



Recovery of Dipolar Sources and Stability Estimates

Ridha Mdimagh^{a,b}

^aUniversity of Jeddah, College of Science and Arts at Khulis, Department of Mathematics, Jeddah, Saudi Arabia.

^bENIT-LAMSIN, BP 37, 1002 Tunis belvédère, University of Tunis El Manar, Tunisia.

Abstract. The inverse problem of identifying dipolar sources with time-dependent moments, located in a bounded domain, via the heat equation is investigated, by applying a heat flux, and from a single lateral boundary measurement of temperature. An uniqueness, and local Lipschitz stability results for this inverse problem are established which are the main contributions of this work. A non-iterative algebraic algorithm based on the reciprocity gap concept is proposed, which permits to determine the number, the spatial locations, and the time-dependent moments of the dipolar sources. Some numerical experiments are given in order to test the efficiency and the robustness of this method.

1. Introduction

The time-dependent heat source identification problem is a classic inverse problem, it has been studied by several authors. In general, this problems is known to be ill-posed, this ill-posedness is mainly the consequence of the absence of continuity and stability, but also of the non-uniqueness [24, 38]. Many theoretical and numerical results concerning the time-dependent heat source identification problems have been established by several authors by using different techniques. Homogenized function technique, to include the initial condition/boundary conditions and supplementary condition, to simplify the governing equations for the recovery of time/space-dependent heat sources was proposed in [30]. Coupled boundary integral equation method was developed to recover a time-dependent heat source under additional measures of temperature at interior points [31, 32]. In [39] a Dirichlet series representation for the boundary observation, and a finite difference approximation method in conjunction with the truncated singular value decomposition was used for the problem inverse of determining the diffusion coefficient, spacewise dependent source term. The reciprocity gap principle [5] was used in many works for the study of the point sources identification problem via elliptic equations [10, 16–19, 23, 33, 34], and was extended in [22] for the monopolar sources identification from fractional diffusion equation. Kang *et al* in [27] considered the problem of identifying simple poles of a meromorphic function by means of the value of the function measured on a circle enclosing those poles, they proposed an algebraic algorithm for this problem with a stability estimate and they applied the method to an electrical impedance tomography problem to detect small inclusions of disk shape via boundary measurements using the asymptotic behavior of the voltage potential in the presence of inclusions satisfying the Laplace equation. Nara *et al* proposed in [35] an algebraic method to reconstruct the positions of multiple poles in meromorphic function field from partial

2010 *Mathematics Subject Classification.* 49N45, 65L09, 65M32, 65N21.

Keywords. Heat equation, Inverse problem, Dipolar sources, Moments, Identifiability, Stability, Reciprocity gap functional.

Received: 20 January 2019; Revised: 29 May 2019; Accepted: 27 September 2019

Communicated by Marko Petković

Email address: ridha.mdimagh@lamsin.rnu.tn, ridha.mdimagh@gmail.com, rmothman1@uj.edu.sa (Ridha Mdimagh)

measurements without an iterative optimization or an estimation of the missing data and applied their algorithm to a 2D electric impedance tomography problem. Ben Abda *et al* [7] formulated the inverse source problem as that of locating the singularities of a meromorphic function from its values on the whole boundary using best rational approximation problems. Clerc *et al* presented in [15] a method, used for EEG source localization via Laplace equation in 3D case by analyzing the sets of planar singularities on the plane sections, based on rational approximation techniques in the complex plane and offered stability. Diverse iterative inversion methods have been applied to heat source identification, see [8, 12, 25]. Their main inconveniences are, as usual for this sort of procedure, the reliance of the result on the choice of initial guess and the elevated computational times derived from repeated forward solutions.

This problem is important in many engineering science disciplines, particularly in the areas of medical imaging and non-destructive testing of materials. The problem of detection of pollution sources on water surfaces is an important motivation to discuss this type of problems, it was studied by El Badia *et al* [17], by using the concept of reciprocity gap that formalizes the comparison of the response of a body containing point sources to the response of safety one of the same physical characteristics, and with an appropriate choice of test function they reduced this problem to the source problem identification from elliptic equation, where the term source was modeled as a linear combination of Dirac distribution at the point sources with L^2 time-varying intensities, and gave an algebraic algorithm to solve the inverse problem.

In this paper, we are interested in the detection of m dipolar sources with $L^2(0, T)^d$ time-dependent moments, located in an open bounded domain of \mathbb{R}^d ($d = 2, 3$) with sufficiently regular boundary $\Gamma := \partial\Omega$ via the heat equation, where $T > 0$ is an arbitrary positive number which denotes the measurement duration. This problem is governed by the following equation:

$$u_t - \Delta u = F \text{ in } \Omega_T := (0, T) \times \Omega, \tag{1.1}$$

where $u := u(t, \xi)$ represents the state variable, and F the unknown term, which has to be determined, and has the following expression:

$$F(t, \xi) = \sum_{j=1}^m M_j(t) \cdot \nabla \delta_{C_j}(\xi).$$

In the above equation, $m \in \mathbb{N}$, $C_j \in \Omega$, and $M_j(t) \in \mathbb{R}^d \setminus \{0\}$, $j = 1, \dots, m$, represent respectively the number, the locations, and the time-dependent moments of the dipolar sources which are all unknown, and δ_{C_j} represents the Dirac distribution at C_j .

We suppose that the dipolar sources C_j are distinct (i.e. $C_n \neq C_j$ for $n \neq j$), inactive after the finite time $T^* < T$ ($M_j(t) = 0$ for $t > T^*$), we suppose that $M_j \in (L^2(0, T))^d$, and

$$\int_0^{T^*} M_j(t) dt \neq 0, \quad j = 1, \dots, m. \tag{1.2}$$

Equation (1.1) is supplemented by the initial condition

$$u(0, \xi) = 0, \text{ for all } \xi \in \Omega, \tag{1.3}$$

and the boundary condition which models the heat transfer with external environment,

$$\frac{\partial u}{\partial \nu} = \varphi, \text{ on } \Gamma_T := (0, T) \times \Gamma \tag{1.4}$$

where ν represents the outward unit normal vector to Γ and $\varphi \in L^2(\Gamma_T)$ which matches with the compatibility condition:

$$\varphi(0, \xi) = 0, \text{ for all } \xi \in \Gamma.$$

The inverse problem consists in identifying the source distribution F in the parabolic problem (1.1)-(1.4) from the space-time boundary data f :

$$u = f, \text{ on } \Gamma_T. \tag{1.5}$$

In the present work, in order to study the identifiability, we show that our inverse problem is well posed from the Cauchy data measurements (in the sense where these measurements are generated by a unique source term). The main contributions of this paper concerning the result of stability obtained for the heat dipolar sources identification problem with time-dependent moments. This stability result, which is valid for two and three dimension, is derived from the Gâteaux differentiability, and by showing that the Gâteaux derivative is not null we obtain a local Lipschitz stability result for this inverse problem. An algebraic method is proposed for the identification of the positions of the dipolar sources and their time-dependent moments inspired from the algorithm given in [17, 18] which is extended by Mdimagh et al in [22] for the problem of identification of points sources via time fractional diffusion equation. The moments of the sources will be recovered via their Fourier transforms. Numerical tests are given to prove the efficiency of the algorithm and evaluate its resolution and robustness with respect to the measurement noise.

The paper is organized as follows:

In section 2, we discuss the question of existence and uniqueness of the direct problem, we state the inverse problem, and we present the reciprocity gap functional associated with the present aim. In section 3, an identifiability result is established. Local Lipschitz stability result is given in section 4. Finally, in section 5, an explicit identification procedure is proposed, numerical experiments are reported using numerically computed synthetic noisy data.

2. Setting problem

A direct variational formulation of problem (1.1)-(1.3) is not possible since the source term F is a distribution with support in Ω which belongs to the Sobolev space $L^2((0, T), H^s(\mathbb{R}^d))$ for $s < -(\frac{d}{2} + 1)$. However, one can consider the function u_0 defined by:

$$u_0 := \sum_{j=1}^m \int_0^t M_j(\tau) \cdot \nabla G(t - \tau, \xi - C_j) d\tau$$

where G represents the fundamental solution of $(\partial_t - \Delta)$ in $[0, \infty) \times \mathbb{R}^d$:

$$G(t, \xi) = \begin{cases} \frac{e^{-\frac{|\xi|^2}{4t}}}{(4\pi t)^{\frac{d}{2}}} & \text{for } t > 0. \\ 0 & \text{for } t = 0. \end{cases}$$

we set $w = u - u_0$, since the support of F is a subset of Ω_T , u_0 is analytic out of Ω_T , and we have:

$$\begin{cases} w_t - \Delta w = 0 & \text{in } \Omega_T, \\ \partial_\nu w = \varphi - \partial_\nu u_0 & \text{on } \Gamma_T, \\ w(0, \cdot) = 0 & \text{in } \Omega. \end{cases} \tag{2.1}$$

The existence and the uniqueness of the problem's solution of (1.1)-(1.3) are deduced from the following result given in [2]:

Theorem 2.1. *If φ belongs to $L^2((0, T), L^2(\Gamma))$, then the problem (2.1) has a unique solution which belongs to $L^2((0, T), H^1(\Omega)) \cap C((0, T), L^2(\Omega))$.*

3. Identifiability

3.1. Recovering of the dipolar sources

In this section, the uniqueness of the source term from boundary measurements will be proved which is one of difficulties of the inverse problem. In what follows, we introduce the reciprocity gap (RG) functional, and we state the inverse problem. Let v an adjoint field of the equation (1.1), it is solution of

$$v_t + \Delta v = 0, \text{ in } \Omega_T.$$

Multiplying equation (1.1) by v , applying a part integration in time over the interval $[0, T]$, and using initial condition (1.3), the below equation is found

$$\int_{\Gamma_T} \left(\frac{\partial v}{\partial \nu} f - v\varphi \right)(s, \xi) + \int_{\Omega} u(T, \xi)v(T, \xi) \, d\xi = \int_{\Omega_T} F v, \tag{3.1}$$

To give the identification procedure of the dipolar sources we need the following lemma

Lemma 3.1. $\lim_{T \rightarrow \infty} \int_{\Omega} u(T, \xi)v(T, \xi) \, d\xi = 0$

Proof. From [2] Prop 8.4.1, we have, for $\xi \in \Omega$, $\lim_{T \rightarrow \infty} u(T, \xi) = 0$, then for $\varepsilon > 0$, it exists $T_0 > 0$ such that for $T > T_0$ $|u(T, \xi)| < \varepsilon$. From Cauchy-Schwartz inequality we obtain

$$\text{for } T > T_0, \left| \int_{\Omega} u(T, \xi)v(T, \xi) \, d\xi \right| \leq c\varepsilon \|v(T, \cdot)\|_{L^2(\Omega)},$$

and if $v(T, \cdot) \in L^2(\Omega)$, then

$$\lim_{T \rightarrow \infty} \int_{\Omega} u(T, \xi)v(T, \xi) \, d\xi = 0.$$

□

Remark 3.2. *There is at least two different ways to remove, or minimize, the non-observable contribution*

$\int_{\Omega} u(T, \xi)v(T, \xi) \, d\xi$ *to the equation (3.1):*

- *Using adjoint fields $v(x, t)$ that vanish at final time $t = T$. Such adjoint fields are available in analytical form involving Fourier-Bessel series.*
- *Exploiting the measurements made over a duration T sufficiently large to make the non-observable component negligible. This approach is followed here as it permits more flexibility in choosing adjoint fields. In particular, harmonic time-independent adjoint fields can be used, allowing a natural extension of previously-proposed algebraic treatments. This essentially requires making measurements until much later than the extinction time T^* of the last active source. For more details see [6].*

For a large T , we can neglect $\int_{\Omega} u(T, \xi)v(T, \xi) \, d\xi$ of the equation (3.1), and we obtain

$$\mathcal{R}(u, v) = \sum_{j=1}^m P_j \cdot \nabla v(C_j), \tag{3.2}$$

where

$$\mathcal{R}(u, v) := \int_{\Gamma_T} \varphi v - \frac{\partial v}{\partial \nu} f,$$

and

$$P_j := \int_0^{\infty} M_j(t) \, dt, \quad j = 1, \dots, m,$$

which are all well defined since all sources are assumed to be inactive for $t \geq T^*$.

In the reciprocity gap (equality (3.2)), when v is given, then the reciprocity gap function $\mathcal{R}(u, v)$ is known while $\sum_{j=1}^m P_j \cdot \nabla v(C_j)$ is not. Based on this remark, equality (3.2) represents the link between the known quantity and the apparent source function. The inverse problem purposes to find the number, the locations

and the moments of the dipolar sources from equation (3.2) using particular test functions v . By exploiting the (RG) values for long time to identify the source term, we set:

$$\Theta(\xi) := \int_0^\infty u(t, \xi) dt,$$

$$\mathcal{F}(\xi) := \int_0^\infty F(t, \xi) dt = \sum_{j=1}^m P_j \cdot \nabla \delta_{C_j}(\xi),$$

Integrating equation (1.1) between 0 and ∞ , have recourse to the initial condition (1.3) and the boundary conditions (1.4)-(1.5), the function Θ satisfies the following Poisson problem:

$$\begin{cases} -\Delta\Theta = \mathcal{F} & \text{in } \Omega, \\ \partial_\nu\Theta = \hat{\varphi} & \text{on } \Gamma, \\ \Theta = \hat{f} & \text{on } \Gamma, \end{cases}$$

where

$$\hat{\varphi}(\xi) = \int_0^\infty \varphi(t, \xi) dt \text{ and } \hat{f}(\xi) = \int_0^\infty f(t, \xi) dt.$$

The problem (1.1)-(1.5) can now be reduced to the problem of identifying dipolar sources C_j with moments $P_j \neq 0$ (see equation (1.2)) via Laplace equation and from the Cauchy data $(\hat{f}, \hat{\varphi})$. This inverse problem can be solved by using the algebraic method introduced by El Badia in [14, 18]. Based on a complex variable formulation of the adjoint field, this approach assumes a spatially two-dimensional setting and proposes a way to extend this approach to three-dimensional configuration by projections of the source points on different planes going down to 2D case. From now on, a spatially two-dimensional configuration is assumed to be complex polynomials used for adjoint fields. Such functions being holomorphic (have harmonic real and imaginary parts). The following test functions are defined by:

$$v_n(z) = z^n, \quad n \in \mathbb{N}$$

where $z = x + i y$ denotes the affix of $\xi = (x, y) \in \mathbb{R}^2$.

The components of the RG equality are then given by:

$$\mathcal{R}(u, v_n) = \sum_{j=1}^m n p_j \sigma_j^{n-1} \tag{3.3}$$

where $\sigma_j := C_{j,1} + i C_{j,2}$ denotes the affix of the j^{th} dipolar sources $C_j := (C_{j,1}, C_{j,2})$ and $p_j = P_{j,1} + i P_{j,2}$ is the affix of the moment $P_j := (P_{j,1}, P_{j,2})$ of C_j . The reconstruction of the dipolar sources consists in finding σ_j and p_j from the equality (3.3).

Let \tilde{M} be an upper bound of the exact number m of the unknown dipolar sources ($\tilde{M} \geq m$), let:

$$\alpha_n := \frac{\mathcal{R}(u, v_n)}{n} = \sum_{j=1}^m p_j \sigma_j^{n-1},$$

$$\mu_n = \begin{pmatrix} \alpha_n \\ \alpha_{n+1} \\ \vdots \\ \alpha_{\tilde{M}+n-1} \end{pmatrix} \in \mathbb{C}^{\tilde{M}}, \quad \Lambda = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{pmatrix} \in \mathbb{C}^m, \tag{3.4}$$

and the matrix

$$A_{n, \tilde{M}} = \begin{pmatrix} \sigma_1^n & \sigma_2^n & \dots & \sigma_m^n \\ \sigma_1^{n+1} & \sigma_2^{n+1} & \dots & \sigma_m^{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_1^{\tilde{M}+n-1} & \sigma_2^{\tilde{M}+n-1} & \dots & \sigma_m^{\tilde{M}+n-1} \end{pmatrix} \in \mathcal{M}_{\tilde{M} \times m}(\mathbb{C}).$$

Following the line of the algorithm given in [18], the unknown m , σ_j , and p_j can then be deduced from the following lemma:

Lemma 3.3. [18]

1. The rank of the family $(\mu_1 \ \mu_2 \ \dots \ \mu_m)$ is $r = m$, and the vectors $\mu_1 \ \mu_2 \ \dots \ \mu_m$ are independent.
2. The affixes σ_j of the dipolar sources C_j are the eigenvalues of the matrix \mathcal{T} which is defined by $\mathcal{T} \mu_j = \mu_{j+1}$, for $j = 1, \dots, m$.
3. p_1, \dots, p_m are solutions of the linear system $A_{1,m} \Lambda = \mu_1$ where $A_{1,m}$ is the Vandermonde matrix of σ_j .

Remark 3.4. 1. In the case where Ω contains a unique dipolar source C_1 , then:

$$p_1 = \alpha_1 \text{ and } \sigma_1 = \frac{\alpha_2}{\alpha_1}.$$

2. In the case where Ω contains two monopolar sources C_1, C_2 , and if (a, b) are the components of the vector μ_3 in the basis (μ_1, μ_2) , then the eigenvalues of the matrix $\mathcal{T} = \begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix}$ are

$$\sigma_1 = \frac{b + \sqrt{b^2 + 4a}}{2}, \quad \sigma_2 = \frac{b - \sqrt{b^2 + 4a}}{2},$$

with moments

$$p_1 = \frac{\sigma_2 \alpha_1 - \alpha_2}{\sigma_1(\sigma_2 - \sigma_1)} \text{ and } p_2 = \frac{\alpha_2 - \alpha_1 \sigma_1}{\sigma_2(\sigma_2 - \sigma_1)}.$$

3.2. Recovering of the moments

The goal of this section is to identify the moments M_j of the dipolar sources C_j . First, we look for test functions ψ_n with separated variables and with exponentially dependent time parts solution of the equation $\psi_t + \Delta \psi = 0$ in Ω_T , we take ψ_n on the following form:

$$\psi_n(t, \xi) = e^{-i\omega t} v_n(\xi),$$

where $\omega \in \mathbb{R}$ and v_n is a solution of the Helmholtz equation

$$\Delta v_n + K^2 v_n = 0 \text{ in } \Omega,$$

where $K = \sqrt{-i\omega}$. We take v_n as the fundamental outgoing solution of the Helmholtz equation originated at some points b_n belonging to $\mathbb{R}^2 \setminus \bar{\Omega}$.

$$\Delta v_n + K^2 v_n = -\delta_{b_n} \text{ in } \mathbb{R}^2,$$

v_n is given explicitly by $v_n(\xi) = \frac{-i}{4} H_0^{(1)}(K|\xi - b_n|)$, where $H_0^{(1)}$ is the Hankel function of first kind and order 0 see [1], and the imaginary part of K is positive.

Multiplying equation (1.1) by ψ_n , integrating with respect t and ξ over $[0, \infty) \times \Omega$, and by using Green's formula, we have

$$\sum_{j=1}^m \widehat{M}_j(\omega) \cdot \nabla v_n(C_j) = \mathcal{R}(u, \psi_n). \tag{3.5}$$

where $\widehat{M}_j(\omega) = \int_0^\infty M_j(t) e^{-i\omega t} dt$ is the Fourier transform of M_j extended out of $(0, T)$ by 0. Now, the reconstructions of the moments is obtained from the knowing of the Fourier transform \widehat{M}_j for sufficiently

many values of w and then use an appropriate algorithm to invert the transform and compute M_j . Let $\widehat{M}_j(w) := (\widehat{\lambda}_{j,1}(w), \widehat{\lambda}_{j,2}(w))$, the equality (3.5) is given by

$$\sum_{j=1}^m \widehat{\lambda}_{j,1}(w) \frac{\partial v_n}{\partial x}(C_j) + \widehat{\lambda}_{j,2}(w) \frac{\partial v_n}{\partial y}(C_j) = \mathcal{R}(u, \psi_n).$$

To determine $\widehat{\lambda}_{j,1}(w)$ and $\widehat{\lambda}_{j,2}(w)$, it is sufficient to find $2m$ points b_1, \dots, b_{2m} in $\mathbb{R}^2 \setminus \bar{\Omega}$ such that the matrix B is invertible, where

$$B = \begin{pmatrix} \frac{\partial v_1}{\partial x}(C_1) & \frac{\partial v_1}{\partial y}(C_1) & \dots & \frac{\partial v_1}{\partial x}(C_m) & \frac{\partial v_1}{\partial y}(C_m) \\ \frac{\partial v_2}{\partial x}(C_1) & \frac{\partial v_2}{\partial y}(C_1) & \dots & \frac{\partial v_2}{\partial x}(C_m) & \frac{\partial v_2}{\partial y}(C_m) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial v_{2m}}{\partial x}(C_1) & \frac{\partial v_{2m}}{\partial y}(C_1) & \dots & \frac{\partial v_{2m}}{\partial x}(C_m) & \frac{\partial v_{2m}}{\partial y}(C_m) \end{pmatrix}.$$

Lemma 3.5. *Let b_1, b_2, \dots, b_{2m} points of $\mathbb{R}^2 \setminus \bar{\Omega}$ assumed to be distinct ($b_j \neq b_n$ if $j \neq n$). Then, the matrix B is invertible.*

Proof. Let us denote by $G(b_1, \dots, b_{2m}) := \det(B)$. Suppose that

$$G(b_1, \dots, b_{2m}) = 0, \text{ for all } b_n \in \mathbb{R}^2 \setminus \bar{\Omega}, n = 1, \dots, 2m.$$

We fix for example b_2, \dots, b_{2m} and consider G as a function of b_1 , this function is an analytic function when extended to the connected domain $\mathbb{R}^2 \setminus \cup\{C_j\}$ and

$$G(b_1, \dots, b_{2m}) = 0, \text{ for all } b_n \in \mathbb{R}^2 \setminus \cup\{C_j\}$$

Let $b_1 = (x, C_{1,2}) \in \mathbb{R}^2 \setminus \cup\{C_j\}$, when x tends to $C_{1,1}$, we obtain a contradiction by pointing out the presence of the singularity C_1 , all the terms of G tend to some finite numbers except

$$B_{1,1} = \frac{iK}{4} \frac{x - C_{1,1}}{|x - C_{1,1}|} H_1^{(1)}(K|x - C_{1,1}|)$$

and

$$\lim_{x \rightarrow C_{1,1}} |B_{1,1}| = \infty.$$

Then, the matrix B is invertible. \square

Remark 3.6. $\widehat{M}_j(w) := (\widehat{\lambda}_{j,1}(w), \widehat{\lambda}_{j,2}(w))$, $j = 1, \dots, m$ are solutions of the linear system $B\tilde{\Lambda} = \tilde{R}$, where

$$\tilde{\Lambda} = \begin{bmatrix} \widehat{\lambda}_{1,1}(w) \\ \widehat{\lambda}_{1,2}(w) \\ \vdots \\ \widehat{\lambda}_{m,1}(w) \\ \widehat{\lambda}_{m,2}(w) \end{bmatrix} \text{ and } \tilde{R} = \begin{bmatrix} \mathcal{R}(u, \psi_1) \\ \mathcal{R}(u, \psi_2) \\ \vdots \\ \mathcal{R}(u, \psi_{2m}) \end{bmatrix}.$$

We can now give our main identifiability result

Theorem 3.7. *Under assumption (1.2), the source F is uniquely determined by the observation data (1.4)-(1.5).*

4. Stability Estimates

In this section, the study concerns the continuity of the unknown source term with respect to the boundary measurements which usually tainted with errors, affect the numerical reconstruction, and make the inverse problem unstable. The question of stability has been the matter of interest of several authors in different contexts. A logarithm-type stability estimate for the 2D case problem of identifying dense masses in the earth from gravimetry data taken at the surface or in the air, assuming that the poles are well separated and their respective strengths are large enough was established in [11]. Hölder-type stability estimate, for the same problem for 3D case problem, was given in [38], and in [20] for time-harmonic Maxwell dipolar-source problem. The notion of local Lipschitz was introduced in [9], and has been used repeatedly by several authors [5, 13], particularly for cracks, boundary recovery and Robin’s coefficient. In many works, local Lipschitz stability results was obtained, derived from algebraic relations for elliptic sources identification problems [6, 10, 16, 19, 21, 33, 34]. A local Lipschitz stability estimate, derived from the Gâteaux differentiability, was established in [22] for the fractional monopolar source identification. A conditional stability for the problem of determining a non linear heat source term was established in [28] using an integral identity.

Our main contribution in this work is to establish a local Lipschitz stability result inspired from the stability result given in [34] for the problem of identification of sources via the Helmholtz equation, which is derived from the Gâteaux differentiability, by establishing that the Gâteaux derivative is not zero.

In the following we recall this result:

We suppose that Ω contains m dipolar sources located at C_j with respectively moments $p_j, j = 1, \dots, m$. We define the perturbed source term F^h by:

$$F^h = - \sum_{j=1}^m p_j^h \cdot \nabla \delta_{C_j^h},$$

where

$$(p_j^h, C_j^h) := (p_j + h q_j, C_j + h R_j), \quad 1 \leq j \leq m, \\ \{(q_j, R_j), 1 \leq j \leq m\} \subset \mathbb{R}^2 \times \mathbb{R}^2,$$

such that $\|q_j\| \leq 1$ and $\|R_j\| \leq 1, h$ being sufficiently small to insure that $C_j + h R_j$ remain in Ω . We denote by u_0 and u_h the solutions of (4.1) with respectively source terms $F = F^0$ and $F = F^h$.

$$\begin{cases} \Delta u + k^2 u = F & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = \varphi & \text{on } \Gamma, \end{cases} \tag{4.1}$$

$\varphi \in H^{-\frac{1}{2}}(\Gamma)$ being the flux on Γ ($\varphi \neq 0$ on Γ), k is the wave number on Ω . We set $u_{0|\Gamma} = f, u_{h|\Gamma} = f^h$.

Theorem 4.1. [34] **(Local Lipschitz stability).** *Assume that k^2 is not an eigenvalue of $-\Delta$ with Neumann condition in the boundary. Then,*

$$\lim_{h \rightarrow 0} \frac{\|f^h - f\|_{L^2(\Gamma)}}{|h|} \text{ exists and is strictly positive.}$$

Now, we are ready to give the main result of this section.

Assuming that the domain Ω contains m dipolar sources C_1, \dots, C_m with respectively moments M_1, \dots, M_m where $M_j \in (L^2(0, T))^2$, let $(\mu_j, R_j) \in (L^2(0, T))^2 \times \mathbb{R}^2$ such that $\mu_j(t) = 0$ for $t \geq T$, and $\|R_j\| \leq 1$ for $j = 1, \dots, m$, we set $\Phi := (M_j, C_j)$,

$$\Phi^h := (M_j^h, C_j^h) := (M_j + h \mu_j, C_j + h R_j),$$

and

$$F^h := \sum_{j=1}^m M_j^h \cdot \nabla \delta_{C_j^h},$$

$h \neq 0$ being sufficiently small to insure that $C_j^h \in \Omega$. Let u_0 and u_h be the solutions of problems (1.1)-(1.4) with respectively sources F^0 and F^h , we set $u_0 = f$ and $u_h = f^h$ on Γ_T . Then, we give the local Lipschitz stability result

Theorem 4.2. (Local Lipschitz stability)
 If $\mu_j(t) \neq 0$, for $0 < t < T$, then

$$\lim_{h \rightarrow 0} \frac{\|f^h - f\|_{L^2(\Gamma_T)}}{h} \neq 0.$$

Proof. Extending the function φ , f and M_j , $j = 1, \dots, m$ by 0 outside the interval $[0, T]$, let $s \in \mathbb{C}$ with $\text{Re}(s) > 0$ and $\text{Im}(is^{\frac{1}{2}}) > 0$, consider the time-integrated quantities

$$\Theta^h(s, \xi) := \int_0^\infty e^{-st} u_h(t, \xi) dt,$$

and

$$\Theta^0(s, \xi) := \int_0^\infty e^{-st} u_0(t, \xi) dt,$$

which are well-defined since all sources are assumed inactive for $t \geq T$, and we assume that

$$\tilde{M}_j(s) := \int_0^\infty e^{-st} M_j(t) dt \neq 0, \quad j = 1, \dots, m.$$

Applying the Laplace transform to the problems (1.1)-(1.4) corresponding respectively to the sources F^h and F^0 , the function Θ^h is solution of the Helmholtz equation with the wave number $k = is^{\frac{1}{2}}$:

$$\begin{cases} \Delta \Theta^h + k^2 \Theta^h = -\tilde{F}^h & \text{in } \Omega \\ \partial_\nu \Theta^h = \tilde{\varphi} & \text{on } \Sigma \\ \Theta^h = \tilde{f}^h & \text{on } \Sigma \end{cases} \tag{4.2}$$

where

$$\tilde{f}^h(s, \xi) = \int_0^\infty e^{-st} f^h(t, \xi) dt, \quad \tilde{f}(s, \xi) = \int_0^\infty e^{-st} f(t, \xi) dt,$$

$$\tilde{F}^h(s, \xi) = \sum_{j=1}^m \tilde{M}_j^h(s) \cdot \nabla \delta_{C_j^h}(\xi), \quad \tilde{F}^0(s, \xi) = \sum_{j=1}^m \tilde{M}_j(s) \cdot \nabla \delta_{S_j}(\xi),$$

$$\tilde{M}_j^h(s) = \tilde{M}_j(s) + h \tilde{\mu}_j(s),$$

and

$$\tilde{\mu}_j(s) = \int_0^\infty e^{-st} \mu_j(t) dt.$$

The source \tilde{F}^h represents the linear perturbation of the source \tilde{F}^0 in the direction $\Psi = \{(\tilde{\mu}_j(s), R_j)_{1 \leq j \leq m}\}$, having the same number m of sources as $\{(\tilde{M}_j(s), C_j)_{1 \leq j \leq m}\}$ for the problem of identifying dipolar sources C_j , located in Ω with respectively moments $\tilde{M}_j(s)$, $j = 1, \dots, m$ via the Helmholtz equation with wave number $k = is^{\frac{1}{2}}$, which is not an eigenvalue of $-\Delta$ with Neumann condition in the boundary, from the given Cauchy data $\tilde{\varphi}$ and \tilde{f} on Σ . We have

$$(\tilde{M}_j^h(s), C_j^h) = (\tilde{M}_j(s) + h \tilde{\mu}_j(s), C_j + h R_j).$$

From Theorem 4.1, which its proof is also valid for the wave number $k = is^{\frac{1}{2}}$ where $s \in \mathbb{C}$ with $\mathcal{R}e(s) > 0$ and $\mathcal{I}m(is^{\frac{1}{2}}) > 0$, we deduce the following result:

$$\lim_{h \rightarrow 0} \frac{|f^{\tilde{h}} - \tilde{f}|_{L^2(\Gamma)}}{h} \neq 0.$$

From Cauchy-Schwartz inequality one has

$$|f^{\tilde{h}} - \tilde{f}|_{L^2(\Gamma)} \leq \frac{1}{\sqrt{2\mathcal{R}e(s)}} |f^h - f|_{L^2(\Sigma_T)},$$

then, we obtain the following local Lipschitz result:

$$\lim_{h \rightarrow 0} \frac{|f^h - f|_{L^2(\Sigma_T)}}{h} \neq 0.$$

□

Remark 4.3. If $\lim_{h \rightarrow 0} \frac{|f^h - f|_{L^2(\Sigma_T)}}{|h|} = \ell \in \mathbb{R}_+^* \cup \{\infty\}$, then there exists $\delta > 0$ and $c > 0$ such that if $|h| < \delta$, then $|h| < c |f^h - f|_{L^2(\Sigma_T)}$, which implies that there exists $\tilde{c} > 0$ such that for $|h| < \delta$

$$\sum_{j=1}^m \|C_j^h - C_j\| + \|M_j^h - M_j\|_{L^2(0, T)^2} \leq \tilde{c} |f^h - f|_{L^2(\Sigma_T)}$$

which gives the local Lipschitz stability result for the identification of dipolar sources problem. The result of the Theorem 4.2 means that we can identify the positions of the dipolar sources and their moments if the measurement error made on the data u on the boundary is of the order $o(h)$.

5. Numerical experiments

In this section, some numerical experiments regarding the detection of dipolar sources are presented. In all numerical tests, the reconstructions of the dipolar sources are computed with noisy data by adding uniformly distributed random variables to the synthetic transient data: $u|_{\Gamma_T} := f^{synth} + \varepsilon r$, where r is a random number in the interval $[-1, 1]$ and ε is the noise level. f^{synth} is defined from the fundamental solution G of (1.1) defined by:

$$G(t, \xi) = \begin{cases} \frac{e^{-\frac{|\xi|^2}{4t}}}{(4\pi t)^{\frac{d}{2}}} & \text{for } t > 0, \\ 0 & \text{for } t = 0. \end{cases}$$

We take

$$u(t, \xi) := \sum_{j=1}^m \int_0^t M_j(\tau) \cdot \nabla G(t - \tau, \xi - C_j) d\tau + \varepsilon r. \tag{5.1}$$

The domain Ω is the disk centered at the origin with unit radius. The simulated measurements of the function (5.1) are evaluated at 240×960 points of $[0, T] \times C(0, 1)$, where $C(0, 1)$ is the circle centered at the origin with unit radius, $T = 4$ is the observation duration, and the values $\alpha_k = \frac{\mathcal{R}(u, \psi_k)}{k}$ are computed for $k = 1, \dots, 2\tilde{M} - 1$ using trapezoidal quadrature rule, where \tilde{M} is an upper bound of the exact number m of the unknown dipolar sources. The Fourier transform and the inverse Fourier transform evaluations of the exact and identified moments again use a trapezoidal quadrature rule.

For the numerical approach of the number m of the dipolar sources, we use the stopping criterion introduced in [27], where the authors remark that for $n = 1, \dots, \tilde{M}$, the matrix:

$$(\mu_1 \ \mu_2 \ \dots \ \mu_n) = \begin{pmatrix} p_1 & p_2 & \dots & p_m \\ p_1\sigma_1 & p_2\sigma_2 & \dots & p_m\sigma_m \\ \vdots & \vdots & & \vdots \\ p_1\sigma_1^{n-1} & p_2\sigma_2^{n-1} & \dots & p_m\sigma_m^{n-1} \end{pmatrix} \begin{pmatrix} 1 & \sigma_1 & \dots & \sigma_1^{n-1} \\ 1 & \sigma_2 & \dots & \sigma_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & \sigma_m & \dots & \sigma_m^{n-1} \end{pmatrix},$$

where μ_j is defined by (3.4).

If $m = n$, then the determinant \mathcal{D}_n of $(\mu_1, \mu_2, \dots, \mu_n)$ is

$$\mathcal{D}_n := \det(\mu_1, \mu_2, \dots, \mu_n) = p_1 p_2 \dots p_n \prod_{k < j} (\sigma_k - \sigma_j)^2$$

If $n > m$ we can reduce the matter to the case $n = m$ by letting $p_{m+1} = \dots = p_n = 0$ and $\mathcal{D}_n = 0$.

In particular

$$|\mathcal{D}_n| \geq \beta^n \alpha^{n^2-n}$$

Refereing to [27] the number of the dipolar sources is the smallest integer m for which

$$|\mathcal{D}_m| > \frac{1}{2} \beta^m \alpha^{m^2-m} \tag{5.2}$$

and

$$|\mathcal{D}_i| \leq \frac{1}{2} \beta^i \alpha^{i^2-i}, \quad i = m + 1, \dots, \tilde{M}. \tag{5.3}$$

where α and β are given such that $|\sigma_k - \sigma_j| \geq \alpha$, $k \neq j$ and $|p_j| \geq \beta$, $j = 1, \dots, m$.

In all subsequent figures, the circle with solid line represents the boundary Γ of Ω , i.e. the measurements location. We, also, indicate in these figures the exact positions of the dipolar source which are represented by \circ , and the reconstructed sources which are represented by \star . The vector which starts from the exact dipolar source represents the Fourier transform at 0 of its moment, and the vector which starts from the identified dipolar source represents the identified moment.

5.1. Identification of the dipolar sources

In the following numerical experiments, the reconstructions of the sources are computed with fix value noise level $\varepsilon = 1\%$ except in one example where the value of the noise level is 20%, and this can show the robustness of our method with respect to this type of noise. Regarding the estimation of the number m of the dipolar sources, we use the stopping criterion (5.2)-(5.3) described in the beginning of this section or we choose m as the smallest natural number k such that \mathcal{D}_k is greater than a fixed small real number $\tilde{\varepsilon}$. In this experiment, we shall vary $\tilde{\varepsilon}$ to test the sensitivity of the identification processing to the maximum number \tilde{M} of sources. The domain Ω contains four dipolar sources $C_1 = (0.4, 0.6)$, $C_2 = (0.5, -0.5)$, $C_3 = (-0.5, -0.5)$ and $C_4 = (-0.5, 0.5)$ with respectively moments:

$$M_1(t) = \begin{cases} (3 \cos(3t) e^{-t^2}, \pi t \cos(2\pi t)) & \text{if } t \in (0, 1) \\ (0, 0) & \text{if } t = 0 \text{ or } t \in [1, 4), \end{cases}$$

$$M_2(t) = \begin{cases} (\frac{1}{2} e^{\sin(\pi t)} \cos(\pi t), \frac{1}{4}) & \text{if } t \in (0, 2) \\ (0, 0) & \text{if } t = 0 \text{ or } t \in [2, 4), \end{cases}$$

$$M_3(t) = \begin{cases} (\frac{t}{4}, \frac{1}{4} e^{-t}) & \text{if } t \in (0, 2.5) \\ (0, 0) & \text{if } t = 0 \text{ or } t \in [2.5, 4), \end{cases}$$

and

$$M_4(t) = \begin{cases} \left(\frac{1}{2} \frac{1}{t^2+1}, \frac{1-\cos(t)}{4} \right) & \text{if } t \in (0,3) \\ (0,0) & \text{if } t = 0 \text{ or } t \in [3,4). \end{cases}$$

The extinction time of the last active dipolar source is $T^* = 3$. In the following numerical tests the exact and the identified dipolar sources are presented with the Fourier transform at 0 of their corresponding moments, the exact and the identified moments of these dipolar sources will be illustrated in the numerical tests of section 5.2. By the trapezoidal quadrature rule the Fourier transform at 0 of the moments M_j of the dipolar sources C_j are respectively $P_1(0.5010, -0.0066)$, $P_2(-0.1038, 0.3734)$, $P_3(0.5010, 0.2147)$ and $P_4(0.5923, 0.4473)$.

In the first numerical tests, we fix the value of the stopping criterion $\tilde{\varepsilon} = 10^{-2}$, and we choose two different values of the upper bound of the unknown dipolar sources $\tilde{M} = 7$ and $\tilde{M} = 10$ which are represented respectively in Figure 1 (Left) and in Figure 1 (Right), the identified dipolar sources in the two cases are $S_1(0.3979, 0.6001)$, $S_2(0.5059, -0.5020)$, $S_3(-0.4999, -0.5058)$ and $S_4(-0.5032, 0.4995)$, with moments respectively are $q_1(0.5198, 0.0210)$, $q_2(-0.1101, 0.3710)$, $q_3(0.5006, 0.2259)$ and $q_4(0.5873, 0.4834)$, we observe that the number m , the locations, and the Fourier transform at 0 of the moments of the dipolar sources are well reconstructed.

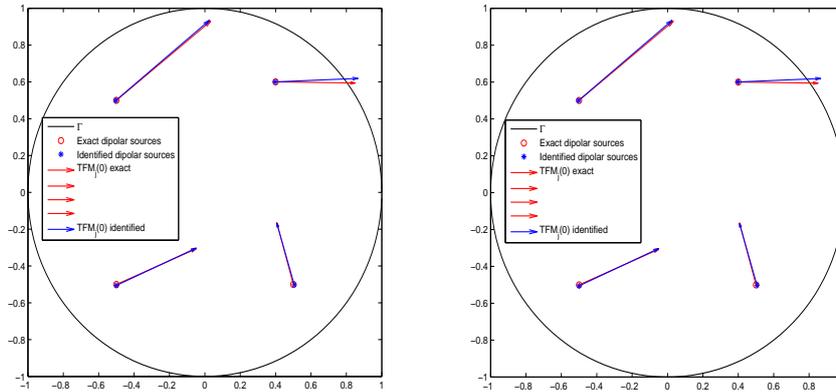


Figure 1: Well reconstruction of the number, the locations and the Fourier transform of the moments at 0 of four dipolar sources for level noise 1% and stopping criterion $\tilde{\varepsilon} = 10^{-2}$. Left: $\tilde{M} = 7$. Right: $\tilde{M} = 10$.

In Figure 2, we take the stopping criterion $\tilde{\varepsilon} = 10^{-3}$, and two different values of the upper bound of the unknown dipolar sources $\tilde{M} = 7$ and $\tilde{M} = 10$ which are represented respectively in Figure 2 (Left) and in Figure 2 (Right). The locations and the Fourier transforms of the moments at 0 of the four dipolar sources are well reconstructed, the same dipolar sources are identified for the two cases: $S_1(0.4013, 0.6020)$, $S_2(0.5071, -0.4997)$, $S_3(-0.4986, -0.5060)$, $S_4(-0.5046, 0.4993)$ with moments respectively are $q_1(0.5139, 0.0251)$, $q_2(-0.1069, 0.3738)$, $q_3(0.5036, 0.2241)$, $q_4(0.5866, 0.4783)$, and an another point appears $S_5(1.1001, 0.5498)$ outside Ω with weak moment $q_5(0.0004, -0.0002)$.

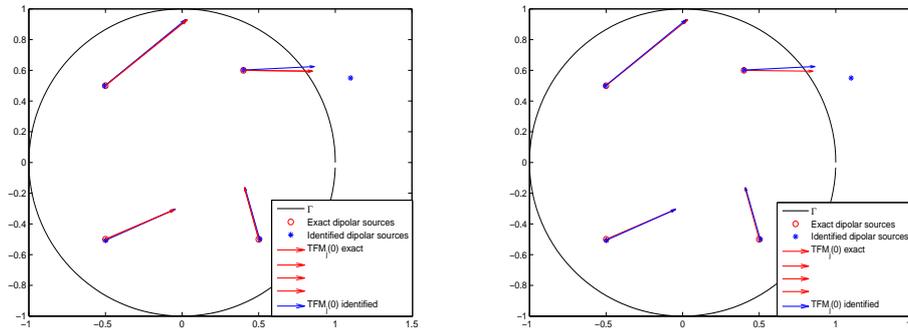


Figure 2: Well reconstruction of the four dipolar sources with their Fourier transform of the moments at 0, and a fifth point appears $S_5(1.1001, 0.5498)$ with moment $q_5(0.0004, -0.0002)$, for level noise 1% and stopping criterion $\varepsilon = 10^{-3}$. Left: $\tilde{M} = 7$. Right: $\tilde{M} = 10$.

These numerical tests show that the identification process is sensitive to the stopping criterion, in subsection 5.2 we show that the time-dependent moment of the fifth dipolar source is zero and this point can be eliminated, and we can conclude the exact number of the dipolar sources.

5.2. Identification of the moments

Once the number m of the dipolar sources is known, their locations can be estimated, the reconstructions of their moments is obtained by knowing the Fourier transform $\widehat{M}_j(w) := (\widehat{\lambda}_{j,1}(w), \widehat{\lambda}_{j,2}(w))$ of M_j extended out of $[0, T]$ by 0. Naturally, to compute the moments M_j , we have to compute $\widehat{M}_j(w)$ for sufficiently many values of w and then use an appropriate algorithm to invert the transform.

Let $b_k = (2 \cos(\frac{k\pi}{m}), 2 \sin(\frac{k\pi}{m}))$, $k = 1, \dots, 2m$, $2m$ distinct points of the circle of center 0 and radius 2. Then, from remark 3.6, $\tilde{\Lambda}$ is solution of the linear system $B \tilde{\Lambda} = \tilde{R}$, where

$$B = \begin{pmatrix} \frac{\partial v_1}{\partial x}(C_1) & \frac{\partial v_1}{\partial y}(C_1) & \dots & \frac{\partial v_1}{\partial x}(C_m) & \frac{\partial v_1}{\partial y}(C_m) \\ \frac{\partial v_2}{\partial x}(C_1) & \frac{\partial v_2}{\partial y}(C_1) & \dots & \frac{\partial v_2}{\partial x}(C_m) & \frac{\partial v_2}{\partial y}(C_m) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial v_{2m}}{\partial x}(C_1) & \frac{\partial v_{2m}}{\partial y}(C_1) & \dots & \frac{\partial v_{2m}}{\partial x}(C_m) & \frac{\partial v_{2m}}{\partial y}(C_m) \end{pmatrix},$$

$$\tilde{\Lambda} = \begin{pmatrix} \widehat{\lambda}_{1,1}(w) \\ \widehat{\lambda}_{1,2}(w) \\ \vdots \\ \widehat{\lambda}_{m,1}(w) \\ \widehat{\lambda}_{m,2}(w) \end{pmatrix},$$

and

$$\tilde{R} = \begin{pmatrix} \mathcal{R}(u, \psi_1) \\ \mathcal{R}(u, \psi_2) \\ \vdots \\ \mathcal{R}(u, \psi_{2m}), \end{pmatrix}$$

$v_k(\xi) = \frac{-i}{4}H_0^{(1)}(K|\xi - b_k|)$, $H_0^{(1)}$ is the Hankel function of first kind and order 0, while the imaginary part of $K = \sqrt{-iw}$ is positive.

In all following experiments, we take 7185 equidistant points $w \in [-30, 30]$ for the numerical evaluations of $\widehat{\lambda}_{j,k}$, $k = 1, 2$ the components of \widehat{M}_j , which are solutions of the linear system $B\widehat{\lambda} = \widehat{R}$, and 240 equidistant points t of the interval $[0, 4]$ to evaluate $\lambda_{j,k}(t) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{\lambda}_{j,k}(w) e^{iwt} dw$, the inverse Fourier transform of $\widehat{\lambda}_{j,k}$.

In the following figures, we give the curves of the real and imaginary parts of the moments of the exact dipolar sources in red color and the curves of the real and imaginary parts of the moments of the identified dipolar sources in blue color.

Figure 3 illustrates the reconstruction of the real and imaginary parts of the exact moments $M_j(t)$ and the identified moments, of the exact and identified dipolar sources presented in Figure 1 (Left), with stopping criterion $\varepsilon = 10^{-2}$, and an upper bound of the unknown dipolar sources $\widetilde{M} = 7$, in this numerical test we obtain a well reconstruction of the four moments.

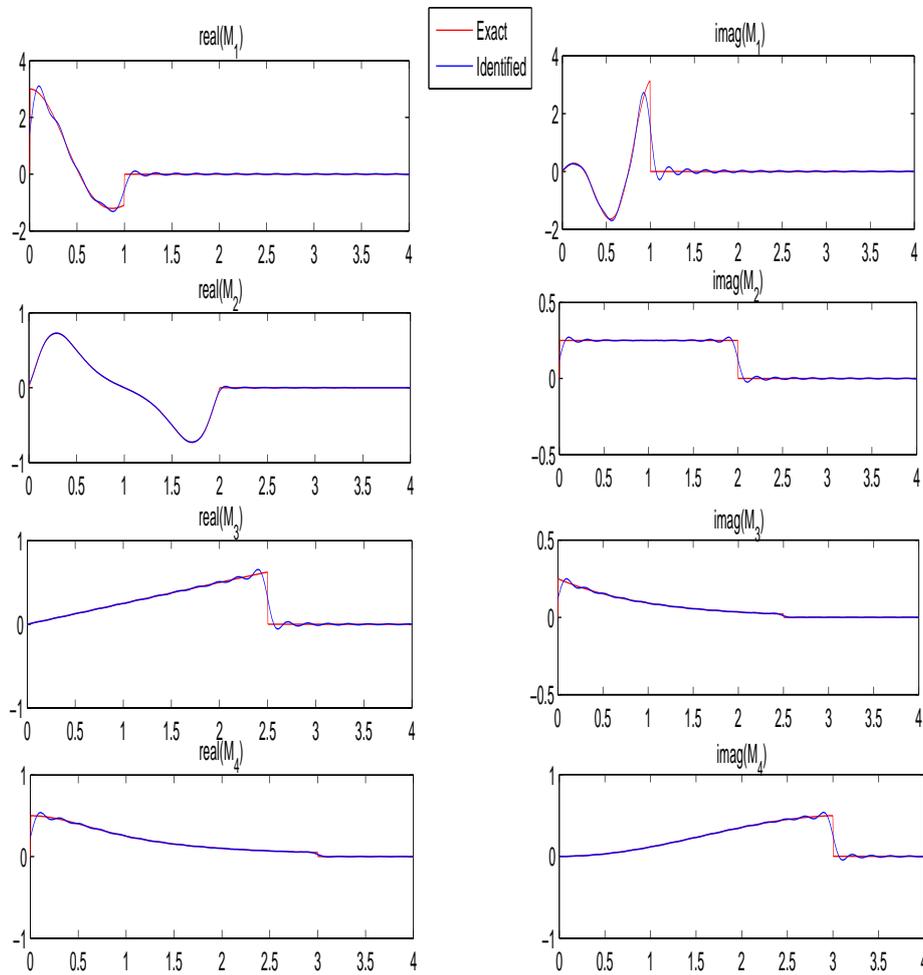


Figure 3: Well reconstruction of the moments of the four dipolar sources with level noise 1%, stopping criterion $\varepsilon = 10^{-2}$ and $\widetilde{M} = 7$.

Figure 4 illustrates the reconstruction of the real and imaginary parts of the exact moments and the identified moments, of the exact and identified dipolar sources presented in Figure 2 (Left), with stopping criterion

$\varepsilon = 10^{-3}$, and an upper bound of the unknown dipolar sources $\widetilde{M} = 7$. The moments of the four dipolar sources are well reconstructed, we observe very low values of the real and imaginary parts of the moment of the fifth dipolar source $S_5(1.1001, 0.5498)$, and this permit to eliminate this dipolar source.

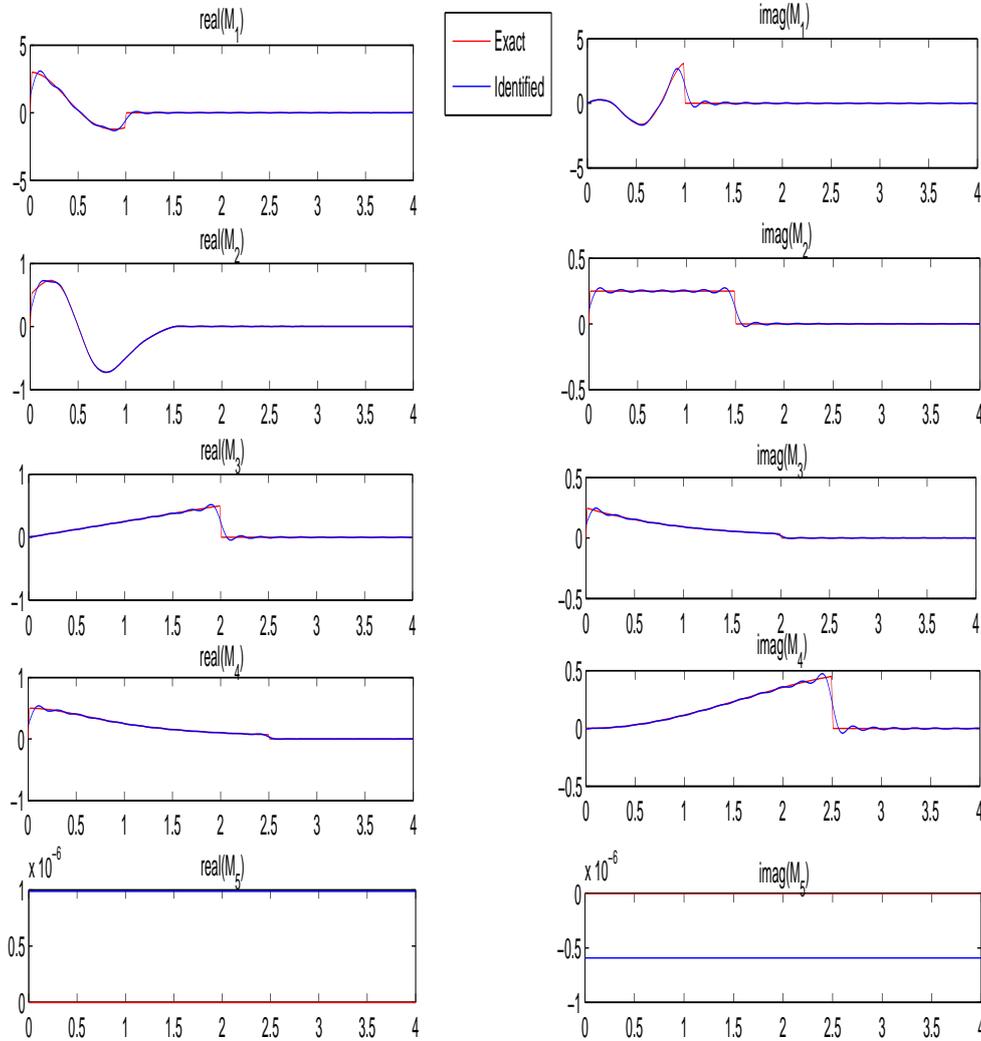


Figure 4: The moments of the four dipolar sources are well reconstructed, and the values of the moment of S_5 is very low, with level noise 1%, stopping criterion is $\varepsilon = 10^{-3}$ and $\widetilde{M} = 7$.

5.2.1. Sensitivity to the relative position of two dipolar sources

In the following numerical experiment (Figure 5), we test the reconstruction of two close dipolar sources $C_1 = (0.49, 0.6)$ and $C_2 = (0.5, 0.6)$ where the distance between them is $d(C_1, C_2) = 0.01$, the Fourier transforms of their moments at 0 are respectively $\widetilde{M}_1(0) = (0.5010, -0.0066)$ and $\widetilde{M}_2(0) = (0.2734, 0.3734)$, where the exact moments M_1 and M_2 are defined by:

$$M_1(t) = \begin{cases} (3e^{-t^2} \cos(3t), \pi t \cos(2\pi t)) & \text{if } t \in (0, 1) \\ (0, 0) & \text{if } t = 0 \text{ or } t \in [1, 4), \end{cases}$$

$$M_2(t) = \begin{cases} \left(\frac{1}{2}e^{\cos(\pi t)} \sin(\pi t), \frac{1}{4} \right) & \text{if } t \in (0, 1.5) \\ (0, 0) & \text{if } t = 0 \text{ or } t \in [1.5, 3), \end{cases}$$

The numerical reconstructions of the sources are computed with the value noise level $\varepsilon = 1\%$, the stopping criterion $\tilde{\varepsilon} = 10^{-3}$, and the upper bound of the unknown dipolar sources $\tilde{M} = 5$, we observe that two dipolar sources are identified: $S_1 = (0.2327, 0.1368)$ with moment $q_1 = (0.0146, 0.0117)$, and $S_2 = (0.492, 0.6059)$ with moment $q_2 = (0.7959, 0.3930)$. We remark that the identified dipolar source S_2 represents the barycenter of the weighted points $(C_1, \frac{\widehat{M}_1(0)}{\widehat{M}_1(0)+\widehat{M}_2(0)})$ and $(C_2, \frac{\widehat{M}_2(0)}{\widehat{M}_1(0)+\widehat{M}_2(0)})$ ($S_2 \simeq \frac{\widehat{M}_1(0)C_1 + \widehat{M}_2(0)C_2}{\widehat{M}_1(0)+\widehat{M}_2(0)}$) with moment $\widehat{M}_1(0) + \widehat{M}_2(0)$.

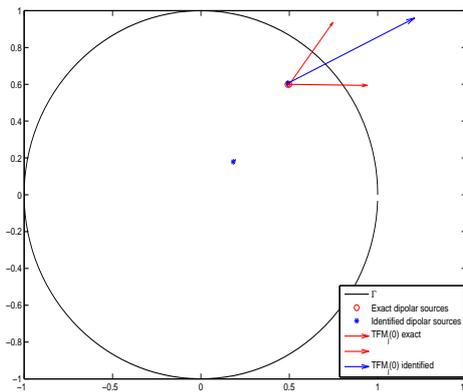


Figure 5: Detection of two close dipolar sources: the identified points are $S_1 = (0.2327, 0.1368)$ with moment $q_1 = (0.0146, 0.0117)$, and $S_2 = (0.492, 0.6059) \simeq \frac{\widehat{M}_1(0)}{\widehat{M}_1(0)+\widehat{M}_2(0)}C_1 + \frac{\widehat{M}_2(0)}{\widehat{M}_1(0)+\widehat{M}_2(0)}C_2$ with moment $q_2 = (0.7959, 0.3930) \simeq \widehat{M}_1(0) + \widehat{M}_2(0)$.

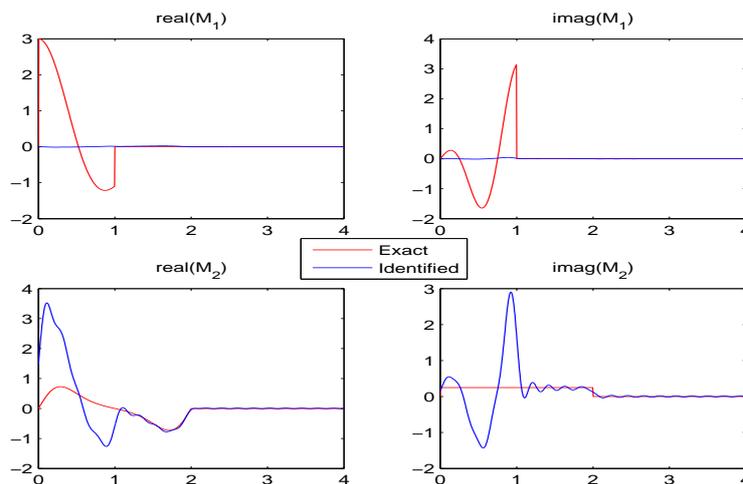


Figure 6: Graphs of the real and imaginary parts of the exact and identified moments of two close points distant by 0.01.

In Figure 6, we display the real and imaginary parts of the exact and identified moments of the exact and

identified dipolar sources presented in Figure 5. This experiment shows that the values of the moment of the identified dipolar source S_1 are very low, and this point can be eliminated, the real and imaginary parts of the exact dipolar source C_2 and the identified dipolar source S_2 are different, we remark that the identified dipolar source S_2 represents the barycenter of the weighted points $(C_1, \frac{\widehat{M}_1(0)}{\widehat{M}_1(0)+\widehat{M}_2(0)})$ and $(C_2, \frac{\widehat{M}_2(0)}{\widehat{M}_1(0)+\widehat{M}_2(0)})$ ($S_2 \simeq \frac{\widehat{M}_1(0)C_1+\widehat{M}_2(0)C_2}{\widehat{M}_1(0)+\widehat{M}_2(0)}$), with moment $M_1(t) + M_2(t)$ see Figure 7, in which we represent the graph of the real and imaginary parts of $M_1 + M_2$ and the moment of S_2 .

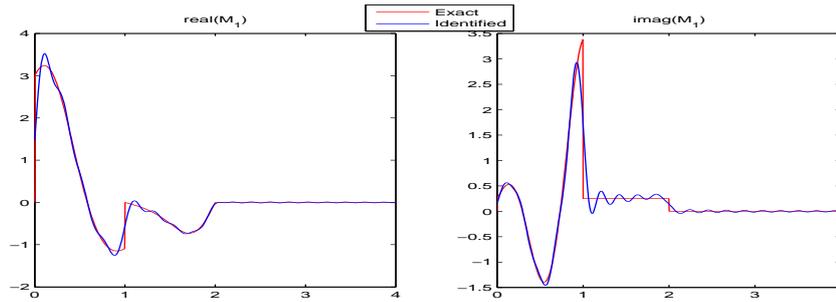


Figure 7: Reconstruction of the graphs of the real parts and imaginary parts of $M_1 + M_2$ and the the graphs of the real and imaginary parts of the moment of S_2 , when the exact dipolar sources are distant by 0.01.

5.2.2. Robustness of the method

The following experiments illustrate the cases of identification of four dipolar sources with data corrupted by 20% pointwise relative random noise, an upper bound of the unknown dipolar sources $\widetilde{M} = 7$, and stopping criterion respectively $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-3}$. In Figure 8 (Left), the stopping criterion $\varepsilon = 10^{-2}$, we observe that the four dipolar sources with their moments are well reconstructed, and for $\varepsilon = 10^{-3}$, see Figure 8 (Right), another point is identified in more outside Ω $S_5(1.0996, 0.5495)$ with moment $q_5(0.0004, -0.0002)$.

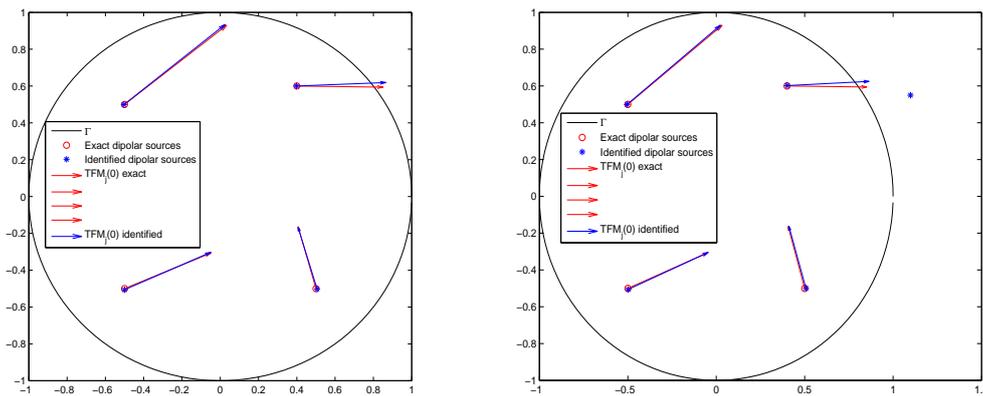


Figure 8: Identification of four dipolar sources with 20% level noise and an upper bound of the unknown dipolar sources $\widetilde{M} = 7$. Left: $\varepsilon = 10^{-2}$, well identification of the dipolar sources. Right: $\varepsilon = 10^{-3}$, a fifth point appears $S_5(1.0996, 0.5495)$ with moment $q_5(0.0004, -0.0002)$.

In Figure 9, we draw the graphs of the real and imaginary parts of the moments of the exact and identified dipolar sources for the case $\tilde{\varepsilon} = 10^{-3}$ illustrated in Figure 8 (Right). We remark that the moments of the four dipolar sources are well reconstructed, and the moment of the fifth dipolar source S_5 is $M_5 \approx (0,0)$, which can be eliminated. One clearly observes that our method is very robust with respect to this type of noise. This is due to the fact that this noise is filtered by the reciprocity gap functional. This has also been observed in previous works on this type of method see [6, 10, 14, 23, 33, 34].

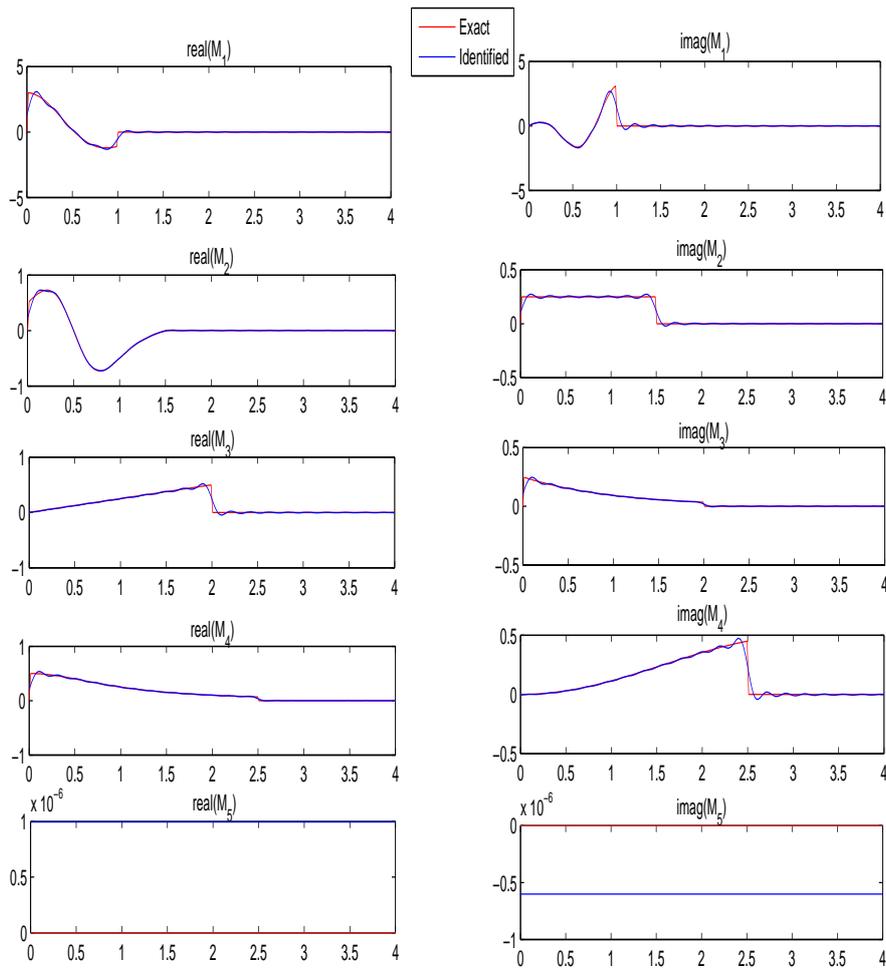


Figure 9: Well reconstruction of the moments with level noise 20%, $\tilde{M} = 7$, and $\tilde{\varepsilon} = 10^{-3}$. Another dipolar source appears $S_5(1.0996, 0.5495)$ with moment $M_5(t) \approx (0,0)$ which can be eliminated.

6. Conclusion

In this work, an identifiability result was proved for the problem of identifying dipolar sources with time-dependent moments. Local Lipschitz stability result was established derived from the Gâteaux differentiability, by establishing that the Gâteaux derivative is not zero. An algebraic algorithm was proposed

using the reciprocity gap concept to identify the locations and the time-dependent moments of the dipolar sources. The proposed method has been verified numerically, it was fast, robust, and gave a good reconstruction of the dipolar sources and their moments, without using an initial estimate or an iterative calculation of any forward solution.

References

- [1] M. Abramowitz and I.A. Stegun. Handbook of Mathematical Functions. New York: Dover Publications. Inc(eds), (1999).
- [2] G. Allaire. Numerical analysis and optimization. Oxford, (2007).
- [3] G. Alessandrini. Stable determination of conductivity by boundary measurements *Appl. Anal.* 27 15372, 1988.
- [4] G. Alessandrini and E. DiBenedetto. Determining 2-dimensional cracks in 3-dimensional bodies: uniqueness and stability *Indiana Univ. Math. J.* 46 181, 1997.
- [5] S. Andrieux, A. Ben Abda. Identification of planar cracks by complete overdetermined data: inversion formulae. *Inverse Problems*, 12, 553-563, (1996).
- [6] N. Auffray and M. Bonnet and S. Pegano. Identification of transient heat sources using the reciprocity gap, *Inverse Problems in Science and Engineering*, Taylor & Francis, 21, pp.721-738, (2013).
- [7] L. Baratchart, A. Ben Abda, F. Ben Hassen and J. Leblond. Recovery of pointwise sources or small inclusions in 2D domains and rational approximation. *Inverse Problems*, 21-51, 2005.
- [8] J. Baumeister and A. Leitao. On iterative methods for solving ill-posed problems modeled by partial differential equations. *J. Inv. Ill-Posed Prob*, 9,13-29, (2001).
- [9] H. Bellout and A. Friedman. Identification problems in potentiel theory. *Arch. Rat. Mech. Anal*, 101, 143-160, 1988.
- [10] A. Ben Abda, R. Mdimagh, A. Saada. Identification of pointwise sources and small size flaws via the reciprocity gap principle; Stability Estimates. *International Journal of Tomography and Statistics (IJTS)*, Vol 4, , No. S06, 24-45, Summer (2006).
- [11] J.R. Cannon and R.E. Ewing. The locations and strengths of point sources improperly posed boundary values problems. *Research Notes in Mathematics*. vol 1 (London: Pitman), 3953, 1975.
- [12] A. Capatina and R. Stavre. Algorithms and convergence results for an inverse problem in heat propagation. *Int.J. Eng. Sc*, 38:575-587, (2000).
- [13] S. Chaabane, F. ElDabaghi and M. Jaoua. On the identification of unknown boundaries submitted to Signorini boundary conditions: identifiability and stability *Math. Methods Appl. Sci*, 21 137998, 1998.
- [14] M. Chafik, A. El Badia and T Ha-Duong. One some inverse EEG problems, *Inverse Problems in engineering mechanics*, 2, 537-544,(2000).
- [15] M. Clerc, J. Leblond, J-P. Marmorat and T. Papadopoulos. Source localization using rational approximation on plane sections. *Inverse Problems*, 28, (2012).
- [16] A. El Badia. Inverse source problem in an anisotropic medium by boundary measurements. *Inverse Problems*, 21, 1487-1506, (2005).
- [17] A. El Badia and T. Ha-Duong. On an inverse source problem for the heat equation. Application to a pollution detection problem, *J. Inv. Ill-Posed Problems*, Vol.10, No.6, pp. 585-599, (2002).
- [18] A. El Badia and T. Ha-Duong An inverse source problem in potential analysis. *Inverse Problems*, 16, 651-663, (2000).
- [19] A. El Badia and T. Nara. An inverse source problem for Helmholtz's equation from the Cauchy data with a single wave number. *Inverse Problems*, 27, 105001, (2011).
- [20] A. El Badia and T. Nara. Inverse dipole source problem for time-harmonic Maxwell equations: algebraic algorithm and Hölder stability. *Inverse Problems*, 29, 015007, (2013).
- [21] A. El Badia., A. ElHajj, M. Jazar and H. Moustafaa. Lipschitz stability estimates for an inverse source problem in an elliptic equation from interior measurements. Taylor & Francis, *Applicable Analysis*, 2015 <http://dx.doi.org/10.1080/00036811.2015.1074680>.
- [22] A. Ghanmi, R. Mdimagh and I. Ben Saad. Identification of points sources via time fractional diffusion equation. *FILOMATH*, Vol 32, No 18, 6189-6201 (2018).
- [23] H. Haddar and R. Mdimagh. Identification of small inclusions from multistatic data using the reciprocity gap concept. *Inverse Problems*, 28, 045011, (2012).
- [24] V. Isakov. *Inverse source problems*. American Mathematical Society, (1989).
- [25] B. Johansson and D. Lesnic. A procedure for determining a spacewise dependent heat source and the initial temperature. *Appl. Anal.*, 87:265-276, (2008).
- [26] D. Kandaswamy, T. Blu and D. Van De Ville. Analytic sensing: Noniterative retrieval of point sources from boundary measurements. *SIAM J. Sci. Comput*, 31,3179-3194, 2009.
- [27] H. Kang and H. Lee. Identification of simple poles via boundary measurements and an application of EIT. *Inverse Problems*, 20, 1853-1863, (2004).
- [28] G. Li. Data compatibility and conditional stability for an inverse source problem in the heat equation. *Applied Mathematics and computation*, 173, 566-581, (2006).
- [29] G.S. Li, Y.J. Tan, J. Cheng, and X.Q. Wang. Determining magnitude of groundwater pollution sources by data compatibility analysis, *Inverse Probl. Sci. Eng*, 14 (3) ,287-300, (2006).
- [30] C.-S. Liu. Homogenized functions to recover $H(t)/H(x)$ by solving a small scale linear system of differencing equations. *International Journal of Heat and Mass Transfer*. 101, 11031110,(2016).
- [31] C.-S. LIU AND D. LIU. Recovering a general space-time-dependent heat source by the coupled boundary integral equation method. *NUMERICAL HEAT TRANSFER, PART B. VOL. 71, NO. 3*, 283297, (2017).

- [32] C.-S. Liu. An integral equation method to recover non-additive and non-separable heat source without initial temperature. *International Journal of Heat and Mass Transfer*. 97, 943953,(2016).
- [33] R. Mdimagh. Identification d'inclusions de petites tailles par des méthodes quasi-explicites: couplage RG-LSM, LAMSIN-ENIT, PhD Thesis, Tunisia, (2009).
- [34] R. Mdimagh, I.B. Saad, Stability estimates for point sources identification problem using reciprocity gap concept via the Helmholtz equation, *Applied Mathematical Modeling*, 40,7844-7861,(2016).
- [35] T. Nara and S. Ando. Direct localization of poles of a meromorphic function from measurements on an incomplete boundary. *Inverse Problems*, 26, 2010.
- [36] K. Ohnaka and K. Uosaki. Boundary element approach for identification of point forces of distributed parameter systems. *Internat. J. Control*. 49(1), 119-127, (1989).
- [37] M. N. Özisik and H.R.B. Orlande. *Inverse Heat Transfer: Fundamentals and Applications*. Hemisphere Pub, (2000).
- [38] S. Vessella. Locations and strengths of point sources: Stability estimates. *Inverse Problems*,8,911-917, (1992).
- [39] Z. X. Zhao, M. K. Banda AND B. Z. Guo. Simultaneous identification of diffusion coefficient, spacewise dependent source and initial value for one-dimensional heat equation. *Math. Meth. Appl. Sci.* DOI: 10.1002/mma.4245, (2016)