied recently (see for example [1, 5, 15]). For $\varphi \in L^2$, a *truncated Toeplitz operator* A_φ^θ is defined, for all $f \in K_\theta$ such that $\varphi f \in L^2$ (and, in particular, for all $f \in K_\theta^\infty$), by

$$A_\varphi^\theta f = P_\theta(\varphi f).$$

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If this operator is bounded, then it can be uniquely extended to a bounded operator on K_θ; in that case we say that $A_\varphi^\theta \in \mathcal{T}(\theta)$.

One can define a conjugation C_θ in L^2, $C_\theta(f) = \theta\bar{z}\bar{f}$ for $f \in L^2$, which preserves the model space K_θ (i.e., $C_\theta P_\theta = P_\theta C_\theta$), and therefore induces a conjugation in K_θ, also denoted by C_θ. This conjugation plays an important role in the study of truncated Toeplitz operators. In fact, the latter are C_θ–symmetric [7], i.e., $C_\theta A C_\theta = A^*$ for $A \in \mathcal{T}(\theta)$ or, equivalently, $AC_\theta - C_\theta A^* = 0$.

More generally, one can consider *asymmetric truncated Toeplitz operators* between two (eventually) different model spaces K_θ and K_α, where α and θ are nonconstant inner functions. For $\varphi \in L^2$, we define

$$A_\varphi^{\theta,\alpha}\colon \mathcal{D} \subset K_\theta \to K_\alpha, \qquad A_\varphi^{\theta,\alpha} f = P_\alpha(\varphi f) \tag{1.1}$$

with domain $\mathcal{D} = \mathcal{D}(A_\varphi^{\theta,\alpha}) = \{f \in K_\theta\colon \varphi f \in L^2\} \supset K_\theta^\infty$. Again, if this operator is bounded, it has a unique bounded extension to K_θ, $A_\varphi^{\theta,\alpha}\colon K_\theta \to K_\alpha$, and the class of all such operators is denoted by $\mathcal{T}(\theta,\alpha)$. Recall after [3] that if $A_\varphi^{\theta,\alpha} \in \mathcal{T}(\theta,\alpha)$, then $(A_\varphi^{\theta,\alpha})^* = A_{\bar{\varphi}}^{\alpha,\theta} \in \mathcal{T}(\alpha,\theta)$. Asymmetric truncated Toeplitz operators were studied in [3] in the context of $H^2(\mathbb{D})$, and in [4] in the context of the Hardy space on the upper half-plane $H^p(\mathbb{C}^+)$ $(1 < p < \infty)$.

When α divides θ ($\alpha \preccurlyeq \theta$), i.e., $\frac{\theta}{\alpha}$ is an inner function, then $K_\alpha \subset K_\theta$ and we have the orthogonal decomposition $K_\theta = K_\alpha \oplus \alpha K_{\frac{\theta}{\alpha}}$. This suggests to define another conjugation in K_θ, besides C_θ, denoted by $C_{\alpha,\frac{\theta}{\alpha}}$ and defined by (3.1). It turns out that these conjugations are unique in the sense that they coincide, on both K_α and $\alpha K_{\frac{\theta}{\alpha}}$, with conjugations on L^2 for which the operator of multiplication by the independent variable, M_z, is C–symmetric (Theorem 4.7).

In this paper we investigate the relations of asymmetric truncated Toeplitz operators with these two conjugations and we show that certain identities of C–symmetric type still hold for these operators when the conjugation C is one of the above mentioned ones, C_θ or $C_{\alpha,\frac{\theta}{\alpha}}$ (Theorem 5.3). Moreover, since we no longer have the equality $A_\varphi^{\theta,\alpha} C - C(A_\varphi^{\theta,\alpha})^* = 0$ in general, we study various differences of that type and we show that they can be expressed in terms of Hankel operators.

2. The actions $\diamond$ and $\boxplus$

In the following section the letters $\mathcal{H}, \mathcal{K}$, with or without indexes, denote complex Hilbert spaces. Let $L(\mathcal{H},\mathcal{K})$ (respectively, $LA(\mathcal{H},\mathcal{K})$) denote the space of all bounded linear (respectively, antilinear) operators from $\mathcal{H}$ to $\mathcal{K}$. Recall that for $X \in LA(\mathcal{H},\mathcal{K})$ there is a unique antilinear operator $X^\sharp$, called the *antilinear adjoint* of X, satisfying the equality

$$\langle Xf, g\rangle = \overline{\langle f, X^\sharp g\rangle}, \tag{2.1}$$

for all $f \in \mathcal{H}$, $g \in \mathcal{K}$. It is easy to see that the antilinear adjoint has the following properties:

Proposition 2.1. 1. *If $X \in LA(\mathcal{H},\mathcal{K})$, then $(X^\sharp)^\sharp = X$.*
2. *If $X_1 \in LA(\mathcal{H}_1,\mathcal{H}_2)$ and $X_2 \in LA(\mathcal{H}_2,\mathcal{H}_3)$, then $X_2X_1 \in L(\mathcal{H}_1,\mathcal{H}_3)$ and $(X_2X_1)^* = X_1^\sharp X_2^\sharp$.*
3. *If $A \in L(\mathcal{H}_1,\mathcal{H})$ and $X \in LA(\mathcal{H},\mathcal{K})$, then $(XA)^\sharp = A^* X^\sharp$.*
4. *If $B \in L(\mathcal{K},\mathcal{K}_1)$ and $X \in LA(\mathcal{H},\mathcal{K})$, then $(BX)^\sharp = X^\sharp B^*$.*

Let $X_1\colon \mathcal{H} \to \mathcal{K}_1$, $X_2\colon \mathcal{H} \to \mathcal{K}_2$, $Y_1\colon \mathcal{K}_1 \to \mathcal{H}$, $Y_2\colon \mathcal{K}_2 \to \mathcal{H}$ be (linear or antilinear) operators. Define the following actions:

$$X_1 \diamond X_2\colon \mathcal{H} \to \mathcal{K}_1 \oplus \mathcal{K}_2,\ (X_1 \diamond X_2)f = X_1 f \oplus X_2 f$$

and

$$Y_1 \boxplus Y_2\colon \mathcal{K}_1 \oplus \mathcal{K}_2 \to \mathcal{H},\ (Y_1 \boxplus Y_2)(f \oplus g) = Y_1 f + Y_2 g.$$

Proposition 2.2. *Let X_1, X_2, Y_1, Y_2 be antilinear operators, then:*

1. $(X_1 \diamond X_2)^\sharp = X_1^\sharp \boxplus X_2^\sharp$;
2. $(X_1 \boxplus X_2)^\sharp = X_1^\sharp \diamond X_2^\sharp$;
3. *if* $A \in L(\mathcal{K}_1 \oplus \mathcal{K}_2, \mathcal{K})$, *then* $(A(X_1 \diamond X_2))^\sharp = (X_1^\sharp \boxplus X_2^\sharp)A^*$;
4. *if* $B \in L(\mathcal{H}_1, \mathcal{K}_1 \oplus \mathcal{K}_2)$, *then* $((Y_1 \boxplus Y_2)B)^\sharp = B^*(Y_1^\sharp \diamond Y_2^\sharp)$.

Proof. To show (1) let us take $f \in \mathcal{H}$, $g_1 \in \mathcal{K}_1$, $g_2 \in \mathcal{K}_2$. Then

$$\langle (X_1 \diamond X_2)f, g_1 \oplus g_2\rangle = \langle X_1 f \oplus X_2 f, g_1 \oplus g_2\rangle = \langle X_1 f, g_1\rangle + \langle X_2 f, g_2\rangle = \overline{\langle f, X_1^\sharp g_1\rangle} + \overline{\langle f, X_2^\sharp g_2\rangle} = \overline{\langle f, (X_1^\sharp \boxplus X_2^\sharp)(g_1 \oplus g_2)\rangle}.$$

The equalities (2), (3) and (4) follow directly from (1) and Proposition 2.1. □

Remark 2.3. *Note that the proposition above holds if we change antilinear operators to linear operators,* $^\sharp$ *to* * *and vice versa.*

Now let us consider two conjugations C_1, C_2 on $\mathcal{H}$. Define the following actions:

$$C_\diamond = \tfrac{1}{\sqrt{2}} C_1 \diamond C_2 \colon \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}, \text{ and } C_\boxplus = \tfrac{1}{\sqrt{2}} C_1 \boxplus C_2 \colon \mathcal{H} \oplus \mathcal{H} \to \mathcal{H}. \tag{2.2}$$

Proposition 2.4. *Let* C_1, C_2 *be conjugations on* $\mathcal{H}$. *Then*

1. $C_\boxplus \circ C_\diamond \colon \mathcal{H} \to \mathcal{H}$ *and* $C_\diamond \circ C_\boxplus \colon \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$ *are linear operators;*
2. $C_\boxplus \circ C_\diamond = I_\mathcal{H}$;
3. $(C_\boxplus)^\sharp = C_\diamond$ *and* $(C_\diamond)^\sharp = C_\boxplus$;
4. $C_\diamond \circ C_\boxplus = Q$, *where* Q *is an orthogonal projection;*
5. $\ker Q = \{C_2 f \oplus -C_1 f : f \in \mathcal{H}\}$;
6. $\operatorname{ran} Q = \{C_2 f \oplus C_1 f : f \in \mathcal{H}\}$.

Proof. The statement (1) is immediate. To prove (2) let us take $f \in \mathcal{H}$. Then we have

$$\tfrac{1}{2}(C_1 \boxplus C_2)(C_1 \diamond C_2)f = \tfrac{1}{2}(C_1 \boxplus C_2)(C_1 f \oplus C_2 f) = \tfrac{1}{2}(C_1^2 f + C_2^2 f) = f.$$

The equalities in (3) follow from Proposition 2.2. Take now $f, g \in \mathcal{H}$, then

$$\tfrac{1}{2}(C_1 \diamond C_2)(C_1 \boxplus C_2)(f \oplus g) = \tfrac{1}{2}((f + C_1 C_2 g) \oplus (g + C_2 C_1 f)). \tag{2.3}$$

Hence

$$(\tfrac{1}{2}(C_1 \diamond C_2)(C_1 \boxplus C_2))^2 (f \oplus g) = \tfrac{1}{2}(C_1 \diamond C_2)(C_1 \boxplus C_2)(\tfrac{1}{2}((f + C_1 C_2 g) \oplus (g + C_2 C_1 f))) = \tfrac{1}{2}((f + C_1 C_2 g) \oplus (g + C_2 C_1 f)).$$

So (4) holds and (5) and (6) follow from (2.3). □

The next proposition is related to (5.6) in the main theorem of the Section 5.

Proposition 2.5. *Let* C_1, C_2 *be conjugations in* $\mathcal{H}$ *and let* $C_\boxplus, C_\diamond$ *be defined as in* (2.2). *Let* $A \in L(\mathcal{H})$ *be* C_1*–symmetric and* C_2*–symmetric. Then*

$$C_\boxplus (A \oplus A) C_\diamond = A^*.$$

Recall that any unitary operator $U \in L(\mathcal{H})$ is a product of two conjugations C_1, C_2 ([9]). Moreover, as it was shown in [6], such a unitary operator is both C_1 and C_2–symmetric. Hence any unitary operator satisfies the assumptions of Proposition 2.5 for suitable conjugations.

3. Conjugations in model spaces: C_θ and $C_{\alpha,\frac{\theta}{\alpha}}$

Let α and θ be nonconstant inner functions such that $\alpha \leqslant \theta$. Then by [5, Lemma 5.10] the model space K_θ can be decomposed as $K_\alpha \oplus \alpha K_{\frac{\theta}{\alpha}}$ or $K_{\frac{\theta}{\alpha}} \oplus \frac{\theta}{\alpha} K_\alpha$. Hence $P_\theta = P_\alpha + \alpha P_{\frac{\theta}{\alpha}} \bar{\alpha}$ and $P_\theta = P_{\frac{\theta}{\alpha}} + \frac{\theta}{\alpha} P_\alpha \frac{\bar{\theta}}{\bar{\alpha}}$.

Proposition 3.1 (Proposition 2.3, [3]). *Let α, θ be nonconstant inner functions such that $\alpha \leqslant \theta$. If $f_1 \in K_\alpha$ and $f_2 \in K_{\frac{\theta}{\alpha}}$, then*

1. $C_\theta(f_1 + \alpha f_2) = C_{\frac{\theta}{\alpha}} f_2 + \frac{\theta}{\alpha} C_\alpha f_1$,
2. $C_\theta(f_2 + \frac{\theta}{\alpha} f_1) = C_\alpha f_1 + \alpha C_{\frac{\theta}{\alpha}} f_2$.

The orthogonal decomposition $K_\theta = K_\alpha \oplus \alpha K_{\frac{\theta}{\alpha}}$ suggests to consider another conjugation $C_{\alpha,\frac{\theta}{\alpha}}$ on K_θ defined as

$$\begin{aligned} C_{\alpha,\frac{\theta}{\alpha}} :=& C_\alpha \oplus \alpha C_{\frac{\theta}{\alpha}} \bar{\alpha}, \\ C_{\alpha,\frac{\theta}{\alpha}}(g_1 + \alpha g_2) =& C_\alpha g_1 + \alpha C_{\frac{\theta}{\alpha}} g_2 = \alpha \bar{z}\, \bar{g_1} + \theta \bar{z}\, \bar{g_2} \end{aligned} \tag{3.1}$$

for $g_1 \in K_\alpha$, $g_2 \in K_{\frac{\theta}{\alpha}}$. To see that $C_{\theta,\frac{\theta}{\alpha}}$ is a conjugation it is enough to show that $C^2_{\theta,\frac{\theta}{\alpha}} = I_{K_\theta}$. Namely,

$$(C_\alpha \oplus \alpha C_{\frac{\theta}{\alpha}} \bar{\alpha})(C_\alpha \oplus \alpha C_{\frac{\theta}{\alpha}} \bar{\alpha}) = P_\alpha \oplus \alpha C_{\frac{\theta}{\alpha}} \bar{\alpha} \alpha C_{\frac{\theta}{\alpha}} \bar{\alpha} = P_\alpha \oplus \alpha I_{K_{\frac{\theta}{\bar{\alpha}}}} \bar{\alpha} = P_\alpha + \alpha P_{\frac{\theta}{\alpha}} \bar{\alpha} = I_{K_\theta}.$$

For any inner function θ and $\lambda \in \mathbb{D}$, denote

$$k^\theta_\lambda(z) = \tfrac{1-\overline{\theta(\lambda)}\,\theta(z)}{1-\bar{\lambda} z} \qquad \text{and} \qquad \tilde{k}^\theta_\lambda(z) = \tfrac{\theta(z)-\theta(\lambda)}{z-\lambda}.$$

Recall that k^θ_λ are reproducing kernel functions for the model space K_θ, i.e., $\langle f, k^\theta_\lambda \rangle = f(\lambda)$ for all $f \in K_\theta$. Assume that $\alpha \leqslant \theta$, the conjugations C_θ and $C_{\alpha,\frac{\theta}{\alpha}}$ act on reproducing kernel functions k^θ_λ as follows:

$$C_\theta\, k^\theta_\lambda = \tilde{k}^\theta_\lambda \qquad \text{and} \qquad C_{\alpha,\frac{\theta}{\alpha}}\, k^\theta_\lambda = \tilde{k}^\alpha_\lambda + \alpha(\lambda)\, \alpha\, \tilde{k}^{\frac{\theta}{\alpha}}_\lambda.$$

We have also the following "reproducing" properties. For any $f \in K_\theta$:

$$\langle f, C_\theta\, k^\theta_\lambda \rangle = \overline{(C_\theta f)(\lambda)} \qquad \text{and} \qquad \langle f, C_{\alpha,\frac{\theta}{\alpha}}\, k^\theta_\lambda \rangle = \overline{(C_{\alpha,\frac{\theta}{\alpha}} f)(\lambda)}.$$

Moreover, $C_{\alpha,\frac{\theta}{\alpha}} C_\theta$ and $C_\theta C_{\alpha,\frac{\theta}{\alpha}}$ are unitary operators (as compositions of two conjugations, see [6], [9]), which are inverses of each other. More precisely:

Proposition 3.2. *Let α, θ be nonconstant inner functions such that $\alpha \leqslant \theta$. Then $C_{\alpha,\frac{\theta}{\alpha}} C_\theta \colon K_\theta = K_{\frac{\theta}{\alpha}} \oplus \frac{\theta}{\alpha} K_\alpha \to K_\theta = K_\alpha \oplus \alpha K_{\frac{\theta}{\alpha}}$ and $C_\theta C_{\alpha,\frac{\theta}{\alpha}} \colon K_\theta = K_\alpha \oplus \alpha K_{\frac{\theta}{\alpha}} \to K_\theta = K_{\frac{\theta}{\alpha}} \oplus \frac{\theta}{\alpha} K_\alpha$ are unitary operators such that*

1. $C_\theta C_{\alpha,\frac{\theta}{\alpha}} = P_{\frac{\theta}{\alpha}} \bar{\alpha} + \frac{\theta}{\alpha} P_\alpha$,
2. $C_{\alpha,\frac{\theta}{\alpha}} C_\theta = P_\alpha \frac{\bar{\theta}}{\bar{\alpha}} + \alpha P_{\frac{\theta}{\alpha}}$.

As a special case of Proposition 2.4 we have:

Proposition 3.3. *Let α, θ be nonconstant inner functions such that $\alpha \leqslant \theta$. Define the following actions:*

$$C_\diamond = \tfrac{1}{\sqrt{2}} C_{\alpha,\frac{\theta}{\alpha}} \diamond C_\theta \colon K_\theta \to K_\theta \oplus K_\theta,\ (C_{\alpha,\frac{\theta}{\alpha}} \diamond C_\theta) f = C_{\alpha,\frac{\theta}{\alpha}} f \oplus C_\theta f$$

and

$$C_\boxplus = \tfrac{1}{\sqrt{2}} C_{\alpha,\frac{\theta}{\alpha}} \boxplus C_\theta \colon K_\theta \oplus K_\theta \to K_\theta,\ (C_{\alpha,\frac{\theta}{\alpha}} \boxplus C_\theta)(f \oplus g) = C_{\alpha,\frac{\theta}{\alpha}} f + C_\theta g.$$

Then

1. $C_\boxplus \circ C_\diamond \colon K_\theta \to K_\theta$ *and* $C_\diamond \circ C_\boxplus \colon K_\theta \oplus K_\theta \to K_\theta \oplus K_\theta$ *are linear operators,*
2. $C_\boxplus \circ C_\diamond = I_{K_\theta}$,
3. $C_\diamond \circ C_\boxplus = Q$, *where Q is an orthogonal projection in $K_\theta \oplus K_\theta$,*
4. $\ker Q = \{C_{\alpha,\frac{\theta}{\alpha}} f \oplus -C_\theta f : f \in K_\theta\}$,
5. $\operatorname{ran} Q = \{C_{\alpha,\frac{\theta}{\alpha}} f \oplus C_\theta f : f \in K_\theta\}$.

4. M_z–conjugations in L^2

In this section we will show that the conjugations C_θ and $C_{\alpha,\frac{\theta}{\alpha}}$ are in a certain sense unique.

Since we are motivated by truncated Toeplitz operators, we will concentrate on conjugations for which the multiplication by the independent variable M_z is C–symmetric. Let J denote the complex conjugation in L^2, that is $J\colon L^2 \to L^2$, $Jf = \bar{f}$ for $f \in L^2$. For $\varphi \in L^\infty$, denote by $M_\varphi\colon L^2 \to L^2$ a multiplication operator $M_\varphi f = \varphi f$, $f \in L^2$. A conjugation C on L^2 will be called an *M–conjugation* if $M_\varphi C = CM_{\bar{\varphi}}$ (i.e., M_φ is C–symmetric) for all $\varphi \in L^\infty$, and C will be called an *M_z–conjugation* if $M_z C = CM_{\bar{z}}$.

The following theorem fully characterizes M–conjugations in L^2. It also says that in fact the definitions of M–conjugation and M_z–conjugation are equivalent.

Theorem 4.1. *Let C be a conjugation in L^2. Then the following are equivalent:*

1. *$M_\varphi C = CM_{\bar{\varphi}}$ for all $\varphi \in L^\infty$ (C is an M–conjugation),*
2. *$M_z C = CM_{\bar{z}}$ (C is an M_z–conjugation),*
3. *there is $\psi \in L^\infty$, with $|\psi| = 1$, such that $C = M_\psi J$.*

Proof. It is enough to show that (2) $\Rightarrow$ (3). Assume that $CM_z = M_{\bar{z}}C$. Then $JCM_z = JM_{\bar{z}}C = M_z JC$. It means that the linear operator JC commutes with M_z. By [14, Theorem 3.2] $JC = M_{\bar{\psi}}$ for some $\psi \in L^\infty$. Hence $C = JM_{\bar{\psi}} = M_\psi J$.

Since C is a conjugation, we have $C^2 = I_{L^2}$. Therefore for all $f \in L^2$ we have

$$f = C^2 f = M_\psi J M_\psi J f = M_\psi J(\psi \bar{f}) = |\psi|^2 f,$$

which implies that $|\psi| = 1$ a.e. □

Now we study the invariant subspaces of M_z–conjugations and their relations with orthogonal decompositions of model spaces.

Theorem 4.2. *Let α, γ, θ be inner functions (α, θ nonconstant) such that $\gamma\alpha \leqslant \theta$. Let C be a conjugation in L^2 such that $M_z C = CM_{\bar{z}}$. Assume that $C(\gamma K_\alpha) \subset K_\theta$. Then there is an inner function β such that $C = C_\beta$, with $\gamma\alpha \leqslant \beta \leqslant \gamma\theta$.*

Proof. Recall the standard notation for the reproducing kernel functions at 0 in K_α, namely, $k_0^\alpha = 1 - \overline{\alpha(0)}\alpha$ and $\tilde{k}_0^\alpha = C_\alpha k_0^\alpha = \bar{z}(\alpha - \alpha(0))$. By Theorem 4.1 we know that $C = M_\psi J$ for some function $\psi \in L^\infty$, $|\psi| = 1$. Hence

$$K_\theta \ni C(\gamma \tilde{k}_0^\alpha) = M_\psi J(\gamma \tilde{k}_0^\alpha) = \psi \overline{\gamma \bar{z}(\alpha - \alpha(0))} = \bar{\gamma}\bar{\alpha} z \psi (1 - \overline{\alpha(0)}\alpha).$$

Thus there is $h \in K_\theta$ such that $h = \bar{\gamma}\bar{\alpha} z\psi(1 - \overline{\alpha(0)}\alpha)$. Since $(1 - \overline{\alpha(0)}\alpha)^{-1}$ is a bounded analytic function, we have

$$\bar{\gamma}\bar{\alpha} z\psi = h(1 - \overline{\alpha(0)}\alpha)^{-1} \in H^2.$$

Since $\beta_1 = \bar{\gamma}\bar{\alpha} z\psi \in H^2$ and $|\bar{\gamma}\bar{\alpha} z\psi| = 1$ a.e. on $\mathbb{T}$, it has to be an inner function.

On the other hand, we have similarly

$$K_\theta \ni C_\theta C(\gamma k_0^\alpha) = C_\theta(\psi \overline{\gamma(1 - \overline{\alpha(0)}\alpha)} = \theta\gamma\bar{z}\bar{\psi}(1 - \overline{\alpha(0)}\alpha),$$

and $\theta\gamma\bar{z}\bar{\psi} \in H^2$. Hence

$$H^2 \ni \theta\gamma\bar{z}\bar{\psi} = \tfrac{\theta}{\alpha}\overline{\bar{\gamma}\,\bar{\alpha} z\psi} = \tfrac{\theta}{\alpha}\,\overline{\beta_1}.$$

But this is only possible when β_1 divides $\frac{\theta}{\alpha}$. Hence $\psi = \gamma\alpha\beta_1\bar{z} = \beta\bar{z}$ with $\gamma\alpha \leqslant \beta \leqslant \gamma\theta$. Finally, we have $C = C_\beta$. □

Taking $\gamma = 1$ and $\alpha = \theta$ we conclude that the conjugation C_θ is the only M_z–conjugation in L^2 which preserves the model space K_θ.

Theorem 4.3. *Let C be an M_z–conjugation in L^2 (i.e., $M_zC = CM_{\bar{z}}$). Assume that $C(K_\theta) \subset K_\theta$ for some nonconstant inner function θ. Then $C = C_\theta$.*

Remark 4.4. *Let us consider nonconstant inner functions α, β, θ such that $\alpha \leqslant \beta \leqslant \theta$. Then we have the decompositions:*

$$K_\theta = K_\beta \oplus \beta K_{\frac{\theta}{\beta}} = K_\alpha \oplus \alpha K_{\frac{\beta}{\alpha}} \oplus \beta K_{\frac{\theta}{\beta}}.$$

Observe that $C_\beta(K_\alpha) \subset K_\beta$. Let $\tilde{C}$ be any conjugation on $\beta K_{\frac{\theta}{\beta}}$. Then $C_\beta|_{K_\beta} \oplus \tilde{C}$ is a conjugation on K_θ.

The following is a consequence of Theorem 4.2 and Remark 4.4.

Proposition 4.5. *Let $\alpha \leqslant \theta$ be some nonconstant inner functions. Let C be an M_z–conjugation in L^2 (i.e., $M_zC = CM_{\bar{z}}$). Assume that $C(K_\alpha) \subset K_\theta$. Let $\tilde{C}$ be a conjugation on K_θ such that $C|_{K_\alpha} = \tilde{C}|_{K_\alpha}$. Then there is an inner function β with $\alpha \leqslant \beta \leqslant \theta$ and a certain conjugation $\tilde{\tilde{C}}$ on $\beta K_{\frac{\theta}{\beta}}$ such that $\tilde{C}_\theta = C_\beta \oplus \tilde{\tilde{C}}$.*

The following lemma will be used to prove the next theorem.

Lemma 4.6. *Let α_1, α_2 be nonconstant inner functions and let γ_1, γ_2 be inner functions such that $\gamma_1 \leqslant \alpha_1$ and $\gamma_2 \leqslant \alpha_2$. Assume that $\gamma_1 K_{\alpha_2} \oplus \gamma_2 K_{\alpha_1} = K_{\alpha_1\alpha_2}$. Then $\gamma_1 = 1$, $\gamma_2 = \alpha_2$ or $\gamma_1 = \alpha_1$, $\gamma_2 = 1$.*

Proof. Recall that inner functions are identified up to multiplication by a constant and let us assume that neither γ_1 nor γ_2 is constant. By [5, Theorem 5.11] we can decompose the space $K_{\alpha_1\alpha_2}$ in two ways

$$K_{\alpha_1\alpha_2} = K_{\gamma_1} \oplus \gamma_1 K_{\alpha_2} \oplus \gamma_1\alpha_2 K_{\frac{\alpha_1}{\gamma_1}} = K_{\gamma_2} \oplus \gamma_2 K_{\alpha_1} \oplus \gamma_2\alpha_1 K_{\frac{\alpha_2}{\gamma_2}}.$$

Since $\gamma_1 K_{\alpha_2} \oplus \gamma_2 K_{\alpha_1} = K_{\alpha_1\alpha_2}$, we have

$$\gamma_1 K_{\alpha_2} = K_{\gamma_2} \oplus \gamma_2\alpha_1 K_{\frac{\alpha_2}{\gamma_2}}.$$

Therefore

$$K_{\gamma_2} \subset \gamma_1 K_{\alpha_2} \subset \gamma_1 H^2.$$

It follows, as in [3, Lemma 4.2] that γ_1 has to be a constant or $K_{\gamma_2} = \{0\}$, i.e., γ_2 is a constant, and so we obtain a contradiction.

If $\gamma_1 = 1$, then, by [5, Theorem 5.11], we have

$$K_{\alpha_2} \oplus \gamma_2 K_{\alpha_1} = K_{\alpha_1\alpha_2} = K_{\alpha_2} \oplus \alpha_2 K_{\alpha_1},$$

hence $\gamma_2 = \alpha_2$. If γ_1 is not a constant, then we obtain $\gamma_1 = \alpha_1$, $\gamma_2 = 1$, analogously. □

The definition of the conjugation $C_{\alpha,\frac{\theta}{\alpha}}$ is natural in view of the orthogonal decomposition $K_\theta = K_\alpha \oplus \alpha K_{\frac{\theta}{\alpha}}$. However, it is easy to see that M_z is not $C_{\alpha,\frac{\theta}{\alpha}}$–symmetric. Moreover, $C_{\alpha,\frac{\theta}{\alpha}}$ is not a restriction to K_θ of any M_z–conjugation C on L^2. On the other hand, the restrictions of $C_{\alpha,\frac{\theta}{\alpha}}$ to the spaces K_α and $K_\theta \ominus K_\alpha$ are equal respectively to the restrictions of some (different) M_z–conjugations. In the following result we show that $C_{\alpha,\frac{\theta}{\alpha}}$ and C_θ are the only conjugations in K_θ with this property.

Theorem 4.7. *Let α, θ be nonconstant inner functions such that $\alpha \leqslant \theta$, and let $\tilde{C}$ be a conjugation on K_θ. Assume that there are conjugations C_i, $i = 1, 2$, on L^2 with $M_z C_i = C_i M_{\bar{z}}$ such that $\tilde{C}|_{K_\alpha} = C_1|_{K_\alpha}$ and $\tilde{C}|_{K_\theta \ominus K_\alpha} = C_2|_{K_\theta \ominus K_\alpha}$. Then $\tilde{C} = C_\theta$ or $\tilde{C} = C_{\alpha,\frac{\theta}{\alpha}} = C_\alpha \oplus \alpha C_{\frac{\theta}{\alpha}} \bar{\alpha}$.*

Proof. Note firstly that $C_1(K_\alpha) = \tilde{C}(K_\alpha) \subset K_\theta$. By Theorem 4.2 there is an inner function γ_1, $1 \leqslant \gamma_1 \leqslant \frac{\theta}{\alpha}$, such that

$$\tilde{C}|_{K_\alpha} = C_1|_{K_\alpha} = C_{\gamma_1\alpha}|_{K_\alpha} : K_\alpha \to \gamma_1 K_\alpha \subset K_\theta.$$

Recall that $C_{\gamma_1\alpha}|_{K_\alpha} f_\alpha = \gamma_1\alpha\bar{z}\bar{f}_\alpha = \gamma_1 C_\alpha f_\alpha$ for $f_\alpha \in K_\alpha$, and note that $C_{\gamma_1\alpha}|_{K_\alpha}$ is a bijection between K_α and $\gamma_1 K_\alpha$. Similarly, $C_2(\alpha K_{\frac{\theta}{\alpha}}) = C_2(K_\theta \ominus K_\alpha) = \tilde{C}(K_\theta \ominus K_\alpha) \subset K_\theta$. Hence there is an inner function γ_2, $1 \leqslant \gamma_2 \leqslant \alpha$, such that

$$\tilde{C}|_{K_\theta \ominus K_\alpha} = C_2|_{\alpha K_{\frac{\theta}{\alpha}}} = C_{\gamma_2\theta}|_{\alpha K_{\frac{\theta}{\alpha}}} : \alpha K_{\frac{\theta}{\alpha}} \to \gamma_2 K_{\frac{\theta}{\alpha}} \subset K_\theta.$$

On the other hand,

$$C_{\gamma_2\theta}|_{\alpha K_{\frac{\theta}{\alpha}}} \alpha f_{\frac{\theta}{\alpha}} = C_{\gamma_2\theta}(\alpha f_{\frac{\theta}{\alpha}}) = \gamma_2\theta\bar{z}\bar{\alpha}\bar{f}_{\frac{\theta}{\alpha}} = \gamma_2\frac{\theta}{\alpha}\bar{z}\bar{f}_{\frac{\theta}{\alpha}} = \gamma_2 C_{\frac{\theta}{\alpha}} f_{\frac{\theta}{\alpha}}$$

for $f_{\frac{\theta}{\alpha}} \in K_{\frac{\theta}{\alpha}}$. Note that $C_{\gamma_2\theta}|_{\alpha K_{\frac{\theta}{\alpha}}}$ is a bijection between $\alpha K_{\frac{\theta}{\alpha}}$ and $\gamma_2 K_{\frac{\theta}{\alpha}}$. Since involution preserves orthogonality and $K_\alpha \oplus \alpha K_{\frac{\theta}{\alpha}} = K_\theta$, we get that $\gamma_1 K_\alpha \oplus \gamma_2 K_{\frac{\theta}{\alpha}} = K_\theta$. By Lemma 4.6 there are now only two possibilities: either $\gamma_1 = 1, \gamma_2 = \alpha$ or $\gamma_1 = \frac{\theta}{\alpha}, \gamma_2 = 1$. In the second case $\tilde{C}|_{K_\alpha} = C_\theta|_{K_\alpha}$ and $\tilde{C}|_{K_\theta \ominus K_\alpha} = C_\theta|_{\alpha K_{\frac{\theta}{\alpha}}}$, hence $\tilde{C} = C_\theta$. In the first case $\tilde{C}|_{K_\alpha} = C_\alpha|_{K_\alpha}$ and $\tilde{C}|_{\alpha K_{\frac{\theta}{\alpha}}} = C_{\alpha\theta}|_{\alpha K_{\frac{\theta}{\alpha}}}$, since for $f_{\frac{\theta}{\alpha}} \in K_{\frac{\theta}{\alpha}}$ we have

$$C_{\alpha\theta}|_{\alpha K_{\frac{\theta}{\alpha}}} \alpha f_{\frac{\theta}{\alpha}} = \alpha\theta\bar{z}\bar{\alpha}\bar{f}_{\frac{\theta}{\alpha}} = \alpha C_{\frac{\theta}{\alpha}} f_{\frac{\theta}{\alpha}} = \alpha C_{\frac{\theta}{\alpha}} \bar{\alpha}\alpha f_{\frac{\theta}{\alpha}} = \alpha C_{\frac{\theta}{\alpha}} \bar{\alpha}|_{\alpha K_{\frac{\theta}{\alpha}}} \alpha f_{\frac{\theta}{\alpha}}.$$

Hence $\tilde{C} = C_\alpha \oplus \alpha C_{\frac{\theta}{\alpha}} \bar{\alpha} = C_{\alpha, \frac{\theta}{\alpha}}$. □

Example 4.8. *Let $\theta = z^5$ and $\alpha = z^3$. The only conjugation, besides C_{z^5}, defined by $C_{z^5}(z_0, z_1, z_2, z_3, z_4) = (\bar{z}_4, \bar{z}_3, \bar{z}_2, \bar{z}_1, \bar{z}_0)$, fulfilling the conditions of Theorem 4.7 is the conjugation C_{z^3,z^2} given by $C_{z^3,z^2}(z_0, z_1, z_2, z_3, z_4) = (\bar{z}_2, \bar{z}_1, \bar{z}_0, \bar{z}_4, \bar{z}_3)$.*

Example 4.9. *Let $\theta(z) = \exp\frac{z+1}{z-1}$ and $\alpha(z) = \exp(a\frac{z+1}{z-1})$ for $0 < a < 1$. Then $\frac{\theta}{\alpha}(z) = \exp((1-a)\frac{z+1}{z-1})$. The only conjugation, besides C_θ (defined by $C_\theta(f) = \theta\bar{z}\bar{f}$ for $f \in K_\theta$), fulfilling the conditions of Theorem 4.7 is the conjugation $C_{\alpha,\frac{\theta}{\alpha}}$ given by $C_{\alpha,\frac{\theta}{\alpha}}(f_\alpha \oplus \alpha f_{\frac{\theta}{\alpha}}) = \alpha\bar{z}\bar{f}_\alpha + \theta\bar{z}\bar{f}_{\frac{\theta}{\alpha}}$ for $f_\alpha \oplus \alpha f_{\frac{\theta}{\alpha}} \in K_\theta = K_\alpha \oplus \alpha K_{\frac{\theta}{\alpha}}$.*

5. C-symmetry of asymmetric truncated Toeplitz operators

Let $C : \mathcal{H} \to \mathcal{H}$ be a conjugation. Note that every conjugation is antilinearly selfadjoint, i.e., $C^\sharp = C$. The next lemma gives simple but important equivalent conditions for an operator to be C–symmetric.

Lemma 5.1. *Let $A \in L(\mathcal{H})$. Then the following are equivalent:*

1. *A is C–symmetric;*
2. *AC is antilinearly selfadjoint, i.e., $(AC)^\sharp = AC$;*
3. *CA is antilinearly selfadjoint, i.e., $(CA)^\sharp = CA$.*

It is well known that truncated Toeplitz operators are C_θ–symmetric, [7], i.e., for $A^\theta_\varphi \in \mathcal{T}(\theta)$ we have

$$A^\theta_\varphi C_\theta = C_\theta A^\theta_{\bar{\varphi}}. \tag{5.1}$$

One may wonder whether A^θ_φ is $C_{\alpha,\frac{\theta}{\alpha}}$–symmetric for all $\alpha \leqslant \theta$ but that is not the case in general, as it is shown by this simple example.

Example 5.2. *Let $\theta = z^2$ and $\alpha = z$. Then $C = C_{z,z} = J$ is the conjugation given by $C(z_0, z_1) = (\bar{z}_0, \bar{z}_1)$, $(z_0, z_1) \in \mathbb{C}^2$. Take a Toeplitz matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Then $AC \neq CA^*$.*

Since an asymmetric truncated Toeplitz operator $A_\varphi^{\theta,\alpha} \in \mathcal{T}(\theta,\alpha)$ reduces to the truncated Toeplitz operator A_φ^θ, and $C_{\alpha,\frac{\theta}{\alpha}} = C_\theta$ when $\alpha = \theta$, we may ask whether the following generalizations of (5.1) hold:

$$A_\varphi^{\theta,\alpha} C_\theta = C_\theta A_{\bar\varphi}^{\alpha,\theta} P_\alpha \qquad \text{or} \tag{5.2}$$

$$A_\varphi^{\theta,\alpha} C_{\alpha,\frac{\theta}{\alpha}} = C_{\alpha,\frac{\theta}{\alpha}} A_{\bar\varphi}^{\alpha,\theta} P_\alpha. \tag{5.3}$$

It is easy to see that neither (5.2) nor (5.3) are true in general. To obtain properties which, in the context of Lemma 5.1, can be regarded in some sense as describing C–symmetric properties of truncated Toeplitz operators, we will consider the whole space K_θ and use the actions $\diamond$, $\boxplus$.

Theorem 5.3. *Let α, θ be nonconstant inner functions such that $\alpha \preccurlyeq \theta$, and let $\varphi \in L^2$ be such that all asymmetric truncated Toeplitz operators below are bounded. Let us consider the conjugations C_θ and $C_{\alpha,\frac{\theta}{\alpha}} = C_\alpha \oplus \alpha C_{\frac{\theta}{\alpha}} \bar\alpha$ in $K_\theta = K_\alpha \oplus \alpha K_{\frac{\theta}{\alpha}}$. Then the following equalities hold:*

$$(A_\varphi^{\theta,\alpha} \diamond \alpha A_{\varphi\bar\alpha}^{\theta,\frac{\theta}{\alpha}}) C_\theta = C_\theta (A_{\bar\varphi}^{\alpha,\theta} \boxplus A_{\bar\varphi\alpha}^{\frac{\theta}{\alpha},\theta} \bar\alpha), \tag{5.4}$$

$$(A_\varphi^{\theta,\alpha} \diamond \alpha A_{\varphi\frac{\theta}{\alpha}}^{\theta,\frac{\theta}{\alpha}}) C_{\alpha,\frac{\theta}{\alpha}} = C_{\alpha,\frac{\theta}{\alpha}} (A_{\bar\varphi}^{\alpha,\theta} \boxplus A_{\varphi\frac{\theta}{\alpha}}^{\frac{\theta}{\alpha},\theta} \bar\alpha), \tag{5.5}$$

$$(A_\varphi^{\theta,\alpha} \oplus \alpha A_\varphi^{\theta,\frac{\theta}{\alpha}})(C_{\alpha,\frac{\theta}{\alpha}} \diamond C_\theta) = (C_{\alpha,\frac{\theta}{\alpha}} \boxplus C_\theta)(A_{\bar\varphi}^{\alpha,\theta} \oplus A_{\bar\varphi}^{\frac{\theta}{\alpha},\theta} \bar\alpha). \tag{5.6}$$

Equivalently, the above operators are antilinearly selfadjoint, i.e.,

$$((A_\varphi^{\theta,\alpha} \diamond \alpha A_{\varphi\bar\alpha}^{\theta,\frac{\theta}{\alpha}}) C_\theta)^\sharp = (A_\varphi^{\theta,\alpha} \diamond \alpha A_{\varphi\bar\alpha}^{\theta,\frac{\theta}{\alpha}}) C_\theta, \tag{5.4a}$$

$$((A_\varphi^{\theta,\alpha} \diamond \alpha A_{\varphi\frac{\theta}{\alpha}}^{\theta,\frac{\theta}{\alpha}}) C_{\alpha,\frac{\theta}{\alpha}})^\sharp = (A_\varphi^{\theta,\alpha} \diamond \alpha A_{\varphi\frac{\theta}{\alpha}}^{\theta,\frac{\theta}{\alpha}}) C_{\alpha,\frac{\theta}{\alpha}}, \tag{5.5a}$$

$$((A_\varphi^{\theta,\alpha} \oplus \alpha A_\varphi^{\theta,\frac{\theta}{\alpha}})(C_{\alpha,\frac{\theta}{\alpha}} \diamond C_\theta))^\sharp = (A_\varphi^{\theta,\alpha} \oplus \alpha A_\varphi^{\theta,\frac{\theta}{\alpha}})(C_{\alpha,\frac{\theta}{\alpha}} \diamond C_\theta). \tag{5.6a}$$

Proof. Let us take $f_1 \in K_\alpha^\infty$, $f_2 \in K_{\frac{\theta}{\alpha}}^\infty$ and $f = f_1 \oplus \alpha f_2$ (recall that $K_\alpha^\infty \oplus \alpha K_{\frac{\theta}{\alpha}}^\infty$ is dense in K_θ – see [5, (5.23)]). To prove (5.4) note that

$$(A_\varphi^{\theta,\alpha} \diamond \alpha A_{\varphi\bar\alpha}^{\theta,\frac{\theta}{\alpha}}) C_\theta f = P_\alpha(\varphi C_\theta f) + \alpha P_{\frac{\theta}{\alpha}} \bar\alpha (\varphi C_\theta f) = P_\theta(\varphi C_\theta f) = P_\theta C_\theta(\bar\varphi f) = C_\theta P_\theta(\bar\varphi f), \tag{5.7}$$

since $P_\alpha + \alpha P_{\frac{\theta}{\alpha}} \bar\alpha = P_\theta$. On the other hand, we obtain

$$C_\theta (A_{\bar\varphi}^{\alpha,\theta} \boxplus A_{\bar\varphi\alpha}^{\frac{\theta}{\alpha},\theta} \bar\alpha)(f_1 \oplus \alpha f_2) = C_\theta(P_\theta(\bar\varphi f_1) + P_\theta(\bar\varphi \alpha f_2)) = C_\theta P_\theta(\bar\varphi f).$$

Now we will show that

$$C_{\alpha,\frac{\theta}{\alpha}} (A_\varphi^{\theta,\alpha} \diamond \alpha A_{\varphi\frac{\theta}{\alpha}}^{\theta,\frac{\theta}{\alpha}}) = (A_{\bar\varphi}^{\alpha,\theta} \boxplus A_{\varphi\frac{\theta}{\alpha}}^{\frac{\theta}{\alpha},\theta} \bar\alpha) C_{\alpha,\frac{\theta}{\alpha}},$$

which is equivalent to (5.5). Note that

$$\begin{aligned} C_{\alpha,\frac{\theta}{\alpha}} (A_\varphi^{\theta,\alpha} \diamond \alpha A_{\varphi\frac{\theta}{\alpha}}^{\theta,\frac{\theta}{\alpha}}) f &= C_{\alpha,\frac{\theta}{\alpha}} P_\alpha(\varphi f) + C_{\alpha,\frac{\theta}{\alpha}} (\alpha P_{\frac{\theta}{\alpha}} (\tfrac{\theta}{\alpha} \varphi f)) \\ &= C_\alpha P_\alpha(\varphi f) + \alpha C_{\frac{\theta}{\alpha}} P_{\frac{\theta}{\alpha}} (\tfrac{\theta}{\alpha} \varphi f) = P_\alpha(\bar\varphi C_\alpha f) + \alpha P_{\frac{\theta}{\alpha}} \bar\alpha (\bar\varphi C_\alpha f) \\ &= P_\theta(\bar\varphi C_\alpha f). \end{aligned}$$

On the other hand, $C_\alpha f = \bar z(\alpha \bar f_1 + \bar f_2)$. Hence by Proposition 3.1 we get

$$\begin{aligned} (A_{\bar\varphi}^{\alpha,\theta} \boxplus A_{\varphi\frac{\theta}{\alpha}}^{\frac{\theta}{\alpha},\theta} \bar\alpha) C_{\alpha,\frac{\theta}{\alpha}} (f_1 \oplus \alpha f_2) &= (A_{\bar\varphi}^{\alpha,\theta} \boxplus A_{\varphi\frac{\theta}{\alpha}}^{\frac{\theta}{\alpha},\theta} \bar\alpha)(C_\alpha f_1 \oplus \alpha C_{\frac{\theta}{\alpha}} f_2) \\ &= P_\theta(\bar\varphi C_\alpha f_1) + P_\theta(\bar\varphi \tfrac{\bar\theta}{\bar\alpha} C_{\frac{\theta}{\alpha}} f_2) = P_\theta(\bar\varphi(\alpha \bar z \bar f_1 + \bar z \bar f_2)) = P_\theta(\bar\varphi C_\alpha f). \end{aligned}$$

To prove (5.6), since $P_\theta = P_\alpha + \alpha P_{\frac{\theta}{\alpha}} \bar{\alpha}$, note that

$$\begin{aligned}
&(A_\varphi^{\theta,\alpha} \oplus \alpha\, A_\varphi^{\theta,\frac{\theta}{\alpha}})((C_\alpha f_1 + \alpha\, C_{\frac{\theta}{\alpha}} f_2) \oplus C_\theta f) \\
&\quad = P_\alpha(\varphi(C_\alpha f_1 + \alpha\, C_{\frac{\theta}{\alpha}} f_2)) + \alpha\, P_{\frac{\theta}{\alpha}}(\varphi C_\theta f) \\
&\quad = P_\alpha(\varphi\alpha\bar{z}\bar{f}_1 + \theta\varphi\bar{z}\bar{f}_2) + \alpha P_{\frac{\theta}{\alpha}}(\varphi\theta\bar{z}\bar{f}_1 + \varphi\theta\bar{z}\bar{\alpha}\bar{f}_2) \\
&\quad = P_\alpha(C_\alpha(\bar{\varphi} f_1)) + P_\alpha(C_{\frac{\theta}{\alpha}}(\bar{\varphi}\bar{\alpha} f_2)) + \alpha P_{\frac{\theta}{\alpha}}(\bar{\alpha} C_{\frac{\theta}{\alpha}}(\bar{\varphi}\bar{\alpha} f_2)) + \alpha P_{\frac{\theta}{\alpha}}(C_{\frac{\theta}{\alpha}}(\bar{\varphi}\bar{\alpha} f_1)) \\
&\quad = P_\alpha(C_\alpha(\bar{\varphi} f_1)) + \alpha P_{\frac{\theta}{\alpha}}(C_{\frac{\theta}{\alpha}}(\bar{\varphi}\bar{\alpha} f_1)) + P_\theta(C_{\frac{\theta}{\alpha}}(\bar{\varphi}\bar{\alpha} f_2)).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&(C_{\alpha,\frac{\theta}{\alpha}} \boxplus C_\theta)(A_{\bar{\varphi}}^{\alpha,\theta} \oplus A_{\bar{\varphi}}^{\frac{\theta}{\alpha},\theta}\bar{\alpha})(f_1 \oplus \alpha f_2) \\
&\qquad = C_\alpha(P_\alpha(\bar{\varphi} f_1)) + \alpha C_{\frac{\theta}{\alpha}}(P_{\frac{\theta}{\alpha}}(\bar{\alpha}\bar{\varphi} f_1)) + C_\theta(P_\theta(\bar{\varphi} f_2)).
\end{aligned}$$

Using $P_\theta = P_{\frac{\theta}{\alpha}} + \frac{\theta}{\alpha} P_\alpha \frac{\bar{\theta}}{\bar{\alpha}}$ we obtain

$$\begin{aligned}
C_\theta(P_\theta(\bar{\varphi} f_2)) &= C_\theta(P_\alpha(\bar{\varphi} f_2)) + \alpha P_{\frac{\theta}{\alpha}}(\bar{\alpha}\bar{\varphi} f_2) \\
&= C_{\frac{\theta}{\alpha}}(P_{\frac{\theta}{\alpha}}(\bar{\alpha}\bar{\varphi} f_2)) + \tfrac{\theta}{\alpha} C_\alpha(P_\alpha(\bar{\varphi} f_2)) \\
&= P_{\frac{\theta}{\alpha}}(C_{\frac{\theta}{\alpha}}(\bar{\alpha}\bar{\varphi} f_2)) + \tfrac{\theta}{\alpha} P_\alpha(C_\alpha(\bar{\varphi} f_2)) \\
&= P_{\frac{\theta}{\alpha}}(C_{\frac{\theta}{\alpha}}(\bar{\alpha}\bar{\varphi} f_2)) + \tfrac{\theta}{\alpha} P_\alpha \tfrac{\bar{\theta}}{\bar{\alpha}}(C_{\frac{\theta}{\alpha}}(\bar{\alpha}\bar{\varphi} f_2)) = P_\theta(C_{\frac{\theta}{\alpha}}(\bar{\alpha}\bar{\varphi} f_2)).
\end{aligned}$$

That completes the proof of (5.6). All calculations were made on a dense subset of K_θ, hence we get all the equalities in the theorem. □

One can also ask for which symbols $\varphi \in L^2$ the equalities (5.2) and (5.3) hold. From Theorem 5.3 and [3, Theorem 4.4] we obtain the following:

Corollary 5.4. *Let α, θ be nonconstant inner functions such that $\alpha \preccurlyeq \theta$, and let $A \in \mathcal{T}(\theta, \alpha)$. Then*

1. *$AC_\theta = C_\theta A^* P_\alpha$ if and only if there is $\varphi \in \overline{\frac{\theta}{\alpha} K_\alpha}$ such that $A = A_\varphi^{\theta,\alpha}$,*
2. *$AC_{\alpha,\frac{\theta}{\alpha}} = C_{\alpha,\frac{\theta}{\alpha}} A^* P_\alpha$ if and only if there is $\varphi \in K_\alpha$ such that $A = A_\varphi^{\theta,\alpha}$.*

Proof. Note that to obtain the desired equality (1) we have to assume that $A_{\varphi\bar{\alpha}}^{\theta,\frac{\theta}{\alpha}} = 0$ in the formula (5.4) of Theorem 5.3, which is equivalent by [3, Theorem 4.4] to $\varphi\bar{\alpha} \in \frac{\theta}{\alpha} H^2 + \overline{\theta H^2}$, i.e., $\varphi \in \theta H^2 + \overline{\frac{\theta}{\alpha} H^2}$. Since for $\varphi \in \alpha H^2 + \overline{\theta H^2}$ the operator $A_\varphi^{\theta,\alpha} = 0$, we may assume that $\varphi \in \overline{K_\theta \cap \frac{\theta}{\alpha} H^2} = \overline{\frac{\theta}{\alpha} K_\alpha}$.

Similarly, the assumption $A_{\varphi\frac{\theta}{\alpha}}^{\theta,\frac{\theta}{\alpha}} = 0$ is equivalent to $\varphi\frac{\theta}{\alpha} \in \frac{\theta}{\alpha} H^2 + \overline{\theta H^2}$. Since for $\varphi \in \alpha H^2 + \overline{\theta H^2}$ the operator $A_\varphi^{\theta,\alpha} = 0$, it is enough to consider $\varphi \in K_\alpha$ for the equality (2). □

Note that if $\varphi \in \overline{\frac{\theta}{\alpha} K_\alpha}$, then $A_\varphi^{\theta,\alpha} f = P_\alpha \varphi P_\theta f = P_\theta \varphi P_\theta f$ for all $f \in K_\theta$, while if $\varphi \in K_\alpha$, then $A_\varphi^{\theta,\alpha} f = P_\alpha \varphi P_\theta f = P_\alpha \varphi P_\alpha f$ for all $f \in K_\theta$. Therefore the conditions in (1) and (2) of the previous corollary are satisfied if and only if $A_\varphi^{\theta,\alpha}$ can be identified with truncated Toeplitz operators A_φ^θ and A_φ^α, respectively.

6. Example with $\theta = z^N$.

To illustrate the equalities in Theorem 5.3 we consider the simplest inner function $\theta = z^N$. Then K_{z^N} is the space of polynomials of degree smaller than N. Hence K_{z^N} can be identified with $\mathbb{C}^N$. Then the conjugation C_{z^N} in $\mathbb{C}^N$ is given by $C_{z^N}(z_0, \ldots, z_N) = (\bar{z}_N, \ldots, \bar{z}_0)$. Let us firstly illustrate Lemma 5.1.

Remark 6.1. *Let $A \in L(\mathbb{C}^N)$ be a truncated Toeplitz operator with matrix $A = (a_{ij})_{i,j=0}^{N-1}$, $a_{ij} = t_{i-j}$ for $i, j = 0, \dots, N$. Recall that A is C_{z^N}–symmetric, i.e., the matrix is symmetric according to the second diagonal (see [7]). On the other hand, by* (2.1)*, an antilinear operator X given by a matrix $(s_{ij})_{i,j=0}^{N-1}$ is antilinearly selfadjoint if its matrix is symmetric, i.e., $s_{ij} = s_{ji}$ for $i, j = 0, \dots, N$. Note that the antilinear operator AC_{z^N} has the Hankel matrix $(b_{ij})_{i,j=0,\dots,N}$, with $b_{i,j} = t_{i+j-N+1}$, which is clearly symmetric ($b_{ij} = b_{ji}$ for $i, j = 0, \dots, N$).*

Now we will illustrate the equations (5.4a), (5.5a), (6.6a).

Example 6.2. *Let $\alpha = z^3$ and $\theta = z^5$. Then any operator in $\mathcal{T}(z^5, z^3)$ has a symbol $\varphi = \sum_{n=-4}^{2} a_k z^k \in K_{z^3} + \overline{K_{z^5}}$ (see [3, Corollary 4.5]). Thus it has a matrix representation $A_{\varphi}^{z^5,z^3} = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & a_{-4} \\ a_1 & a_0 & a_{-1} & a_{-2} & a_{-3} \\ a_2 & a_1 & a_0 & a_{-1} & a_{-2} \end{bmatrix}$. To illustrate the equality* (5.4a) *in Theorem 5.3 note that $A_{\bar{\alpha}\varphi}^{z^5,z^2} = \begin{bmatrix} 0 & a_2 & a_1 & a_0 & a_{-1} \\ 0 & 0 & a_2 & a_1 & a_0 \end{bmatrix}$, so $A_{\varphi}^{z^5,z^3} \diamond z^3 A_{\varphi\bar{z}^3}^{z^5,z^2}$ is simply the Toeplitz matrix in $\mathbb{C}^5$ with the symbol $\varphi = \sum_{n=-4}^{2} a_k z^k \in K_{z^3} + \overline{K_{z^5}} \subsetneq K_{z^5} + \overline{K_{z^5}}$, and its C_{z^5}–symmetry or the symmetry of the Hankel matrix $(A_{\varphi}^{z^5,z^3} \diamond z^3 A_{\varphi\bar{z}^3}^{z^5,z^2})\, C_{z^5}$ is easily satisfied. Now to obtain equality (1) in Corollary 5.4 in our case we have to assume that $\varphi = a_{-4}\bar{z}^4 + a_{-3}\bar{z}^3 + a_{-2}\bar{z}^2$, so $a_{-1} = a_0 = a_1 = a_2 = 0$.*

To illustrate (5.5a)*, besides the involution C_{z^5}, we consider another involution $C_{z^3,z^2}(z_0, z_1, z_2, z_3, z_4) = (\bar{z}_2, \bar{z}_1, \bar{z}_0, \bar{z}_4, \bar{z}_3)$. Note that $A_{\varphi z^2}^{z^5,z^2} = \begin{bmatrix} a_{-2} & a_{-3} & a_{-4} & 0 & 0 \\ a_{-1} & a_{-2} & a_{-3} & a_{-4} & 0 \end{bmatrix}$. Hence*

$$(A_{\varphi}^{z^5,z^3} \diamond \alpha A_{\varphi z^2}^{z^5,z^2}) C_{z^3,z^2}(z_0, z_1, z_2, z_3, z_4) = \left[\begin{array}{ccc:cc} a_{-2} & a_{-1} & a_0 & a_{-4} & a_{-3} \\ a_{-1} & a_0 & a_1 & a_{-3} & a_{-2} \\ a_0 & a_1 & a_2 & a_{-2} & a_{-1} \\ \hdashline a_{-4} & a_{-3} & a_{-2} & 0 & 0 \\ a_{-3} & a_{-2} & a_{-1} & 0 & a_{-4} \end{array}\right] \begin{bmatrix} \bar{z}_0 \\ \bar{z}_1 \\ \bar{z}_2 \\ \bar{z}_3 \\ \bar{z}_4 \end{bmatrix}. \tag{6.1}$$

Note that to obtain the equality (2) in Corollary 5.4 we have to take $\varphi = a_0 + a_1 z + a_2 z^2$.

In the equality (5.6a) *$A_{\varphi}^{z^5,z^2} = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & a_{-4} \\ a_1 & a_0 & a_{-1} & a_{-2} & a_{-3} \end{bmatrix}$. Hence*

$$(A_{\varphi}^{z^5,z^3} \oplus A_{\varphi}^{z^5,z^2})(C_{z^3,z^2} \diamond C_{z^5})(z_0, z_1, z_2, z_3, z_4) = \left[\begin{array}{ccc:cc} a_{-2} & a_{-1} & a_0 & a_{-4} & a_{-3} \\ a_{-1} & a_0 & a_1 & a_{-3} & a_{-2} \\ a_0 & a_1 & a_2 & a_{-2} & a_{-1} \\ \hdashline a_{-4} & a_{-3} & a_{-2} & a_{-1} & a_0 \\ a_{-3} & a_{-2} & a_{-1} & a_0 & a_1 \end{array}\right] \begin{bmatrix} \bar{z}_0 \\ \bar{z}_1 \\ \bar{z}_2 \\ \bar{z}_3 \\ \bar{z}_4 \end{bmatrix}. \tag{6.2}$$

The equations (5.5a) *and* (5.6a) *say that the antilinear operators $(A_{\varphi}^{z^5,z^3} \diamond \alpha A_{\varphi z^2}^{z^5,z^2}) C_{z^3,z^2}$ and $(A_{\varphi}^{z^5,z^3} \oplus A_{\varphi}^{z^5,z^2})(C_{z^3,z^2} \diamond C_{z^5})$ are antilinearly selfadjoint. If we write, in both cases, the above matrices by blocks $\left[\begin{array}{c:c} H_{11} & H_{12} \\ \hdashline H_{21} & H_{22} \end{array}\right]$, then each block is a Hankel matrix and the whole matrix is symmetric, moreover, H_{12} is symmetric to H_{21}. In the first case some part of H_{22} annihilates. The above should be also seen in the context of Remark 6.1.*

7. Connections with Hankel operators

In light of Theorem 5.3 it is natural to ask about the differences $A^{\theta,\alpha}_{\varphi}C_\theta - C_\theta A^{\alpha,\theta}_{\bar\varphi}P_\alpha$ and $A^{\theta,\alpha}_{\varphi}C_{\alpha,\frac{\theta}{\alpha}} - C_{\alpha,\frac{\theta}{\alpha}}A^{\alpha,\theta}_{\bar\varphi}P_\alpha$, which have to become zero when $\alpha = \theta$. It turns out that these differences can be expressed in terms of certain Hankel operators.

Let P denote the orthogonal projection from L^2 onto H^2, and P^- denote the orthogonal projection from L^2 onto $\overline{H^2_0} = L^2 \ominus H^2$. For $\varphi \in L^2$ we define:

$$H_\varphi \colon H^2 \to \overline{H^2_0}, \quad H_\varphi f = P^-(\varphi f);$$

for $f \in H^2$ such that $\varphi f \in L^2$. Similarly, for $\theta \in L^\infty$,

$$\widetilde{H}_\theta \colon \overline{H^2_0} \to H^2, \quad \widetilde{H}_\theta f = P(\theta f) \text{ for } f \in \overline{H^2_0}.$$

Let θ be a nonconstant inner function. Recall firstly the following:

Proposition 7.1. *Let θ be a nonconstant inner function and let $K_\theta = H^2 \ominus \theta H^2$ be the associated model space. Then*

1. $P_\theta = \theta P^- \bar\theta P = \theta P^- \bar\theta - P^-$,
2. $P_\theta f = \theta P^- \bar\theta f = f - \theta P \bar\theta f$ *for all* $f \in H^2$,
3. $P_\theta \bar f = P_\theta P \bar f = \overline{f(0)} P_\theta 1 = \overline{f(0)}(1 - \overline{\theta(0)}\theta)$ *for all* $f \in H^2$.

Using Proposition 7.1 it it easy to see that, for $A^\theta_\varphi \in \mathcal{T}(\theta)$, both $A^\theta_\varphi C_\theta$ and $C_\theta A^\theta_\varphi$ can be expressed in terms of Hankel operators. In fact we have

$$A^\theta_\varphi C_\theta = \widetilde{H}_\theta H_{\bar\theta\varphi} C_\theta \;\text{ and }\; C_\theta A^\theta_\varphi = \widetilde{H}_\theta H_{\bar\theta\bar\varphi} C_\theta\,,$$

which is another way to see that $A^\theta_\varphi C_\theta = C_\theta A^\theta_{\bar\varphi}$, i.e., A^θ_φ is C_θ-symmetric.

In the asymmetric case ($\alpha < \theta$) we no longer have, in general, either

$$A^{\theta,\alpha}_{\varphi} C_\theta = C_\theta A^{\alpha,\theta}_{\bar\varphi} \qquad \text{or} \tag{7.1}$$

$$A^{\theta,\alpha}_{\varphi} C_{\alpha,\frac{\theta}{\alpha}} = C_{\alpha,\frac{\theta}{\alpha}} A^{\alpha,\theta}_{\bar\varphi}, \tag{7.2}$$

where, for simplicity, we identify $A^{\theta,\alpha}_\varphi$ and $A^{\alpha,\theta}_{\bar\varphi}$ with the operators $P_\alpha \varphi P_\theta$ and $P_\theta \bar\varphi P_\alpha$, respectively. Thus it is natural to ask about the differences between the operators on the left and on the right hand sides of the equalities (7.1) and (7.2). In the following theorem we characterize those differences in terms of Hankel operators. This will later provide, in particular, another way to prove (5.6).

Theorem 7.2. *Let α, θ be nonconstant inner functions and $\alpha \leqslant \theta$. If $A^{\theta,\alpha}_\varphi \in \mathcal{T}(\theta,\alpha)$ for $\varphi \in L^2$, then the following equalities hold:*

$$(A^{\theta,\alpha}_{\varphi} C_\theta - C_\theta A^{\alpha,\theta}_{\bar\varphi} P_\alpha) f = (\widetilde{H}_\alpha H_{\bar\alpha\varphi} C_{\frac{\theta}{\alpha}} P_{\frac{\theta}{\alpha}} \bar\alpha - \alpha \widetilde{H}_{\frac{\theta}{\alpha}} H_{\bar\theta\varphi} C_\theta P_\alpha) f; \tag{7.3}$$

$$(A^{\theta,\alpha}_{\varphi} C_{\alpha,\frac{\theta}{\alpha}} - C_{\alpha,\frac{\theta}{\alpha}} A^{\alpha,\theta}_{\varphi} P_\alpha) f = (\widetilde{H}_\alpha H_\varphi C_\theta - \widetilde{H}_\theta H_\varphi C_\alpha P_\alpha) f; \tag{7.4}$$

$$(\tfrac{\theta}{\alpha} A^{\theta,\alpha}_{\varphi} C_\theta - C_\theta A^{\alpha,\theta}_{\bar\varphi} P_\alpha \tfrac{\bar\theta}{\bar\alpha}) f = (\widetilde{H}_\theta H_\varphi C_{\frac{\theta}{\alpha}} P_{\frac{\theta}{\alpha}} - \widetilde{H}_{\frac{\theta}{\alpha}} H_\varphi C_\theta) f \tag{7.5}$$

for $f \in K_\theta$.

Proof. As in the proof of Theorem 5.3, it is enough to consider $f = f_\alpha + \alpha f_{\frac{\theta}{\alpha}}$, $f_\alpha \in K^\infty_\alpha$, $f_{\frac{\theta}{\alpha}} \in K^\infty_{\frac{\theta}{\alpha}}$. To prove (7.3) note that by Proposition 3.1 and by Proposition 7.1, we have

$$\begin{aligned} A^{\theta,\alpha}_{\varphi} C_\theta f = A^{\theta,\alpha}_{\varphi}(C_{\frac{\theta}{\alpha}} f_{\frac{\theta}{\alpha}} + \tfrac{\theta}{\alpha} C_\alpha f_\alpha) &= P_\alpha(\varphi C_{\frac{\theta}{\alpha}} f_{\frac{\theta}{\alpha}}) + P_\alpha(\varphi \tfrac{\theta}{\alpha} \alpha \bar z \bar f_\alpha) \\ &= P(\alpha P^-(\bar\alpha \varphi C_{\frac{\theta}{\alpha}} f_{\frac{\theta}{\alpha}})) + P_\alpha(\theta \varphi \bar z \bar f_\alpha) = \widetilde{H}_\alpha H_{\bar\alpha\varphi} C_{\frac{\theta}{\alpha}} f_{\frac{\theta}{\alpha}} + P_\alpha(\theta \varphi \bar z \bar f_\alpha) \end{aligned}$$

and

$$\begin{aligned} C_\theta A_{\bar\varphi}^{\alpha,\theta} P_\alpha f &= C_\theta P_\theta(\bar\varphi f_\alpha) = P_\theta(\theta\varphi\bar z\bar f_\alpha) \\ &= P_\alpha(\theta\varphi\bar z\bar f_\alpha) + \alpha P_{\frac{\theta}{\alpha}}(\bar\alpha\varphi C_\theta f_\alpha) \\ &= P_\alpha(\theta\varphi\bar z\bar f_\alpha) + \alpha P(\tfrac{\theta}{\alpha}P^-(\bar\theta\varphi C_\theta f_\alpha)) \\ &= P_\alpha(\theta\varphi\bar z\bar f_\alpha) + \alpha\widetilde{H}_{\frac{\theta}{\alpha}}H_{\bar\theta\varphi}C_\theta f_\alpha. \end{aligned}$$

To prove (7.4) note firstly that $A_\varphi^{\theta,\alpha} = A_\varphi^\alpha P_\alpha + P_\alpha(\varphi\alpha P_{\frac{\theta}{\alpha}}(\bar\alpha I_{K_\theta}))$. So we have

$$A_\varphi^\alpha C_{\alpha,\frac{\theta}{\alpha}|K_\alpha} = A_\varphi^\alpha C_\alpha = C_\alpha A_{\bar\varphi}^\alpha = C_{\alpha,\frac{\theta}{\alpha}}A_{\bar\varphi}^\alpha$$

on K_α. On the other hand,

$$P_\alpha(\varphi\alpha P_{\frac{\theta}{\alpha}}(\bar\alpha C_{\alpha,\frac{\theta}{\alpha}}f)) = P_\alpha(\varphi\alpha P_{\frac{\theta}{\alpha}}(\bar\alpha(\alpha\bar z\,\bar f_\alpha + \alpha C_{\frac{\theta}{\alpha}}(f_{\frac{\theta}{\alpha}})))) = P_\alpha(\varphi\alpha C_{\frac{\theta}{\alpha}}(f_{\frac{\theta}{\alpha}})) = P\alpha P^-\varphi C_{\frac{\theta}{\alpha}}f_{\frac{\theta}{\alpha}}.$$

Thus, $A_\varphi^{\theta,\alpha}C_{\alpha,\frac{\theta}{\alpha}} = C_{\alpha,\frac{\theta}{\alpha}}A_{\bar\varphi}^\alpha + \widetilde{H}_\alpha H_\varphi C_{\frac{\theta}{\alpha}}\bar\alpha$. Analogously, $A_{\bar\varphi}^{\alpha,\theta} = A_{\bar\varphi}^\alpha + \alpha P_{\frac{\theta}{\alpha}}\bar\alpha\,\bar\varphi P_\alpha$ and

$$\begin{aligned} &P\alpha P^-(\varphi C_{\frac{\theta}{\alpha}}f_{\frac{\theta}{\alpha}}) - C_{\alpha,\frac{\theta}{\alpha}}(\alpha P_{\frac{\theta}{\alpha}}(\bar\alpha\,\bar\varphi P_\alpha f)) \\ &= P\alpha P^-(\varphi C_{\frac{\theta}{\alpha}}f_{\frac{\theta}{\alpha}}) + P\alpha P^-(\varphi\tfrac{\theta}{\alpha}C_\alpha f_\alpha) - P\alpha P^-(\varphi\tfrac{\theta}{\alpha}C_\alpha f_\alpha) - \alpha C_{\frac{\theta}{\alpha}}(P_{\frac{\theta}{\alpha}}(\bar\alpha\,\bar\varphi f_\alpha)) \\ &= P\alpha P^-(\varphi C_\theta f) - (P\alpha P^-\bar\alpha + \alpha P_{\frac{\theta}{\alpha}}\bar\alpha)(\theta\varphi C_\alpha f_\alpha) \\ &= \widetilde{H}_\alpha H_\varphi C_\theta f - P\theta P^-(\varphi C_\alpha f_\alpha), \end{aligned}$$

since $P\alpha P^-\bar\alpha + \alpha P_{\frac{\theta}{\alpha}}\bar\alpha = P_\theta = P\theta P^-\bar\theta$. Hence $(A_\varphi^{\theta,\alpha}C_{\alpha,\frac{\theta}{\alpha}} - C_{\alpha,\frac{\theta}{\alpha}}A_{\bar\varphi}^{\alpha,\theta})f = (\widetilde{H}_\alpha H_\varphi C_\theta - \widetilde{H}_\theta H_\varphi C_\alpha P_\alpha)f$ for $f \in K_\theta$.

To show (7.5) consider $g = g_{\frac{\theta}{\alpha}} + \frac{\theta}{\alpha}g_\alpha$, $g_\alpha \in K_\alpha^\infty$, $g_{\frac{\theta}{\alpha}} \in K_{\frac{\theta}{\alpha}}^\infty$. Then by Proposition 3.1 we have

$$\tfrac{\theta}{\alpha}A_\varphi^{\theta,\alpha}C_\theta g = \tfrac{\theta}{\alpha}A_\varphi^{\theta,\alpha}(C_\alpha g_\alpha + \alpha C_{\frac{\theta}{\alpha}}g_{\frac{\theta}{\alpha}}) = \tfrac{\theta}{\alpha}P_\alpha(\varphi C_\alpha g_\alpha) + \tfrac{\theta}{\alpha}P_\alpha(\varphi\alpha C_{\frac{\theta}{\alpha}}g_{\frac{\theta}{\alpha}}) = \tfrac{\theta}{\alpha}P_\alpha(\varphi\alpha\bar z\bar g_\alpha) + \tfrac{\theta}{\alpha}P_\alpha(\varphi\alpha C_{\frac{\theta}{\alpha}}g_{\frac{\theta}{\alpha}}),$$

and

$$C_\theta A_{\bar\varphi}^{\alpha,\theta}P_\alpha(\tfrac{\bar\theta}{\bar\alpha}g_{\frac{\theta}{\alpha}} + g_\alpha) = C_\theta P_\theta\bar\varphi g_\alpha = P_\theta C_\theta\bar\varphi g_\alpha = (P_{\frac{\theta}{\alpha}} + \tfrac{\theta}{\alpha}P_\alpha\tfrac{\bar\theta}{\bar\alpha})(\theta\varphi\bar z\bar g_\alpha) = P_{\frac{\theta}{\alpha}}(\theta\varphi\bar z\bar g_\alpha) + \tfrac{\theta}{\alpha}P_\alpha(\alpha\varphi\bar z\bar g_\alpha).$$

Hence

$$\begin{aligned} &\tfrac{\theta}{\alpha}A_\varphi^{\theta,\alpha}C_\theta g - C_\theta A_{\bar\varphi}^{\alpha,\theta}P_\alpha(\tfrac{\bar\theta}{\bar\alpha}g) = \tfrac{\theta}{\alpha}P_\alpha(\alpha\varphi C_{\frac{\theta}{\alpha}}g_{\frac{\theta}{\alpha}}) - P_{\frac{\theta}{\alpha}}(\theta\varphi\bar z\bar g_\alpha) \\ &= P\tfrac{\theta}{\alpha}P_\alpha(\alpha\varphi C_{\frac{\theta}{\alpha}}g_{\frac{\theta}{\alpha}}) + P\tfrac{\theta}{\alpha}P^-(\alpha\varphi C_{\frac{\theta}{\alpha}}g_{\frac{\theta}{\alpha}}) - (P\tfrac{\theta}{\alpha}P^-(\alpha\varphi\bar z\bar g_\alpha) + P\tfrac{\theta}{\alpha}P^-(\alpha\varphi C_{\frac{\theta}{\alpha}}g_{\frac{\theta}{\alpha}})) \\ &= P\tfrac{\theta}{\alpha}(P_\alpha + P^-)(\alpha\varphi C_{\frac{\theta}{\alpha}}g_{\frac{\theta}{\alpha}}) - P\tfrac{\theta}{\alpha}P^-(\varphi(C_\alpha g_\alpha + \alpha C_{\frac{\theta}{\alpha}}g_{\frac{\theta}{\alpha}})) \\ &= P\theta P^-(\varphi C_{\frac{\theta}{\alpha}}g_{\frac{\theta}{\alpha}}) - P\tfrac{\theta}{\alpha}P^-(\varphi C_\theta g) = \widetilde{H}_\theta H_\varphi C_{\frac{\theta}{\alpha}}g_{\frac{\theta}{\alpha}} - \widetilde{H}_{\frac{\theta}{\alpha}}H_\varphi C_\theta g, \end{aligned}$$

since by Proposition 7.1 $P_\alpha + P^- = \alpha P^-\bar\alpha$. □

From (7.5) we can obtain in particular the following:

Corollary 7.3. *Let α, θ be nonconstant inner functions such that $\alpha \leqslant \theta$. If $A_\varphi^{\theta,\frac{\theta}{\alpha}} \in \mathcal{T}(\theta,\frac{\theta}{\alpha})$ for $\varphi \in L^2$, then*

$$\alpha A_\varphi^{\theta,\frac{\theta}{\alpha}}C_\theta - C_\theta A_{\bar\varphi}^{\frac{\theta}{\alpha},\theta}P_{\frac{\theta}{\alpha}}\bar\alpha = \widetilde{H}_\theta H_\varphi C_\alpha P_\alpha - \widetilde{H}_\alpha H_\varphi C_\theta. \tag{7.6}$$

Note that comparing (7.4) with (7.6) we get:

$$A_\varphi^{\theta,\alpha}C_{\alpha,\frac{\theta}{\alpha}} + \alpha A_\varphi^{\theta,\frac{\theta}{\alpha}}C_\theta = C_{\alpha,\frac{\theta}{\alpha}}A_{\bar\varphi}^{\alpha,\theta}P_\alpha + C_\theta A_{\bar\varphi}^{\frac{\theta}{\alpha},\theta}P_{\frac{\theta}{\alpha}}\bar\alpha, \tag{7.7}$$

which is equivalent to (5.6). Hence we obtained another proof of (5.6).

8. Examples with Hankel matrices

To illustrate the equalities in Theorem 7.2 let us consider the following examples.

Example 8.1. *Let $\alpha = z^3$, $\theta = z^5$ and $\varphi = \sum_{n=-4}^{2} a_n z^n \in \overline{K_{z^5}} + K_{z^3}$. Then for $f = (z_0, z_1, z_2, z_3, z_4) \in K_{z^5}$ we have, regarding the left hand side of* (7.3),

$$A_{\varphi}^{z^5,z^3} C_{z^5} f = \begin{bmatrix} \boxed{\begin{matrix} a_{-4} & a_{-3} & a_{-2} \\ a_{-3} & a_{-2} & a_{-1} \\ a_{-2} & a_{-1} & a_0 \end{matrix}} \begin{matrix} a_{-1} & a_0 \\ a_0 & a_1 \\ a_1 & a_2 \end{matrix} \end{bmatrix} \begin{bmatrix} \bar{z}_0 \\ \bar{z}_1 \\ \bar{z}_2 \\ \bar{z}_3 \\ \bar{z}_4 \end{bmatrix}$$

and

$$C_{z^5} A_{\overline{\varphi}}^{z^3,z^5} P_{z^3} f = \begin{bmatrix} \boxed{\begin{matrix} a_{-4} & a_{-3} & a_{-2} \\ a_{-3} & a_{-2} & a_{-1} \\ a_{-2} & a_{-1} & a_0 \end{matrix}} \\ \begin{matrix} a_{-1} & a_0 & a_1 \\ a_0 & a_1 & a_2 \end{matrix} \end{bmatrix} \begin{bmatrix} \bar{z}_0 \\ \bar{z}_1 \\ \bar{z}_2 \end{bmatrix}.$$

The right hand side is given by Hankel matrices

$$\widetilde{H}_{z^3} H_{\bar{z}^3 z^5} C_{z^2} P_{z^2} (\bar{z}^3 f) = \begin{bmatrix} a_{-1} & a_0 \\ a_0 & a_1 \\ a_1 & a_2 \end{bmatrix} \begin{bmatrix} \bar{z}_3 \\ \bar{z}_4 \end{bmatrix}$$

and

$$z^3 \widetilde{H}_{z^2} H_{\bar{z}^5 \varphi} C_{z^5} P_{z^3} f = \begin{bmatrix} a_{-1} & a_0 & a_1 \\ a_0 & a_1 & a_2 \end{bmatrix} \begin{bmatrix} \bar{z}_0 \\ \bar{z}_1 \\ \bar{z}_2 \end{bmatrix}.$$

Example 8.2. *The equation* (7.4) *will be illustrated with the same data as before. Hence*

$$A_{\varphi}^{z^5,z^3} C_{z^3,z^2} f = \begin{bmatrix} \boxed{\begin{matrix} a_{-2} & a_{-1} & a_0 \\ a_{-1} & a_0 & a_1 \\ a_0 & a_1 & a_2 \end{matrix}} \begin{matrix} a_{-4} & a_{-3} \\ a_{-3} & a_{-2} \\ a_{-2} & a_{-1} \end{matrix} \end{bmatrix} \begin{bmatrix} \bar{z}_0 \\ \bar{z}_1 \\ \bar{z}_2 \\ \bar{z}_3 \\ \bar{z}_4 \end{bmatrix}$$

and

$$C_{z^3,z^2} A_{\overline{\varphi}}^{z^3,z^5} P_{z^3} f = \begin{bmatrix} \boxed{\begin{matrix} a_{-2} & a_{-1} & a_0 \\ a_{-1} & a_0 & a_1 \\ a_0 & a_1 & a_2 \end{matrix}} \\ \begin{matrix} a_{-4} & a_{-3} & a_{-2} \\ a_{-3} & a_{-2} & a_{-1} \end{matrix} \end{bmatrix} \begin{bmatrix} \bar{z}_0 \\ \bar{z}_1 \\ \bar{z}_2 \end{bmatrix}.$$

On the other hand,

$$\widetilde{H}_{z^3} H_{\varphi} C_{z^5} f = \begin{bmatrix} \boxed{\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & a_{-4} \\ 0 & a_{-4} & a_{-3} \end{matrix}} \begin{matrix} a_{-4} & a_{-3} \\ a_{-3} & a_{-2} \\ a_{-2} & a_{-1} \end{matrix} \end{bmatrix} \begin{bmatrix} \bar{z}_0 \\ \bar{z}_1 \\ \bar{z}_2 \\ \bar{z}_3 \\ \bar{z}_4 \end{bmatrix}$$

and

$$\widetilde{H}_{z^5} H_{\varphi} C_{z^3} P_{z^3} f = \begin{bmatrix} \boxed{\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & a_{-4} \\ 0 & a_{-4} & a_{-3} \end{matrix}} \\ \begin{matrix} a_{-4} & a_{-3} & a_{-2} \\ a_{-3} & a_{-2} & a_{-1} \end{matrix} \end{bmatrix} \begin{bmatrix} \bar{z}_0 \\ \bar{z}_1 \\ \bar{z}_2 \end{bmatrix}.$$

Example 8.3. *Using the same data again we obtain for the equation* (7.5)

$$z^2 A_{\varphi}^{z^5,z^3} C_{z^5} f = z^2 \begin{bmatrix} a_{-4} & a_{-3} & \boxed{a_{-2} & a_{-1} & a_0} \\ a_{-3} & a_{-2} & a_{-1} & a_0 & a_1 \\ a_{-2} & a_{-1} & a_0 & a_1 & a_2 \end{bmatrix} \begin{bmatrix} \bar{z}_0 \\ \bar{z}_1 \\ \bar{z}_2 \\ \bar{z}_3 \\ \bar{z}_4 \end{bmatrix}$$

and

$$C_{z^5} A_{\varphi}^{z^3,z^5} P_{z^3} \alpha \bar{z}^2 f = \begin{bmatrix} a_{-4} & a_{-3} & a_{-2} \\ a_{-3} & a_{-2} & a_{-1} \\ \boxed{a_{-2} & a_{-1} & a_0} \\ a_{-1} & a_0 & a_1 \\ a_0 & a_1 & a_2 \end{bmatrix} \begin{bmatrix} \bar{z}_2 \\ \bar{z}_3 \\ \bar{z}_4 \end{bmatrix}.$$

On the other hand,

$$\widetilde{H}_{z^5} H_{\varphi} C_{z^2} P_{z^2} f = \begin{bmatrix} \boxed{0 & 0} \\ \boxed{0 & a_{-4}} \\ a_{-4} & a_{-3} \\ a_{-3} & a_{-2} \\ a_{-2} & a_{-1} \end{bmatrix} \begin{bmatrix} \bar{z}_0 \\ \bar{z}_1 \end{bmatrix}$$

and

$$\widetilde{H}_{z^2} H_{\varphi} C_{z^5} f = \begin{bmatrix} \boxed{0 & 0} & a_{-4} & a_{-3} & a_{-2} \\ \boxed{0 & a_{-4}} & a_{-3} & a_{-2} & a_{-1} \end{bmatrix} \begin{bmatrix} \bar{z}_0 \\ \bar{z}_1 \\ \bar{z}_2 \\ \bar{z}_3 \\ \bar{z}_4 \end{bmatrix}.$$

References

[1] A. Baranov, I. Chalendar, E. Fricain, J. Mashreghi, and D. Timotin, Bounded symbols and reproducing kernels thesis for truncated Toeplitz operators. J. Funct. Anal., 259 (2010), 2673–2701.
[2] R. V. Bessonov, Truncated Toeplitz operators of finite rank. Proc. Amer. Math. Soc., 142 (2014), no. 4, 1301–1313.
[3] C. Câmara, J. Jurasik, K. Kliś-Garlicka, and M. Ptak, Characterizations of asymmetric truncated Toeplitz operators, Banach J. Math. Anal., 11 (2017), 899–922.
[4] M. Cristina Câmara and J. R. Partington, Asymmetric truncated Toeplitz operators and Toeplitz operators with matrix symbol, J. Operator Theory, 77 (2017), 455–479.
[5] S. R. Garcia, J. Mashreghi, and W. T. Ross, Introduction to Model Spaces and their Operators, Cambridge University Press 2016.
[6] S. R. Garcia, E. Prodan, and M. Putinar, Mathematical and phisical aspects of complex symmetric operators, J. Phis. A, Math Theor., 47 (2014), 1–54.
[7] S. R. Garcia and M. Putinar, Complex symmetric operators and applications, Trans. Amer. Math.Soc., 358 (2006), 1285–1315.
[8] S. R. Garcia and M. Putinar, Complex symmetric operators and applications II, Trans. Amer. Math.Soc., 359 (2007), 3913–3931.
[9] V. I. Godič and I. E. Lucenko, On the representation of a unitary operator in the form of a product of two involutions, Uspiehi Mat. Nauk, 126 (2006), 64–65.
[10] Ch. G. Li and T. T. Zhou, Skew symmetry of a class of operators, Banach J. Math. Anal. 8 (2014), 279–294.
[11] K. Kliś-Garlicka, B. Łanucha, and M. Ptak, Characterization of truncated Toeplitz operators by conjugations, Oper. Matrices 11 (2017), 807–822.
[12] K. Kliś-Garlicka and M. Ptak, C-symmetric operators and reflexivity, Operators and Matrices, 9 (2015), 225–232.
[13] N. K. Nikolski, Hardy, Hankel, and Toeplitz, AMS, Mathematical Surveys and Monographs, vol. 92 AMS, 2002.
[14] H. Radjavi and P. Rosenthal, Invariant Subspaces, Springer, New York, 1973.
[15] D. Sarason, Algebraic properties of truncated Toeplitz operators, Oper. Matrices, 1 (2007), 491–526.
[16] J. Stochel and J. B. Stochel, Composition operators on Hilbert spaces of entire functions with analytic symbols, J. Math. Anal. Appl., 454 (2017), 1019–1066.