Filomat 33:13 (2019), 4115–4120
https://doi.org/10.2298/FIL1913115S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some New Characterizations of Normal Elements

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Abstract. In this paper, we shall give some new characterizations of normal elements in a ring with involution by the solutions of related equations.

1. Introduction

Throughout this paper, let R be an associative ring with 1. An involution in R is an anti-isomorphism $* : R \to R, a \to a^*$ of degree 2, that is,

$$(a^*)^* = a, (a+b)^* = a^* + b^*, (ab)^* = b^*a^*.$$

An element $a^\dagger$ is called the Moore-Penrose inverse (or MP-inverse) of a, if

$$a = aa^\dagger a, a^\dagger = a^\dagger aa^\dagger, (aa^\dagger)^* = aa^\dagger, (a^\dagger a)^* = a^\dagger a.$$

If $a^\dagger$ exists, then it is unique [1]. Denote by $R^\dagger$ the set of all MP-invertible elements of R.

An element $a \in R$ is said to be group invertible if there exists $a^\# \in R$ such that

$$a = aa^\# a, a^\# = a^\# aa^\#, aa^\# = a^\# a.$$

$a^\#$ is called a group inverse of a, and it is uniquely determined by the above condition [2]. We write $R^\#$ for the set of all group invertible elements of R.

The element $a \in R^\# \cap R^\dagger$ satisfying $a^\# = a^\dagger$ is said to be EP [3]. The set of all EP elements of R will be denoted by R^{EP}.

If $a^*a = aa^*$, then the element $a \in R$ is called normal. Mosić and Djordjević in [4, Lemma 1.2] proved for an element $a \in R^\dagger$ that a is normal if and only if $aa^\dagger = a^\dagger a$ and $a^*a^\dagger = a^\dagger a^*$. It is known by [5, Corollary 2.8, Lemma 2.7] $a \in R^\dagger$ is normal if and only if $a^\dagger(a^\dagger)^* = (a^\dagger)^* a^\dagger$ or $a \in R^{EP}$ and $a^*a^\dagger = a^\dagger a^*$. More results on normal elements are given in [5].

Following the fore study, this paper provide some equivalent conditions for an element to be normal in a ring with involution.

The following results are frequently used in this paper.

THEOREM 1.1 [5]. For any $a \in R^\# \cap R^\dagger$, the following are satisfied:

2010 *Mathematics Subject Classification*. 15A09; 16W10, 16U99
Keywords. normal element, EP element, Mooer-Penrose inverse, group inverse, solutions of equation.
Received: 22 January 2019; Accepted: 17 June 2019
Communicated by Dijana Mosić
Research supported by the National Natural Science Foundation of China (No. 11471282)
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(1) $(a^\dagger)^*R = aR$, $(a^\#)^*R = a^*R$;
(2) $aR = aa^\dagger R = aa^*R$, $a^*R = a^\dagger R = a^*aR = a^\dagger aR$;
(3) $aR = a^\# R = a^2R = aa^\# R$, $(a^*)^2R = a^*R$;
(4) $aR = aa^*a^\# R = a^\# a^*R$, $a^*R = a^*a^\# R$.
THEOREM 1.2 [2]. $a \in R^\#$ if and only if $a \in a^2R \cap Ra^2$.

2. Characterizations of normal elements

Proposition 2.1. *Let $a \in R^\# \cap R^\dagger$. Then a is normal if and only if $aa^*a^\dagger a^\dagger = a^*a^\dagger$.*

Proof. " ⇒ " *Since $a \in R^\dagger$ and a is normal, we have $a^*a = aa^*$ and $aa^\dagger = a^\dagger a$. Hence $aa^*a^\dagger a^\dagger = a^*a^\dagger aa^\dagger = a^*a^\dagger$.*
" ⇐ " *If $a \in R^\# \cap R^\dagger$, then by Theorem 1.1, we get*

$$a^\dagger R = a^*R = (a^*)^2R = a^*a^\dagger R = aa^*a^\dagger a^\dagger R \subseteq aR = aa^\# aR = a^\# a^2R \subseteq a^\# R,$$

which gives $(1 - a^\# a)a^\dagger \in (1 - a^\# a)a^\dagger R \subseteq (1 - a^\# a)a^\# R = 0$. Thus $a^\dagger = a^\# aa^\dagger$, and then we have $a^\dagger a = a^\# aa^\dagger a = a^\# a = aa^\#$, implies imediatly that $aa^\dagger = a^\dagger a$. Since $aa^*a^\dagger a^\dagger = a^*a^\dagger$, $a^*a^\dagger a^\dagger = a^\dagger aa^*a^\dagger a^\dagger = a^\dagger a^*a^\dagger$. It follows that $a^*a^\dagger = a^*a^\dagger aa^\dagger = a^*a^\dagger a^\dagger a = a^\dagger a^*a^\dagger a = a^\dagger a^*aa^\dagger = a^\dagger a^*$. This means a is normal. □

We alrady know that $a \in R^\# \cap R^\dagger$ satisfying $a^\# = a^\dagger$ is said to be EP. So we have the following corollary.

Corollary 2.2. *Let $a \in R^\# \cap R^\dagger$. Then a is normal if and only if $aa^*a^\# a^\dagger = a^*a^\#$.*

Proof. " ⇒ " *It is evident.*
" ⇐ " *Since $a \in R^\# \cap R^\dagger$ and $aa^*a^\# a^\dagger = a^*a^\#$, then by Theorem 1.1, we get $a^\dagger R = a^*R = a^*a^\# R = aa^*a^\# a^\dagger R = aa^*a^\# a^*R = aa^*aR = aa^*R = aR$. It follows that $aa^\dagger = a^\dagger a$. This gives that $a^*a = a^*a^\# a^2 = aa^*a^\# a^\dagger a^2 = aa^*a^\# a^2a^\dagger = aa^*aa^\dagger = aa^*$. Therefore a is normal.* □

Proposition 2.3. *Let $a \in R^\# \cap R^\dagger$. Then a is normal if and only if $(aa^*)^2 = a^*a^2a^*$.*

Proof. " ⇒ " *Assume that a is normal, then $a^*a = aa^*$. Hence $(aa^*)^2 = a^*a^2a^*$.*
" ⇐ " *Since $a \in R^\# \cap R^\dagger$ and $(aa^*)^2 = a^*a^2a^*$, then by Theorem 1.1, one obtains that $a^\dagger R = a^*aR = a^*a^2R = a^*a^2a^*R = (aa^*)^2R = aa^*aR = aa^\dagger R = aR$. So we arrive at $aa^\dagger = a^\dagger a$. This gives that $aa^*a = aa^*aa^\dagger a = aa^*aa^*(a^\dagger)^* = a^*a^2a^*(a^\dagger)^* = a^*a^2$. Multiplying the equality on the right by $a^\dagger$, we have $aa^* = a^*a$. Therefore a is normal.* □

Corollary 2.4. *Let $a \in R^\# \cap R^\dagger$. Then a is normal if and only if $a^* = aa^*a^\dagger$.*

Corollary 2.5. *Let $a \in R^\# \cap R^\dagger$. Then a is normal if and only if $\begin{pmatrix} a^* \\ a \end{pmatrix}$ is regular and $\begin{pmatrix} a^* \\ a \end{pmatrix}^- = \begin{pmatrix} 1 - aa^\dagger & (a^\dagger)^*a^\dagger a^* \end{pmatrix}$.*

Proof. " ⇒ " *If a is normal, then $aa^* = a^*a$. By [4, Theorem 2.2(xi)], we get $a^* = a^\dagger a^*a$. Thus*

$$\begin{pmatrix} a^* \\ a \end{pmatrix}\begin{pmatrix} 1 - aa^\dagger & (a^\dagger)^*a^\dagger a^* \end{pmatrix} = \begin{pmatrix} a^*(1 - aa^\dagger) & a^*(a^\dagger)^*a^\dagger a^* \\ a - a^2a^\dagger & a(a^\dagger)^*a^\dagger a^* \end{pmatrix} = \begin{pmatrix} 0 & a^\dagger aa^\dagger a^* \\ a - a^2a^\dagger & a(a^\dagger)^*a^\dagger a^* \end{pmatrix}.$$

By [5, Lemma 2.7], we have $a \in R^{EP}$, which gives $a = a^2a^\dagger$. By [5, Corollary 2.8], we get $(a^\dagger)^*a^\dagger = a^\dagger(a^\dagger)^*$. So we arrive at $a(a^\dagger)^*a^\dagger a^* = aa^\dagger(a^\dagger)^*a^* = aa^\dagger$. It follows that

$$\begin{pmatrix} a^* \\ a \end{pmatrix}\begin{pmatrix} 1 - aa^\dagger & (a^\dagger)^*a^\dagger a^* \end{pmatrix} = \begin{pmatrix} 0 & a^\dagger a^* \\ 0 & aa^\dagger \end{pmatrix},$$

meaning that

$$\begin{pmatrix} a^* \\ a \end{pmatrix}\begin{pmatrix} 1 - aa^\dagger & (a^\dagger)^*a^\dagger a^* \end{pmatrix}\begin{pmatrix} a^* \\ a \end{pmatrix} = \begin{pmatrix} 0 & a^\dagger a^* \\ 0 & aa^\dagger \end{pmatrix}\begin{pmatrix} a^* \\ a \end{pmatrix} = \begin{pmatrix} a^\dagger a^*a \\ aa^\dagger a \end{pmatrix} = \begin{pmatrix} a^* \\ a \end{pmatrix}.$$

" ⇐ " If $\begin{pmatrix} a^* \\ a \end{pmatrix}^- = \begin{pmatrix} 1 - aa^\dagger & (a^\dagger)^* a^\dagger a^* \end{pmatrix}$. *Then we have*

$$\begin{pmatrix} a^* \\ a \end{pmatrix} = \begin{pmatrix} a^* \\ a \end{pmatrix} \begin{pmatrix} 1 - aa^\dagger & (a^\dagger)^* a^\dagger a^* \end{pmatrix} \begin{pmatrix} a^* \\ a \end{pmatrix} = \begin{pmatrix} a^\dagger a^* a \\ (a - a^2 a^\dagger) a^* + a (a^\dagger)^* a^\dagger a^* a \end{pmatrix},$$

one obtains that $a^* = a^\dagger a^* a$, *therefore a is normal by [4, Theorem 2.2(xi)].* □

Note that if a is normal, then $(aa^*)^2 = aa^*aa^* = a^2a^*a^*$.

Conversely, we can ask if $a \in R^\# \cap R^\dagger$ with $(aa^*)^2 = a^2a^*a^*$, is it still a normal element?

The following example illustrates that this conclusion does not necessarily hold.

Example 2.6. *Let* $R = M_3(Z_2)$, *with the involution is the transpose of matrix. Suppose that* $a = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in R$. *So* $a^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = a^*$ *since*

$$aa^\dagger a = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = a,$$

$$a^\dagger aa^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = a^\dagger,$$

$$(aa^\dagger)^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = aa^\dagger,$$

$$(a^\dagger a)^* = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}^* = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = a^\dagger a.$$

Noting that $(aa^*)^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = aa^* = a^2a^*a^*$. *Nevertheless,* $a^*a = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \neq aa^*$. *We obtain a is not normal.*

Corollary 2.7. *Let* $a \in R^\dagger$. *Then a is normal if and only if* $(aa^*)^2 = a^2a^*a^*$ *and* $aa^\dagger = a^\dagger a$.

Proof. *" ⇒ " It is evident.*

" ⇐ " Suppose that $(aa^*)^2 = a^2a^*a^*$ *and* $aa^\dagger = a^\dagger a$. *Now, we get* $aa^* = aa^*aa^*(a^\dagger)^*a^\dagger = a^2a^*a^*(a^\dagger)^*a^\dagger = a^2a^*a^\dagger$. *Multiplying this equality by* $a^\dagger$ *from the left side, it follows* $a^* = a^\dagger a^2a^*a^\dagger = aa^\dagger aa^*a^\dagger = aa^*a^\dagger$. *Furthermore, we obtain* $a^\dagger a^* = a^\dagger aa^*a^\dagger = a^*a^\dagger$, *which implies that a is normal.* □

It is well known that $a \in R^{EP}$ if and only if $a \in R^\dagger$ and $aa^\dagger = a^\dagger a$. Hence we get following corollary.

Corollary 2.8. *Let* $a \in R^\dagger$. *Then a is normal if and only if* $a \in R^{EP}$ *and* $(aa^*)^2 = a^2a^*a^*$.

Corollary 2.9. *Let* $a \in R^\dagger$. *Then a is normal if and only if* $(aa^*)^2 = a^2a^*a^*$ *and* $a^* = a^*a^\dagger a$.

Proof. *" ⇐ " Let* $a^* = a^*a^\dagger a$ *and* $(aa^*)^2 = a^2a^*a^*$, *then* $a^*(1 - a^\dagger a) = 0$, *by taking the involution, gives* $(1 - a^\dagger a)a = 0$, *thus we get* $a = a^\dagger a^2$. *Since* $(aa^*)^2 = a^2a^*a^*$, *which yields* $a^\dagger aa^*aa^*(a^\dagger)^*a^\dagger = a^\dagger a^2a^*a^*(a^\dagger)^*a^\dagger$, *this shows that* $a^* = a^\dagger a^2a^*a^\dagger = aa^*a^\dagger$, *hence* $a^\dagger a^* = a^\dagger aa^*a^\dagger = a^*a^\dagger$. *Note that* $aR = a^\dagger a^2R \subseteq a^\dagger R = a^*R = aa^*a^\dagger R \subseteq aR$, *so we arrive that* $a \in R^{EP}$. *Therefore a is normal.*

" ⇒ " It is routine verification. □

Similarly, we have the following corollary.

Corollary 2.10. *Let $a \in R^\dagger$. Then a is normal if and only if $(aa^*)^2 = a^*a^2a^*$ and $a^* = aa^\dagger a^*$.*

Let $a \in R^\dagger \cap R^\#$. We write $comm(a) = \{x \in R | xa = ax\}$ and $\chi_a = \{a, a^\#, a^\dagger, a^*, (a^\dagger)^*, (a^\#)^*\}$. Now we consider the relations between normal elements and the solutions of certain equations.

Theorem 2.11. *Let $a \in R^\dagger$. Then a is normal if and only if the system of equations* (1)

$$\begin{cases} a^* = a^*ax \\ a^\dagger = a^\dagger ax \end{cases} \tag{1}$$

has at least one solution in $comm(a) \cap comm(a^*)$.

Proof. " $\Rightarrow$ " For any $a \in R^\dagger$ and a is normal, we deduce that $aa^\dagger = a^\dagger a$ and $a^\dagger a^* = a^*a^\dagger$, so $a^\dagger \in comm(a) \cap comm(a^*)$. Hence $x = a^\dagger$ is a solution of the system (1).

" $\Leftarrow$ " Assume that $x = c$ is a solution of the system (1), which belongs to $comm(a) \cap comm(a^*)$. Then we have $a^* = a^*ac$, $a^\dagger = a^\dagger ac$ and $ca = ac$, $ca^* = a^*c$. It follows that $a^* = a^*ac = (a^*ac)ac = a^*a^2c^2$ and $aa^\dagger = (a^\dagger)^*a^* = (a^\dagger)^*a^*a^2c^2 = aa^\dagger a^2c^2 = a^2c^2 = c^2a^2$, that is, $a = a^2c^2a = c^2a^3 \in a^2R \cap Ra^2$, then $a \in R^\#$ by Theorem 1.2. Furthermore, it implies that $aa^\dagger = c^2a^2 = c^2a(aa^\#a) = c^2a^2a^\#a = aa^\dagger aa^\# = aa^\#$, thus $a \in R^{EP}$. Noting that $a^*a^\dagger = (a^*ac)a^\dagger = ca^*aa^\dagger = ca^* = a^*c = a^\dagger aa^*c = a^\dagger aca^* = a^\dagger a^*$. Then we get a is normal. □

Theorem 2.12. *Let $a \in R^\dagger$. Then a is normal if and only if the following equation* (2) *has at least one solution in $comm(a) \cap comm(a^*)$.*

$$a^\dagger = a^*x(a^\dagger)^* \tag{2}$$

Proof. " $\Rightarrow$ " If $a \in R^\dagger$ and a is normal, then $aa^\dagger = a^\dagger a$, $a^\dagger a^* = a^*a^\dagger$ and $a^\dagger(a^\dagger)^* = (a^\dagger)^*a^\dagger$, gives $a^\dagger = a^\dagger aa^\dagger = a^*(a^\dagger)^*a^\dagger = a^*a^\dagger(a^\dagger)^*$. Thus $x = a^\dagger$ is a solution of the equation (2) and $a^\dagger \in comm(a) \cap comm(a^*)$.

" $\Leftarrow$ " If $x = c$ is a solution of the equation (2), which belongs to $comm(a) \cap comm(a^*)$, then $a^\dagger = a^*c(a^\dagger)^*$, $ca = ac$, $ca^* = a^*c$. Now we get $a = aa^\dagger a = a(a^*c(a^\dagger)^*)a = aca^*(a^\dagger)^*a = aca^\dagger aa = caa^\dagger aa = ca^2 = a^2c \in a^2R \cap Ra^2$. By Theorem 1.2, we have $a \in R^\#$. Then $aa^\dagger = a(a^*c(a^\dagger)^*) = aca^*(a^\dagger)^* = aca^\dagger a = caa^\dagger a = ca = caa^\#a = ca^2a^\# = aa^\#$, this means that $a \in R^{EP}$ and $aa^\dagger = a^\dagger a$. Note that

$$a^\dagger a^* = a^*c(a^\dagger)^*a^* = ca^*aa^\dagger = ca^*,$$

$$a^*a^\dagger = a^*a^*c(a^\dagger)^* = ca^*a^\dagger a = ca^*aa^\dagger = ca^*.$$

Combining these two equalities, we get $a^\dagger a^* = a^*a^\dagger$. Thus a is normal. □

It is know by [6, Theorem 2.3], if $a \in R^\dagger$, then $a^\dagger = a^*$ and a is normal if and only if $a \in R^\#$ and $aa^* = a^\dagger a$. The following theorem 2.13 shows that, in the conditions $a \in R^\#$ and $aa^* = a^\dagger a$, $a \in R^\#$ can be removed.

Theorem 2.13. *Let $a \in R^\dagger$. Then $a^\dagger = a^*$ and a is normal if and only if $aa^* = a^\dagger a$.*

Proof. " $\Rightarrow$ " *Assume that $a \in R^\dagger$ and a is normal, then $aa^\dagger = a^\dagger a$. Since $a^\dagger = a^*$, then $aa^* = a^\dagger a$.*

" $\Leftarrow$ " *Since $a \in R^\dagger$ and $aa^* = a^\dagger a$, $aR = aa^*R = a^\dagger aR = a^\dagger R$, that is $a \in R^{EP}$ and therefore $aa^\dagger = a^\dagger a$. Multiplying $aa^* = a^\dagger a$ by $a^\dagger$ from the left side, it follows $a^* = a^\dagger$, then $aa^* = a^*a$. So a is normal.* □

Note that if $a \in R^\# \cap R^\dagger$ and $a = a^2a^*$, multiplying this equality on the left by $a^\dagger$, it follows $a^\dagger a = a^\dagger a^2a^*$, by taking the involution, then $a^\dagger a = aa^*a^\dagger a$, thus, $a^\dagger R = a^\dagger aR = aa^*a^\dagger aR = aa^*a^*R = aa^*R = aR$ by Theorem 1.1, that is, $a \in R^{EP}$ and therefore $aa^\dagger = a^\dagger a$. It implies that $a^\dagger a = a^\dagger a^2a^* = aa^\dagger aa^* = aa^*$. Hence we can get the following lemma.

Lemma 2.14. *Let $a \in R^\dagger$. Then $a^\dagger = a^*$ and a is normal if and only if $a = a^2a^*$ and $a \in R^\#$.*

Theorem 2.15. *Let $a \in R^{\dagger}$. Then $a^{\dagger} = a^*$ and a is normal if and only if $a \in R^{\#}$ and the following equation* (3) *has at least one solution in χ_a.*

$$axa^* = xa^{\dagger}a \tag{3}$$

Proof. " $\Leftarrow$ " (1) $x = a$ is a solution of the equation (3), then $a^2a^* = aa^{\dagger}a = a$. By Lemma 2.14, a is normal and $a^{\dagger} = a^*$.

(2) If $x = a^{\dagger}$ is a solution of the equation (3), one has that $aa^{\dagger}a^* = a^{\dagger}a^{\dagger}a$. By taking the involution, we have $a^2a^{\dagger} = a^{\dagger}a(a^{\dagger})^*$. By Theorem 1.1, we get $aR = a^2R = a^2a^{\dagger}R = a^{\dagger}a(a^{\dagger})^*R = a^{\dagger}a^2R = a^{\dagger}R$. This shows that $aa^{\dagger} = a^{\dagger}a$. Then $a^* = a^{\dagger}aa^* = aa^{\dagger}a^* = a^{\dagger}a^{\dagger}a = a^{\dagger}aa^{\dagger} = a^{\dagger}$, which implies that $aa^* = a^*a$. We see that a is normal.

(3) If $x = a^*$ is a solution of the equation (3), then $aa^*a^* = a^*a^{\dagger}a$. Thus $aR = aa^*R = aa^*a^*R = a^*a^{\dagger}aR \subseteq a^*R = a^{\dagger}R$ by Theorem 1.1. It follows that $(1 - a^{\dagger}a)a \in (1 - a^{\dagger}a)aR \subseteq (1 - a^{\dagger}a)a^{\dagger}R = 0$, that is $(1 - a^{\dagger}a)a = 0$, then $a = a^{\dagger}a^2$. One concludes that $aa^{\#} = a^{\dagger}a^2a^{\#} = a^{\dagger}a$, which implies that $aa^{\dagger} = a^{\dagger}a$. Hence $aa^*a^* = a^*aa^{\dagger} = a^*$, multiplying the equality on the right by $(a^{\dagger})^*$, we get $aa^* = a^{\dagger}a$. So a is normal and $a^{\dagger} = a^*$ by Theorem 2.13.

(4) If $x = a^{\#}$ is a solution of the equation (3), then we have $a^{\#}a^{\dagger}a = aa^{\#}a^* = a^{\#}aa^*$. By taking the involution, $aa^*(a^{\#})^* = a^{\dagger}a(a^{\#})^*$, thus $aR = aa^*a^*R = aa^*(a^{\#})^*R = a^{\dagger}a(a^{\#})^*R = a^{\dagger}aa^*R = a^{\dagger}aR = a^{\dagger}R$ by Theorem 1.1, this means that $a^{\dagger} = a^{\#}$. According to the proof of (2), we get a is normal and $a^{\dagger} = a^*$.

(5) If $x = (a^{\dagger})^*$ is a solution of the equation (3), then we can show $a(a^{\dagger})^*a^* = (a^{\dagger})^*a^{\dagger}a$, that is, $a^2a^{\dagger} = (a^{\dagger})^*a^{\dagger}a$. Multiplying the equality on the right by $aa^{\#}a^{\dagger}$, we obtain $aa^{\dagger} = (a^{\dagger})^*a^{\dagger}$, then multiplying this equality by a^* from the left side, we have $a^* = a^{\dagger}$, hence $a = aa^{\dagger}a = (a^*)^*a^{\dagger}a = (a^{\dagger})^*a^{\dagger}a = a^2a^{\dagger} = a^2a^*$. It follows that a is normal by Lemma 2.14.

(6) If $x = (a^{\#})^*$ is a solution of the equation (3), then $a(a^{\#})^*a^* = (a^{\#})^*a^{\dagger}a$, gives $a(aa^{\#})^* = (a^{\#})^*a^{\dagger}a$. Multiplying this equality by a^* from the right side, it follows $a(a^2a^{\#})^* = (a^{\#})^*a^*$, that is, $aa^* = (a^{\#})^*a^*$. By Theorem 1.1, we have $aR = aa^*R = (a^{\#})^*a^*R \subseteq (a^{\#})^*R = a^*R = a^{\dagger}R$, this means that $a^{\dagger} = a^{\#}$. Thus a is normal and $a^{\dagger} = a^*$ by (5).

" $\Rightarrow$ " By Lemma 2.14, we know that $x = a$ is a solution of the equation (3). □

Theorem 2.16. *Let $a \in R^{\dagger} \cap R^{\#}$. Then a is normal if and only if the following equation* (4) *has at least one solution in χ_a.*

$$aa^*x = a^*ax \tag{4}$$

Proof. " $\Rightarrow$ " Since a is normal, then $aa^* = a^*a$, hence $aa^*a^{\dagger} = a^*aa^{\dagger}$. It turns out that $x = a^{\dagger}$ is a solution of the equation (4).

" $\Leftarrow$ " (1) If $x = a$ is a solution of the equation (4), then we have $aa^*a = a^*a^2$. By Theorem 1.1, we have $a^{\dagger}R = a^*aR = a^*a^2R = aa^*aR = aa^*R = aR$, which forces that $a \in R^{EP}$ and therefore $aa^{\dagger} = a^{\dagger}a$. Multiplying $aa^*a = a^*a^2$ on the right by $a^{\dagger}$, we get $aa^* = a^*a$. Thus a is normal.

(2) If $x = a^{\dagger}$ is a solution of the equation (4), then we know that $aa^*a^{\dagger} = a^*aa^{\dagger} = a^*$. By Theorem 1.1, we have $a^{\dagger}R = a^*R = aa^*a^{\dagger}R = a(a^*)^2R = aa^*R = aR$. This means that $aa^{\dagger} = a^{\dagger}a$, that is, $aa^* = aa^*aa^{\dagger} = aa^*a^{\dagger}a = a^*a$. Which implies that a is normal.

(3) If $x = a^*$ is a solution of the equation (4), then $aa^*a^* = a^*aa^*$. Multiplying this equality by $(a^{\dagger})^*$ from the right side, we have $aa^*a^{\dagger}a = a^*a$. So $a^{\dagger}R = a^*aR = aa^*a^{\dagger}aR = a(a^*)^2R = aa^*R = aR$ by Theorem 1.1, it follows that $a \in R^{EP}$ and therefore $aa^{\dagger} = a^{\dagger}a$. Which implying that $a^*a = aa^*a^{\dagger}a = aa^*aa^{\dagger} = aa^*$. Hence a is normal.

(4) If $x = a^{\#}$ is a solution of the equation (4), then we get that $aa^*a^{\#} = a^*aa^{\#}$. By Theorem 1.1, we know that $a^{\dagger}R = a^*a^2R = a^*aa^{\#}R = aa^*a^{\#}R = aa^*aR = aa^{\dagger}R = aR$. Hence $a^{\dagger} = a^{\#}$, which implies that a is normal by (2).

(5) If $x = (a^{\dagger})^*$ is a solution of the equation (4), then one has that $a^*a(a^{\dagger})^* = aa^*(a^{\dagger})^* = aa^{\dagger}a = a$. By Theorem 1.1, we get $aR = a^*a(a^{\dagger})^*R = a^*a^2R = a^*aR = a^{\dagger}R$, thus $a \in R^{EP}$ and therefore $aa^{\dagger} = a^{\dagger}a$. Then $aa^* = a^*a(a^{\dagger})^*a^* = a^*aaa^{\dagger} = a^*aa^{\dagger}a = a^*a$. Giving that a is normal.

(6) If $x = (a^{\#})^*$ is a solution of the equation (4), then we know that $aa^*(a^{\#})^* = a^*a(a^{\#})^*$. Multiplying the equality on the right by a^*, we have $aa^* = a^*a(a^{\#})^*a^*$. So, $aR = aa^*R = a^*a(a^{\#})^*a^*R \subseteq a^*R = a^{\dagger}R$ by Theorem 1.1, it follows that $a^{\dagger} = a^{\#}$. We see that a is normal by (5). □

Theorem 2.17. *Let $a \in R^{\dagger} \cap R^{\#}$. Then a is normal if and only if the following equation* (5) *has at least one solution in χ_a.*

$$aa^*a^{\dagger}x = a^*x \tag{5}$$

Proof. " $\Rightarrow$ " Assume that a is normal, then $aa^* = a^*a$, one obtains that $a^*a = a^*aa^{\dagger}a = aa^*a^{\dagger}a$. Thus $x = a$ is a solution of the equation (5).

" $\Leftarrow$ " (1) If $x = a^{\dagger}$ is a solution of the equation (5), then we have $aa^*a^{\dagger}a^{\dagger} = a^*a^{\dagger}$. Hence, by Proposition 2.1, a is normal.

(2) If $x = a$ is a solution of the equation (5), then we get $a^*a = aa^*a^{\dagger}a$, by Theorem 1.1, $a^{\dagger}R = a^*aR = aa^*a^{\dagger}aR = aa^*a^*R = aa^*R = aR$, it follows that $aa^{\dagger} = a^{\dagger}a$. This implies $a^*a = aa^*a^{\dagger}a = aa^*aa^{\dagger} = aa^*$. Thus a is normal.

(3) If $x = a^{\#}$ is a solution of the equation (5), then $aa^*a^{\dagger}a^{\#} = a^*a^{\#}$. By Theorem 1.1, we have $a^{\dagger}R = a^*aR = a^*a^{\#}R = aa^*a^{\dagger}a^{\#}R = aa^*a^{\dagger}aR = aa^*a^*R = aR$, one obtains that $a^{\#} = a^{\dagger}$. Hence, a is normal by (1).

(4) If $x = a^*$ is a solution of the equation (5), then we have $aa^*a^{\dagger}a^* = a^*a^*$. So $a^{\dagger}R = a^*R = a^*a^*R = aa^*a^{\dagger}a^*R \subseteq aR = a^{\#}a^2R \subseteq a^{\#}R$ by Theorem 1.1. It follows that $(1 - a^{\#}a)a^{\dagger} \in (1 - a^{\#}a)a^{\dagger}R \subseteq (1 - a^{\#}a)a^{\#}R = 0$, that is $(1 - a^{\#}a)a^{\dagger} = 0$, then $a^{\dagger} = a^{\#}aa^{\dagger}$. We obtain $a^{\dagger}a = a^{\#}aa^{\dagger}a = a^{\#}a$, which implies that $aa^{\dagger} = a^{\dagger}a$. Multiplying $aa^*a^{\dagger}a^* = a^*a^*$ on the right by $(a^{\dagger})^*$, we have $aa^*a^{\dagger} = a^*$. Hence, by [4, Theorem 2.2(x)], a is normal.

(5) If $x = (a^{\dagger})^*$ is a solution of the equation (5), then $aa^*a^{\dagger}(a^{\dagger})^* = a^*(a^{\dagger})^* = a^{\dagger}a$. By Theorem 1.1, we get $a^{\dagger}R = a^{\dagger}aR = aa^*a^{\dagger}(a^{\dagger})^*R = aa^*a^{\dagger}aR = aa^*a^*R = aR$, we know that $aa^{\dagger} = a^{\dagger}a$. This means $a^* = a^{\dagger}aa^* = aa^*a^{\dagger}(a^{\dagger})^*a^* = aa^*a^{\dagger}aa^{\dagger} = aa^*a^{\dagger}$. Hence, by [4, Theorem 2.2(x)], a is normal.

(6) If $x = (a^{\#})^*$ is a solution of the equation (5), then we get $aa^*a^{\dagger}(a^{\#})^* = a^*(a^{\#})^*$, by Theorem 1.1, $a^{\dagger}R = a^*R = a^*a^*R = a^*(a^{\#})^*R = aa^*a^{\dagger}(a^{\#})^*R \subseteq aR$, so we arrive at $a^{\#} = a^{\dagger}$. According to the proof of (5), a is normal. □

References

[1] R. Penrose, A generalized inverse for matrices, Proc. Cambridge Philos. Soc 51 (1955) 406–413.
[2] A. Ben-Israel, T. N. E. Greville, Generalized Inverses: Theory and Applications, (2nd edition), Springer, New York, 2003.
[3] R. E. Hartwig, Block generalized inverses, Arch. Rational Mech. Anal 61 (1976) 197-251.
[4] D. Mosić, D. S. Djordjević, Moore-Penrose invertible normal and Hermitian elements in rings, Linear Algebra Appl 431 (2009) 732-745.
[5] Y. Qu, J. Wei, H. Yao, Characterizations of normal elements in rings with involution, Acta. Math. Hungar 156(2) (2018) 459-464.
[6] D. Mosić, D. S. Djordjević, Partial isometries and EP elememts in rings with involution, Electronic J. Linear Algebra 18 (2009) 761-772.