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Symmetrized GA-convexity and Related Some Integral Inequalities

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Abstract. In this paper, we introduce a new the concept called the symmetrized GA-convex function and give Hermite-Hadamard's inequalities for symmetrized GA-convex functions. Furthermore, we establish Hermite-Hadamard type inequalites for the product of a GA-convex function with a symmetrized GA-convex function and also for two symmetrized GA-convex functions.

1. Introduction

Let real function f be defined on some nonempty interval I of real line $\mathbb{R}$. The function f is said to be convex on I if inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1}$$

holds for all $x, y \in I$ and $t \in [0,1]$.

The following definitions are well known in the literature.

Definition 1.1 ([9, 10]). *A function $f : I \subseteq (0,\infty) \to \mathbb{R}$ is said to be GA-convex (geometric-arithmatically convex) if*

$$f(x^t y^{1-t}) \leq tf(x) + (1-t)\, f(y) \tag{2}$$

for all $x, y \in I$ and $t \in [0,1]$.

Since the condition (2) can be written as

$$(f \circ \exp)(t \ln x + (1-t) \ln y) \leq t\,(f \circ \exp)(\ln x) + (1-t)\,(f \circ \exp)(\ln y),$$

then we observe that $f : I \subseteq (0,\infty) \to \mathbb{R}$ is GA-convex on I if and only if $f \circ \exp$ is convex on $lnI := \{lnz, z \in I\}$. If $I = [a,b]$ then $lnI = [lna, lnb]$.

In [8], Latif et al. established the following inequality which is the weighted generalization of Hermite-Hadamard inequality for GA-convex functions as follows:

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Theorem 1.2. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a GA-convex function and $a, b \in I$ with $a < b$. Let $g : [a,b] \to [0,\infty)$ be continuous positive mapping and geometrically symmetric with respect to $\sqrt{ab}$ (i.e. $g(x) = g\left(\frac{ab}{x}\right)$ for all $x \in [a,b]$). Then*

$$f\left(\sqrt{ab}\right)\int_a^b \frac{g(x)}{x}dx \le \int_a^b \frac{g(x)f(x)}{x}dx \le \frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x}dx.$$

We will now give definitions of the right-sided and left-sided Hadamard fractional integrals which are used throughout this paper.

Definition 1.3. *Let $f \in L[a,b]$. The left-sided and right-sided Hadamard fractional integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of oder $\alpha > 0$ with $b > a \ge 0$ are defined by*

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \left(\ln\frac{x}{t}\right)^{\alpha-1} f(t)\frac{dt}{t},\ a < x < b$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \left(\ln\frac{t}{x}\right)^{\alpha-1} f(t)\frac{dt}{t},\ a < x < b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}dt$ (see [7]).

Hermite-Hadamard's inequalities can be represented for GA-convex functions in fractional integral forms as follows.

Theorem 1.4 ([5]). *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a function such that $f \in L[a,b]$, where $a, b \in I$ with $a < b$. If f is a GA-convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\sqrt{ab}\right) \le \frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left\{J^{\alpha}_{a+}f(b) + J^{\alpha}_{b-}f(a)\right\} \le \frac{f(a)+f(b)}{2} \tag{3}$$

with $\alpha > 0$.

In [6], the authors presented Hermite–Hadamard-Fejer inequalities for GA-convex functions in fractional integral forms as follows:

Theorem 1.5. *Let $f : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ be a GA-convex function on $[a,b]$ with $a < b$ and $f \in L[a,b]$. If $g : [a,b] \to \mathbb{R}$ is nonnegative, integrable and geometrically symmetric with respect to $\sqrt{ab}$, then the following inequalities for fractional integrals holds:*

$$f\left(\sqrt{ab}\right)\left[J^{\alpha}_{a+}g(b) + J^{\alpha}_{b-}g(a)\right] \le \left[J^{\alpha}_{a+}(fg)(b) + J^{\alpha}_{b-}(fg)(a)\right] \le \frac{f(a)+f(b)}{2}\left[J^{\alpha}_{a+}g(b) + J^{\alpha}_{b-}g(a)\right] \tag{4}$$

with $\alpha > 0$.

For a function $f : [a,b] \to \mathbb{R}$ we consider the symmetrical transform of f on the interval , denoted by $\breve{f}_{[a,b]}$ or simply $\breve{f}$, when the interval $[a,b]$ is implicit, which

is defined by

$$\breve{f}(x) := \frac{1}{2}\left[f(x) + f\left(a+b-x\right)\right],\ x \in [a,b].$$

The anti symmetrical transform of f on the interval $[a,b]$ is denoted by $\tilde{f}_{[a,b]}$or simply $\tilde{f}$ as defined by

$$\tilde{f}(x) := \frac{1}{2}\left[f(x) - f\left(a+b-x\right)\right],\ x \in [a,b].$$

It is obvious that for any function f we have $\breve{f} + \tilde{f} = f$.

If f is convex on $[a,b]$, then $\breve{f}$ is also convex on $[a,b]$. But, when $\breve{f}$ is onvex on $[a,b]$, f may not be convex on $[a,b]$ ([1]).

In [1], Dragomir introduced symmetrized convexity concept as follow:

Definition 1.6. *A function $f : [a,b] \to \mathbb{R}$ is said to be symmetrized convex (concave) on $[a,b]$ if symmetrical transform $\breve{f}$ is convex (concave) on $[a,b]$.*

Dragomir extended the Hermite-Hadamard inequality to the class of symmetrized convex functions as follow:

Theorem 1.7 ([1]). *Assume that $f : [a,b] \to \mathbb{R}$ is symmetrized convex on the interval $[a,b]$, then we have the Hermite-Hadamard inequalities*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{5}$$

Theorem 1.8 ([1]). *Assume that $f : [a,b] \to \mathbb{R}$ is symmetrized convex on the interval $[a,b]$. Then for any $x \in [a,b]$ we have the bounds*

$$f\left(\frac{a+b}{2}\right) \leq \breve{f}(x) = \frac{1}{2}\left[f(x) + f(a+b-x)\right] \leq \frac{f(a)+f(b)}{2} \tag{6}$$

Corollary 1.9 ([1]). *If $f : [a,b] \to \mathbb{R}$ is symmetrized convex on the interval $[a,b]$ and $g : [a,b] \to [0,\infty)$ is integrable on $[a,b]$, then*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leq \int_a^b g(x)\breve{f}(x)\,dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx.$$

Moreover, if g is symmetric with respect to $\frac{a+b}{2}$ on $[a,b]$, i.e. $g(x) = g(a+b-x)$ for almost every $x \in [a,b]$, then

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leq \int_a^b g(x)f(x)\,dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx.$$

Theorem 1.10 ([1]). *Assume that $f : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ is symmetrized convex on the interval $[a,b]$. Then for any $x, y \in [a,b]$ with $x \neq y$ we have the Hermite-Hadamard inequalities*

$$\begin{aligned}\frac{1}{2}\left[f\left(\frac{x+y}{2}\right) + f\left(a+b-\frac{x+y}{2}\right)\right] &\leq \frac{1}{2(y-x)}\left[\int_x^y f(t)dt + \int_{a+b-y}^{a+b-x} f(t)dt\right] \\ &\leq \frac{1}{4}\left[f(x)+f(y)+f\left(a+b-x\right)+f\left(a+b-y\right)\right].\end{aligned} \tag{7}$$

In [2], Dragomir establish some Hermite-Hadamard type inequalities for the product of a convex function with a symmetrized convex function and also for two convex functions. In [3], the author also we obtain some Hermite-Hadamard type inequalities for product of symmetrized convex functions.

Motivated by the above results, in this paper we introduces the concept of the symmetrized GA-convex function and establish some Hermite-Hadamard type inequalities for product of symmetrized GA-convex functions. Some examples of interest are provided as well. Furthermore, we establish Hermite-Hadamard type inequalites for the product of a GA-convex function with a symmetrized GA-convex function and also for two symmetrized GA-convex functions.

2. Symmetrized GA-Convexity

For a function $f : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ we consider the geometric symmetrical transform of f on the interval , denoted by $\breve{G}_{f,[a,b]}$ or simply $\breve{G}_f$, when the interval $[a,b]$ is implicit, which

is defined by

$$\breve{G}_f(x) := \frac{1}{2}\left[f(x) + f\left(\frac{ab}{x}\right)\right],\ x \in [a,b].$$

The anti-geometric symmetrical transform of f on the interval $[a,b]$ is denoted by $\hat{G}_{f,[a,b]}$ or simply $\hat{G}_f$ as defined by

$$\hat{G}_f(x) := \frac{1}{2}\left[f(x) - f\left(\frac{ab}{x}\right)\right],\ x \in [a,b].$$

It is obvious that for any function f we have $\breve{G}_f + \hat{G}_f = f$.

If f is GA-convex on $[a,b]$, then $\breve{G}_f$ is also GA-convex on $[a,b]$. Indeed, for any $x,y \in [a,b]$ and $t \in [0,1]$ we have

$$\begin{aligned}
\breve{G}_f(x^t y^{1-t}) &= \frac{1}{2}\left[f(x^t y^{1-t}) + f\left(\frac{ab}{x^t y^{1-t}}\right)\right] \\
&= \frac{1}{2}\left[f(x^t y^{1-t}) + f\left(\left(\frac{ab}{x}\right)^t \left(\frac{ab}{y}\right)^{1-t}\right)\right] \\
&\le t\frac{1}{2}\left[f(x) + f\left(\frac{ab}{x}\right)\right] + (1-t)\frac{1}{2}\left[f(y) + f\left(\frac{ab}{y}\right)\right] \\
&= t\breve{G}_f(x) + (1-t)\breve{G}_f(y).
\end{aligned}$$

Remark 2.1. *Consider the real numbers $a < 0 < b$ with $(a+b)/2 > 0$ and define the function $f_* : \left[e^a, e^b\right] \to \mathbb{R}$, $f_*(x) = \ln^3 x$. Since the function $(f_* \circ \exp)(x) = x^3$ is not convex on $[a,b]$, f_* is not GA-convex on $\left[e^a, e^b\right]$. On the other hand, since the function $\left(\breve{G}_{f_*} \circ \exp\right)(x) = \frac{1}{2}\left[x^3 + (a+b-x)^3\right]$ is convex on $[a,b]$, $\breve{G}_{f_*}(x) = \frac{1}{2}\left[\ln^3 x + (a+b-\ln x)^3\right]$ is GA-convex on $\left[e^a, e^b\right]$.*

Definition 2.2. *A function $f : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ is said to be symmetrized GA-convex (GA-concave)on $[a,b]$ if geometric symmetrical transform $\breve{G}_f$ is GA-convex (GA-concave) on $[a,b]$.*

Example 2.3. *Let $a,b \in \mathbb{R}$ with $0 < a < b$ and $\alpha \ge 2$. Then the function $f : [a,b] \to \mathbb{R}$, $f(x) = \left(\ln \frac{x}{a}\right)^{\alpha-1}$ is GA-convex on $[a,b]$. Indeed, for any $u,v \in [a,b]$ and $t \in [0,1]$ by convexity of the function $g(\varsigma) = \varsigma^{\alpha-1}, \zeta \ge 0$, we*

have

$$\begin{aligned} f(u^t v^{1-t}) &= \left(\ln \frac{u^t v^{1-t}}{a}\right)^{\alpha-1} \\ &= \left(t \ln \frac{u}{a} + (1-t) \ln \frac{v}{a}\right)^{\alpha-1} \\ &\leq t\left(\ln \frac{u}{a}\right)^{\alpha-1} + (1-t)\left(\ln \frac{v}{a}\right)^{\alpha-1} \\ &= tf(u) + (1-t)f(v). \end{aligned}$$

Thus $\breve{G}_f$ is also GA-convex on $[a,b]$. Therefore f is symmetrized GA-convex function.

Example 2.4. *Let $\alpha \geq 2$. Then the function $f : [a,b] \subseteq (0,\infty) \to \mathbb{R}$, $f(x) = \left(\ln \frac{b}{x}\right)^{\alpha-1}$, is GA-convex on $[a,b]$ .Therefore f is symmetrized GA-convex function.*

Example 2.5. *Let $\alpha \geq 2$. Then the function $f : [a,b] \subseteq (0,\infty) \to \mathbb{R}$, $f(x) = \left(\ln \frac{x}{a}\right)^{\alpha-1} + \left(\ln \frac{b}{x}\right)^{\alpha-1}$, is symmetrized GA-convex function.*

Now if $GAC[a,b]$ is the class of GA-convex functions defined on I and $SGAC[a,b]$ is the class of symmetrized GA-convex functions on $[a,b]$ then

$$GAC[a,b] \subsetneq SGAC[a,b].$$

Also, if $[c,d] \subset [a,b]$ and $f \in SGAC[a,b]$, then this does not imply in general that $f \in SGAC[c,d]$.

Proposition 2.6. *Let $f : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ be a function. f is symmetrized GA-convex on the interval $[a,b]$ if and only if $f \circ \exp$ is symmetrized convex on the interval $[\ln a, \ln b]$.*

Proof. Let f be a symmetrized GA-convex function on the interval $[a,b]$. If we take arbitrary $x, y \in [\ln a, \ln b]$, then there exist $u, v \in [a,b]$ such that $x = \ln u$ and $y = \ln v$

$$\begin{aligned} & (f \breve{\circ} \exp)(tx + (1-t)y) \qquad (8) \\ &= \frac{1}{2}\left[(f \circ \exp)(tx + (1-t)y) + (f \circ \exp)(\ln a + \ln b - tx - (1-t)y)\right] \\ &= \frac{1}{2}\left[(f \circ \exp)(tx + (1-t)y) + (f \circ \exp)(t\left[\ln(ab) - x\right] + (1-t)\left[\ln(ab) - y\right])\right] \\ &= \frac{1}{2}\left[(f \circ \exp)(\ln u^t v^{1-t}) + (f \circ \exp)\left(\ln\left(\frac{ab}{u}\right)^t \left(\frac{ab}{v}\right)^{1-t}\right)\right] \\ &= \frac{1}{2}\left[f(u^t v^{1-t}) + f\left(\frac{ab}{u^t v^{1-t}}\right)\right] \\ &= \breve{G}_f(u^t v^{1-t}). \end{aligned}$$

Since f is a symmetrized GA-convex function on the interval $[a,b]$, we have

$$\begin{aligned} \breve{G}_f(u^t v^{1-t}) &\leq t\breve{G}_f(u) + (1-t)\breve{G}_f(v) \qquad (9) \\ &= t\frac{1}{2}\left[f(u) + f\left(\frac{ab}{u}\right)\right] + (1-t)\frac{1}{2}\left[f(v) + f\left(\frac{ab}{v}\right)\right] \\ &= t\breve{f}(u) + (1-t)\breve{f}(v) = t\breve{f}(e^x) + (1-t)\breve{f}(e^y) \\ &= t\,(f \breve{\circ} \exp)(x) + (1-t)(f \breve{\circ} \exp)(y) \end{aligned}$$

By (8) and (9), we obtain that $f \circ \exp$ is symmetrized convex on the interval $[\ln a, \ln b]$.

Conversely, if $f \circ \exp$ is symmetrized convex on the interval $[\ln a, \ln b]$ then it is easily seen that f is symmetrized GA-convex on the interval $[a,b]$ by a similar procedure. The details are omitted. □

Theorem 2.7. *If $f : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ is symmetrized GA-convex on the interval $[a,b]$, then we have the Hermite-Hadamard inequalities*

$$f\left(\sqrt{ab}\right) \leq \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx \leq \frac{f(a)+f(b)}{2}. \tag{10}$$

Proof. Since $f : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ is symmetrized GA-convex on the interval $[a,b]$, then by writing the Hermite-Hadamard inequality for the function $\breve{G}_f$ we have

$$\breve{G}_f\left(\sqrt{ab}\right) \leq \frac{1}{\ln b - \ln a}\int_a^b \frac{\breve{G}_f(x)}{x}dx \leq \frac{\breve{G}_f(a)+\breve{G}_f(b)}{2}, \tag{11}$$

where, it is easily seen that

$$\breve{G}_f\left(\sqrt{ab}\right) = f\left(\sqrt{ab}\right),\ \frac{\breve{G}_f(a)+\breve{G}_f(b)}{2} = \frac{f(a)+f(b)}{2},$$

and

$$\frac{1}{\ln b - \ln a}\int_a^b \frac{\breve{G}_f(x)}{x}dx = \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx$$

Then by (11) we get required inequalities. □

Remark 2.8. *By helping Theorem 1.7 and Proposition 2.6, the proof of Theorem 2.7 can also be given as follows :*

Since $f : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ is symmetrized GA-convex on the interval $[a,b]$, $f \circ \exp$ is symmetrized convex on the interval $[\ln a, \ln b]$. So, by Theorem 1.7 we have

$$(f\circ\exp)\left(\frac{\ln a + \ln b}{2}\right) \leq \frac{1}{\ln b - \ln a}\int_{\ln a}^{\ln b}(f\circ\exp)(x)dx \leq \frac{(f\circ\exp)(\ln a)+(f\circ\exp)(\ln b)}{2},$$

i.e.

$$f\left(\sqrt{ab}\right) \leq \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx \leq \frac{f(a)+f(b)}{2}.$$

Theorem 2.9. *If $f : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ is symmetrized GA-convex on the interval $[a,b]$. Then for any $x \in [a,b]$ we have the bounds*

$$f\left(\sqrt{ab}\right) \leq \breve{G}_f(x) \leq \frac{f(a)+f(b)}{2}. \tag{12}$$

Proof. Since$\breve{G}_f$ is GA-convex on $[a,b]$ then for any $x \in [a,b]$ we have

$$f\left(\sqrt{ab}\right) = \breve{G}_f\left(\sqrt{ab}\right) \leq \frac{\breve{G}_f(x)+\breve{G}_f(\frac{ab}{x})}{2} = \breve{G}_f(x).$$

This give us the first inequality in (12).

Also, for any $x \in [a,b]$ there exist a number $t_0 \in [0,1]$ such that $x = a^{t_0}b^{1-t_0}$. By the GA-convexity of $\breve{G}_f$ we have

$$\begin{aligned}\breve{G}_f(x) &\leq t_0\breve{G}_f(a) + (1-t_0)\breve{G}_f(b)\\ &= \breve{G}_f(a) = \frac{f(a)+f(b)}{2}\end{aligned}$$

which gives the second inequality in (12). □

Remark 2.10. *By helping Theorem 1.8 and Proposition 2.6, the proof of Theorem 2.9 can also be given as follows :*

Since $f : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ is symmetrized GA-convex on the interval $[a,b]$, $f \circ \exp$ is symmetrized convex on the interval $[\ln a, \ln b]$. So, by Theorem 1.8 we have

$$(f \circ \exp)\left(\frac{\ln a + \ln b}{2}\right) \leq (f \circ \breve{\exp})(\ln x) \leq \frac{(f \circ \exp)(\ln a) + (f \circ \exp)(\ln b)}{2},$$

i.e.

$$f\left(\sqrt{ab}\right) \leq \breve{G}_f(x) \leq \frac{f(a) + f(b)}{2}$$

for any $x \in [a,b]$.

Remark 2.11. *If $f : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ is symmetrized GA-convex on the interval $[a,b]$, then we have the bounds*

$$\inf_{x\in[a,b]} \breve{G}_f(x) = f\left(\sqrt{ab}\right)$$

and

$$\sup_{x\in[a,b]} \breve{G}_f(x) = \frac{f(a) + f(b)}{2}.$$

Corollary 2.12. *If $f : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ is symmetrized GA-convex on the interval $[a,b]$ and $g : [a,b] \to [0,\infty)$ is integrable on $[a,b]$, then*

$$f\left(\sqrt{ab}\right)\int_a^b \frac{g(x)}{x}dx \leq \int_a^b \frac{g(x)\breve{G}_f(x)}{x}dx \leq \frac{f(a) + f(b)}{2}\int_a^b \frac{g(x)}{x}dx. \tag{13}$$

Moreover, if g is geometrically symmetric with respect to $\sqrt{ab}$ on $[a,b]$, i.e. $g(x) = g(ab/x)$ for all every $x \in [a,b]$, then

$$f\left(\sqrt{ab}\right)\int_a^b \frac{g(x)}{x}dx \leq \int_a^b \frac{g(x)f(x)}{x}dx \leq \frac{f(a) + f(b)}{2}\int_a^b \frac{g(x)}{x}dx. \tag{14}$$

Proof. The inequality (13) follows by (12) multiplying by $g(x)/x \geq 0$ and integrating over x on $[a,b]$.

By changing the variable, we have

$$\int_a^b \frac{g(x)f\left(\frac{ab}{x}\right)}{x}dx = \int_a^b \frac{g(\frac{ab}{x})f(x)}{x}dx.$$

Since g is geometrically symmetric with respect to $\sqrt{ab}$, then

$$\int_a^b \frac{g(\frac{ab}{x})f(x)}{x}dx = \int_a^b \frac{g(x)f(x)}{x}dx.$$

Thus

$$\begin{aligned}\int_a^b \frac{g(x)\breve{G}_f(x)}{x}dx &= \frac{1}{2}\left[\int_a^b \frac{g(x)f(x)}{x}dx + \int_a^b \frac{g(x)f\left(\frac{ab}{x}\right)}{x}dx\right] \\ &= \int_a^b \frac{g(x)f(x)}{x}dx\end{aligned}$$

and by (13) we get (14). □

Remark 2.13. *The inequality (14) is known as weighted generalization of Hermite-Hadamard inequality for GA-convex functions. It has been shown now that this inequality remains valid for the larger class of symmetrized GA-convex functions f on the interval $[a,b]$.*

Remark 2.14. *We note that by helping Corollary 1.9 and Proposition 2.6, the proof of Corollary 2.12 can also be given. The details is omitted.*

Remark 2.15. *Let $a,b,\alpha \in \mathbb{R}$ with $0 < a < b$ and $\alpha \geq 2$. Then the function $f : [a,b] \to \mathbb{R}$, $f(x) = \left(\ln \frac{x}{a}\right)^{\alpha-1}$ is symmetrized GA-convex on $[a,b]$, then*

i.) If we consider the function

$$f(x) = \left(\ln \frac{x}{a}\right)^{\alpha-1}$$

which is symmetrized GA-convex on $[a,b]$ in the inequality (13), then we have

$$\frac{1}{2^{\alpha-1}} \int_a^b \frac{g(x)}{x} dx \leq \frac{\Gamma(\alpha+1)}{2\left(\ln \frac{b}{a}\right)^{\alpha-1}} \left[J_{a+}^{\alpha} g(b) + J_{b-}^{\alpha} g(a)\right] \leq \frac{1}{2} \int_a^b \frac{g(x)}{x} dx$$

for any $g : [a,b] \to [0,\infty)$ is integrable on $[a,b]$

ii.) If we consider the function

$$g(x) = \left(\ln \frac{x}{a}\right)^{\alpha-1} + \left(\ln \frac{b}{x}\right)^{\alpha-1}$$

which is geometrically symmetric with respect to $\sqrt{ab}$ in the inequality (14), then we have the following inequalities

$$f\left(\sqrt{ab}\right) \leq \frac{\Gamma(\alpha+1)}{2\left(\ln \frac{b}{a}\right)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \leq \frac{f(a)+f(b)}{2}$$

which are the same of inequalities in (3).

iii.) Let w be geometrically symmetric with respect to $\sqrt{ab}$. If we consider the function

$$g(x) = \left[\left(\ln \frac{x}{a}\right)^{\alpha-1} + \left(\ln \frac{b}{x}\right)^{\alpha-1}\right] w(x)$$

which is geometrically symmetric with respect to $\sqrt{ab}$ in the inequality (14), then we have the following inequalities

$$\begin{aligned} f\left(\sqrt{ab}\right)\left[J_{a+}^{\alpha} w(b) + J_{b-}^{\alpha} w(a)\right] &\leq \left[J_{a+}^{\alpha} f w(b) + J_{b-}^{\alpha} f w(a)\right] \\ &\leq \frac{f(a)+f(b)}{2}\left[J_{a+}^{\alpha} w(b) + J_{b-}^{\alpha} w(a)\right] \end{aligned}$$

which are the same of inequalities in (4).

Theorem 2.16. *Assume that $f : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ is symmetrized GA-convex on the interval $[a,b]$. Then for any $x,y \in [a,b]$ with $x \neq y$ we have the Hermite-Hadamard inequalities*

$$\begin{aligned} \frac{1}{2}\left[f(\sqrt{xy}) + f\left(\frac{ab}{\sqrt{xy}}\right)\right] &\leq \frac{1}{2\ln(y/x)}\left[\int_x^y \frac{f(t)}{t} dt + \int_{ab/y}^{ab/x} \frac{f(t)}{t} dt\right] \\ &\leq \frac{1}{4}\left[f(x) + f(y) + f\left(\frac{ab}{x}\right) + f\left(\frac{ab}{y}\right)\right]. \end{aligned} \tag{15}$$

Proof. Since $\breve{G}_{f,[a,b]}$ is GA-convex on $[a,b]$, then $\breve{G}_{f,[a,b]}$ is also GA-convex on any subinterval $[x,y]$ (or $[y,x]$) where $x,y\in[a,b]$.

By Hermite-Hadamard inequalities for GA-convex functions we have

$$\breve{G}_{f,[a,b]}\left(\sqrt{xy}\right)\leq\frac{1}{\ln y-\ln x}\int_x^y\frac{\breve{G}_{f,[a,b]}(t)}{t}dt\leq\frac{\breve{G}_{f,[a,b]}(x)+\breve{G}_{f,[a,b]}(y)}{2} \tag{16}$$

for any $x,y\in[a,b]$ with $x\neq y$.

By definition of $\breve{G}_f$, we have

$$\breve{G}_{f,[a,b]}\left(\sqrt{xy}\right)=\frac{1}{2}\left[f(\sqrt{xy})+f\left(\frac{ab}{\sqrt{xy}}\right)\right],$$

$$\begin{aligned}\int_x^y\frac{\breve{G}_{f,[a,b]}(t)}{t}dt &= \frac{1}{2}\int_x^y\frac{1}{t}\left[f(t)+f\left(\frac{ab}{t}\right)\right]dt\\ &= \frac{1}{2}\int_x^y\frac{f(t)}{t}dt+\frac{1}{2}\int_x^y\frac{f\left(\frac{ab}{t}\right)}{t}dt\\ &= \frac{1}{2}\int_x^y\frac{f(t)}{t}dt+\frac{1}{2}\int_{ab/y}^{ab/x}\frac{f(t)}{t}dt\end{aligned}$$

and

$$\frac{\breve{G}_{f,[a,b]}(x)+\breve{G}_{f,[a,b]}(y)}{2}=\frac{1}{4}\left[f(x)+f(y)+f\left(\frac{ab}{x}\right)+f\left(\frac{ab}{y}\right)\right].$$

Thus by (16) we obtain the desired result (15). □

Remark 2.17. *We note that by helping Theorem 1.10 and Proposition 2.6, the proof of Theorem 2.16 can also be given. The details is omitted.*

Remark 2.18. *If we take $x=a$ and $y=b$ in (15), then we get (10). If, for a given $x\in[a,b]$, we take $y=ab/x$, then from (15) we get*

$$f\left(\sqrt{ab}\right)\leq\frac{1}{2\ln\left(\sqrt{ab}/x\right)}\int_x^{ab/x}\frac{f(t)}{t}dt\leq\frac{1}{2}\left[f(x)+f(\frac{ab}{x})\right], \tag{17}$$

where $x\neq\sqrt{ab}$, provided that $f:[a,b]\subseteq(0,\infty)\to\mathbb{R}$ is symmetrized GA-convex on the interval $[a,b]$.

Multiplying the inequalities (17) by $1/x$, then integrating the resulting inequality over x we get the following refinement of the first part of (10)

$$f\left(\sqrt{ab}\right)\leq\frac{1}{2\ln b/a}\int_a^b\left[\frac{1}{x\ln\left(\sqrt{ab}/x\right)}\int_x^{ab/x}\frac{f(t)}{t}dt\right]dx\leq\frac{1}{\ln b/a}\int_a^b\frac{f(x)}{x}dx,$$

provided that $f:[a,b]\subseteq(0,\infty)\to\mathbb{R}$ is symmetrized GA-convex on the interval $[a,b]$.

When the function is GA-convex, we have the following inequalities as well:

Remark 2.19. *If $f:[a,b]\subseteq(0,\infty)\to\mathbb{R}$ is GA-convex, then from (15) we have the inequalities*

$$\begin{aligned} f(\sqrt{ab}) &\leq \frac{1}{2}\left[f(\sqrt{xy})+f\left(\frac{ab}{\sqrt{xy}}\right)\right] \\ &\leq \frac{1}{2\ln(y/x)}\left[\int_x^y \frac{f(t)}{t}dt+\int_{ab/y}^{ab/x}\frac{f(t)}{t}dt\right] \\ &\leq \frac{1}{4}\left[f(x)+f(y)+f\left(\frac{ab}{x}\right)+f\left(\frac{ab}{y}\right)\right], \end{aligned} \tag{18}$$

for any $x,y\in[a,b]$ with $x\neq y$.

If we multiply the inequalities(18) by $1/(xy)$ and integrate (18) over (x,y) on the square $[a,b]^2$ and divide by $(\ln b/a)^2$, then we get the following refinement of the first Hermite-Hadamard inequality for convex functions

$$\begin{aligned} f(\sqrt{ab}) &\leq \frac{1}{2(\ln b/a)^2}\left[\int_a^b\int_a^b \frac{f(\sqrt{xy})}{xy}dxdy+\int_a^b\int_a^b\frac{1}{xy}f\left(\frac{ab}{\sqrt{xy}}\right)dxdy\right] \\ &\leq \frac{1}{2(\ln b/a)^2}\int_a^b\int_a^b\frac{1}{xy\ln(y/x)}\left[\int_x^y\frac{f(t)}{t}dt+\int_{ab/y}^{ab/x}\frac{f(t)}{t}dt\right]dxdy \\ &\leq \frac{1}{\ln b/a}\int_a^b\frac{f(x)}{x}dx. \end{aligned}$$

3. The Case of One GA-convex and the other Symmetrized GA-convex functions

In this section, we analyze the case in which one function is GA-convex (concave) in the classical sense and the other is symmetrized GA-convex (concave) on an interval $[a,b]$.

Theorem 3.1. *Assume that $g:[a,b]\subseteq(0,\infty)\to\mathbb{R}$ is GA-convex (concave) and $f:[a,b]\subseteq(0,\infty)\to\mathbb{R}$ is symmetrized GA-convex (concave) and integrable on the interval $[a,b]$. Then we have*

$$\begin{aligned} &\frac{1}{\ln b/a}\int_a^b\frac{\breve{G}_f(x)\,g(x)}{x}dx \\ &\geq \frac{f(a)+f(b)}{2}\frac{1}{\ln b/a}\int_a^b\frac{g(x)}{x}dx+\frac{g(a)+g(b)}{2}\frac{1}{\ln b/a}\int_a^b\frac{f(x)}{x}dx-\frac{f(a)+f(b)}{2}\frac{g(a)+g(b)}{2}, \end{aligned} \tag{19}$$

and

$$\begin{aligned} &\frac{1}{\ln b/a}\int_a^b\frac{\breve{G}_f(x)\,g(x)}{x}dx \\ &\leq \frac{g(a)+g(b)}{2}\frac{1}{\ln b/a}\int_a^b\frac{f(x)}{x}dx+f\left(\sqrt{ab}\right)\frac{1}{\ln b/a}\int_a^b\frac{g(x)}{x}dx-f\left(\sqrt{ab}\right)\frac{g(a)+g(b)}{2}. \end{aligned} \tag{20}$$

Proof. Assume that $g:[a,b]\to\mathbb{R}$ is GA-convex and $f:[a,b]\to\mathbb{R}$ is symmetrized GA-convex on $[a,b]$, then for any $t\in[0,1]$

$$g(a^tb^{1-t})\leq tg(a)+(1-t)g(b) \tag{21}$$

and by (12) we have

$$f\left(\sqrt{ab}\right)\leq\breve{G}_f\left(a^tb^{1-t}\right)\leq\frac{f(a)+f(b)}{2}. \tag{22}$$

By (21) and (22), we get

$$\begin{aligned} 0 &\leq \left[tg(a)+(1-t)g(b)-g(a^tb^{1-t})\right]\left[\frac{f(a)+f(b)}{2}-\breve{G}_f\left(a^tb^{1-t}\right)\right] \\ &= \left[tg(a)+(1-t)g(b)\right]\frac{f(a)+f(b)}{2}-g(a^tb^{1-t})\frac{f(a)+f(b)}{2} \\ &\quad -\left[tg(a)+(1-t)g(b)\right]\breve{G}_f\left(a^tb^{1-t}\right)+\breve{G}_f\left(a^tb^{1-t}\right)g(a^tb^{1-t}). \end{aligned}$$

That is equivalent to

$$\begin{aligned} &\left[tg(a)+(1-t)g(b)\right]\frac{f(a)+f(b)}{2}+\breve{G}_f\left(a^tb^{1-t}\right)g(a^tb^{1-t}) \\ \geq\ & g(a^tb^{1-t})\frac{f(a)+f(b)}{2}+\left[tg(a)+(1-t)g(b)\right]\breve{G}_f\left(a^tb^{1-t}\right). \end{aligned}$$

Integrating over t on $[0,1]$, we get

$$\begin{aligned} &\frac{f(a)+f(b)}{2}\int_0^1\left[tg(a)+(1-t)g(b)\right]dt+\int_0^1\breve{G}_f\left(a^tb^{1-t}\right)g(a^tb^{1-t})dt \\ \geq\ &\frac{f(a)+f(b)}{2}\int_0^1 g(a^tb^{1-t})dt+\int_0^1\left[tg(a)+(1-t)g(b)\right]\breve{G}_f\left(a^tb^{1-t}\right)dt. \end{aligned} \tag{23}$$

Observe that

$$\begin{aligned} \int_0^1\left[tg(a)+(1-t)g(b)\right]dt &= \frac{g(a)+g(b)}{2}, \\ \int_0^1 g(a^tb^{1-t})dt &= \frac{1}{\ln b-\ln a}\int_a^b\frac{g(x)}{x}dx, \\ \int_0^1\breve{G}_f\left(a^tb^{1-t}\right)g(a^tb^{1-t})dt &= \frac{1}{\ln b-\ln a}\int_a^b\frac{\breve{G}_f(x)\,g(x)}{x}dx, \end{aligned}$$

and

$$\begin{aligned} &\int_0^1\left[tg(a)+(1-t)g(b)\right]\breve{G}_f\left(a^tb^{1-t}\right)dt \\ =\ & g(a)\int_0^1 t\breve{G}_f\left(a^tb^{1-t}\right)dt+g(b)\int_0^1(1-t)\breve{G}_f\left(a^tb^{1-t}\right)dt. \end{aligned} \tag{24}$$

Since $\breve{G}_f$ is geometrically symmetric with respect to $\sqrt{ab}$, we have

$$\int_0^1(1-t)\breve{G}_f\left(a^tb^{1-t}\right)dt=\int_0^1 t\breve{G}_f\left(a^tb^{1-t}\right)dt,$$

thus by (24), we get

$$\int_0^1\left[tg(a)+(1-t)g(b)\right]\breve{G}_f\left(a^tb^{1-t}\right)dt=\left[g(a)+g(b)\right]\int_0^1 t\breve{G}_f\left(a^tb^{1-t}\right)dt.$$

Further,

$$\begin{aligned}\int_0^1 t\breve{G}_f\left(a^tb^{1-t}\right)dt &= \frac{1}{2}\left[\int_0^1 tf(a^tb^{1-t})dt+\int_0^1 tf(a^{1-t}b^t)dt\right]\\ &= \frac{1}{2}\left[\int_0^1 tf(a^tb^{1-t})dt+\int_0^1 (1-t)f(a^tb^{1-t})dt\right]\\ &= \frac{1}{2}\int_0^1 f(a^tb^{1-t})dt=\frac{1}{2(\ln b-\ln a)}\int_a^b \frac{f(x)}{x}dx.\end{aligned}$$

By the inequality (23), we get

$$\begin{aligned}&\frac{f(a)+f(b)}{2}\frac{g(a)+g(b)}{2}+\frac{1}{\ln b-\ln a}\int_a^b \frac{\breve{G}_f(x)\,g(x)}{x}dx\\ \geq\ &\frac{f(a)+f(b)}{2}\frac{1}{\ln b-\ln a}\int_a^b \frac{g(x)}{x}dx+\frac{g(a)+g(b)}{2}\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx\end{aligned}$$

and the inequality (19) is proved.

By (21) and (22), we also have

$$\begin{aligned}0 &\leq \left[tg(a)+(1-t)g(b)-g(a^tb^{1-t})\right]\left[\breve{G}_f\left(a^tb^{1-t}\right)-f\left(\sqrt{ab}\right)\right]\\ &= [tg(a)+(1-t)g(b)]\,\breve{G}_f\left(a^tb^{1-t}\right)+g(a^tb^{1-t})f\left(\sqrt{ab}\right)\\ &\quad -[tg(a)+(1-t)g(b)]\,f\left(\sqrt{ab}\right)-g(a^tb^{1-t})\breve{G}_f\left(a^tb^{1-t}\right)\end{aligned}$$

for any $t\in[0,1]$,which is equivalent to

$$\begin{aligned}&[tg(a)+(1-t)g(b)]\,f\left(\sqrt{ab}\right)+g(a^tb^{1-t})\breve{G}_f\left(a^tb^{1-t}\right)\\ \leq\ &[tg(a)+(1-t)g(b)]\,\breve{G}_f\left(a^tb^{1-t}\right)+g(a^tb^{1-t})f\left(\sqrt{ab}\right)\end{aligned}$$

for any $t\in[0,1]$.

Taking the integral over $t\in[0,1]$, we get

$$\begin{aligned}&f\left(\sqrt{ab}\right)\int_0^1 [tg(a)+(1-t)g(b)]\,dt+\int_0^1 g(a^tb^{1-t})\breve{G}_f\left(a^tb^{1-t}\right)dt\\ =\ &f\left(\sqrt{ab}\right)\frac{g(a)+g(b)}{2}+\frac{1}{\ln b-\ln a}\int_a^b \frac{\breve{G}_f(x)\,g(x)}{x}dx\\ \leq\ &\int_0^1 [tg(a)+(1-t)g(b)]\,\breve{G}_f\left(a^tb^{1-t}\right)dt+f\left(\sqrt{ab}\right)\int_0^1 g(a^tb^{1-t})dt\\ =\ &\frac{g(a)+g(b)}{2}\frac{1}{(\ln b-\ln a)}\int_a^b \frac{f(x)}{x}dx+f\left(\sqrt{ab}\right)\frac{1}{\ln b-\ln a}\int_a^b \frac{g(x)}{x}dx,\end{aligned}$$

and the inequality (20) is proved. □

4. Inequalities for Product of Symmetrized GA-convex functions

Theorem 4.1. *Assume that both $f,g:[a,b]\subseteq(0,\infty)\to\mathbb{R}$ are symmetrized GA-convex and integrable on the interval $[a,b]$. Then we have*

$$\frac{1}{\ln b/a}\int_a^b \frac{\breve{G}_f(x)\,\breve{G}_g(x)}{x}dx\geq\{S_{mid}(f,g;a,b),S_{tra}(f,g;a,b)\}, \tag{25}$$

and

$$\frac{1}{\ln b/a}\int_a^b \frac{\check{G}_f(x)\,\check{G}_g(x)}{x}dx \le \{S_{mix}(f,g;a,b), S_{mix}(g,f;a,b)\} \tag{26}$$

where

$$\begin{aligned} S_{mid}(f,g;a,b) &= g\left(\sqrt{ab}\right)\frac{1}{\ln b/a}\int_a^b \frac{f(x)}{x}dx + f\left(\sqrt{ab}\right)\frac{1}{\ln b/a}\int_a^b \frac{g(x)}{x}dx - f\left(\sqrt{ab}\right)g\left(\sqrt{ab}\right), \\ S_{tra}(f,g;a,b) &= \frac{f(a)+f(b)}{2}\frac{1}{\ln b/a}\int_a^b \frac{g(x)}{x}dx + \frac{g(a)+g(b)}{2}\frac{1}{\ln b/a}\int_a^b \frac{f(x)}{x}dx - \frac{f(a)+f(b)}{2}\frac{g(a)+g(b)}{2}, \end{aligned}$$

and

$$S_{mix}(f,g;a,b) = \frac{f(a)+f(b)}{2}\frac{1}{\ln b/a}\int_a^b \frac{g(x)}{x}dx + g\left(\sqrt{ab}\right)\frac{1}{\ln b/a}\int_a^b \frac{f(x)}{x}dx - \frac{f(a)+f(b)}{2}g\left(\sqrt{ab}\right).$$

Proof. By (12) we have

$$\left(\check{G}_f(x) - f\left(\sqrt{ab}\right)\right)\left(\check{G}_g(x) - g\left(\sqrt{ab}\right)\right) \ge 0$$

for any $x \in [a,b]$.

This is equivalent to

$$\check{G}_f(x)\,\check{G}_g(x) + f\left(\sqrt{ab}\right)g\left(\sqrt{ab}\right) \ge \check{G}_f(x)\,g\left(\sqrt{ab}\right) + \check{G}_g(x)\,f\left(\sqrt{ab}\right) \tag{27}$$

for any $x \in [a,b]$.

If we multiply the inequality (27) by $1/x$ and integrate over x on $[a,b]$ and divide by $\ln b/a$, then we get

$$\begin{aligned} &\frac{1}{\ln b/a}\int_a^b \frac{\check{G}_f(x)\,\check{G}_g(x)}{x}dx + f\left(\sqrt{ab}\right)g\left(\sqrt{ab}\right) \\ \ge\ & g\left(\sqrt{ab}\right)\frac{1}{\ln b/a}\int_a^b \frac{\check{G}_f(x)}{x}dx + f\left(\sqrt{ab}\right)\frac{1}{\ln b/a}\int_a^b \frac{\check{G}_g(x)}{x}dx \end{aligned}$$

and since

$$\begin{aligned} \frac{1}{\ln b/a}\int_a^b \frac{\check{G}_f(x)}{x}dx &= \frac{1}{2}\left[\frac{1}{\ln b/a}\int_a^b \frac{f(x)}{x}dx + \frac{1}{\ln b/a}\int_a^b \frac{f\left(\frac{ab}{x}\right)}{x}dx\right] \\ &= \frac{1}{\ln b/a}\int_a^b \frac{f(x)}{x}dx, \end{aligned}$$

the inequality

$$\frac{1}{\ln b/a}\int_a^b \frac{\check{G}_f(x)\,\check{G}_g(x)}{x}dx \ge S_{mid}(f,g;a,b)$$

is easily obtained.

Also, by (12) we have

$$\left(\frac{f(a)+f(b)}{2} - \check{G}_f(x)\right)\left(\frac{g(a)+g(b)}{2} - \check{G}_g(x)\right) \ge 0$$

for any $x \in [a,b]$, which by the same procedure produces the inequality

$$\frac{1}{\ln b/a}\int_a^b \frac{\check{G}_f(x)\,\check{G}_g(x)}{x}dx \ge S_{tra}(f,g;a,b).$$

Finally, if both of them are symmetrized convex, then by (12) we have

$$\left(\frac{f(a)+f(b)}{2}-\breve{G}_f(x)\right)\left(\breve{G}_g(x)-g\left(\sqrt{ab}\right)\right)\geq 0 \tag{28}$$

for any $x\in[a,b]$, which is equivalent to

$$\breve{G}_f(x)\,\breve{G}_g(x)\leq \frac{f(a)+f(b)}{2}\breve{G}_g(x)+\breve{G}_f(x)\,g\left(\sqrt{ab}\right)-\frac{f(a)+f(b)}{2}g\left(\sqrt{ab}\right)$$

for any $x\in[a,b]$. If we multiply the last inequality by $1/x$ and integrate over x on $[a,b]$ and divide by $\ln b/a$, then we get the desired result

$$\frac{1}{\ln b/a}\int_a^b\frac{\breve{G}_f(x)\,\breve{G}_g(x)}{x}dx\leq S_{mix}(f,g;a,b).$$

The inequality

$$\frac{1}{\ln b/a}\int_a^b\frac{\breve{G}_f(x)\,\breve{G}_g(x)}{x}dx\leq S_{mix}(g,f;a,b)$$

follows from (28) by replacing f with g. □

Remark 4.2. *Observe that*

$$\begin{aligned}\int_a^b\frac{\breve{G}_f(x)\,\breve{G}_g(x)}{x}dx &= \frac{1}{4}\int_a^b\frac{\left[f(x)+f\left(\frac{ab}{x}\right)\right]\left[g(x)+g\left(\frac{ab}{x}\right)\right]}{x}dx\\ &= \frac{1}{4}\left[\int_a^b\frac{f(x)g(x)}{x}dx+\int_a^b\frac{f\left(\frac{ab}{x}\right)g(x)}{x}dx+\int_a^b\frac{f(x)g\left(\frac{ab}{x}\right)}{x}dx+\int_a^b\frac{f\left(\frac{ab}{x}\right)g\left(\frac{ab}{x}\right)}{x}dx\right]\\ &= \frac{1}{2}\left[\int_a^b\frac{f(x)g(x)}{x}dx+\int_a^b\frac{f\left(\frac{ab}{x}\right)g(x)}{x}dx\right]\\ &= \int_a^b\frac{\breve{G}_f(x)\,g(x)}{x}dx\end{aligned}$$

and

$$\int_a^b\frac{\breve{G}_f(x)\,\breve{G}_g(x)}{x}dx=\int_a^b\frac{f(x)\,\breve{G}_g(x)}{x}dx.$$

From (25) and (26) we then have

$$\begin{aligned}\{S_{mid}(f,g;a,b),S_{tra}(f,g;a,b)\} &\leq \frac{1}{\ln b/a}\int_a^b\frac{\breve{G}_f(x)\,g(x)}{x}dx=\int_a^b\frac{f(x)\,\breve{G}_g(x)}{x}dx\\ &\leq \{S_{mix}(f,g;a,b),S_{mix}(g,f;a,b)\}.\end{aligned}$$

The following particular case when one of the functions is geometrically symmetric with respect to $\sqrt{ab}$ on $[a,b]$ holds:

Corollary 4.3. *Assume that both $f:[a,b]\subseteq(0,\infty)\to\mathbb{R}$ is GA-convex and geometrically symmetric with respect to $\sqrt{ab}$. If $g:[a,b]\subseteq(0,\infty)\to\mathbb{R}$ is symmetrized GA-convex and integrable on the interval $[a,b]$, then*

$$\begin{aligned}\{S_{mid}(f,g;a,b),S_{tra}(f,g;a,b)\} &\leq \frac{1}{\ln b/a}\int_a^b\frac{f(x)\,g(x)}{x}dx\\ &\leq \{S_{mix}(f,g;a,b),S_{mix}(g,f;a,b)\}.\end{aligned}$$

Theorem 4.4. *Assume that both $f, g : [a, b] \subseteq (0, \infty) \to [0, \infty)$ are symmetrized GA-convex and integrable on the interval $[a, b]$. Then we have*

$$\begin{aligned} f\left(\sqrt{ab}\right) g\left(\sqrt{ab}\right) &\leq f\left(\sqrt{ab}\right) \frac{1}{\ln b - \ln a} \int_a^b \frac{g(x)}{x} dx \\ &\leq \frac{1}{\ln b/a} \int_a^b \frac{\breve{G}_f(x)\, g(x)}{x} dx \\ &\leq \frac{f(a) + f(b)}{2} \frac{1}{\ln b/a} \int_a^b \frac{g(x)}{x} dx \\ &\leq \frac{f(a) + f(b)}{2} \frac{g(a) + g(b)}{2}, \end{aligned} \tag{29}$$

and

$$\begin{aligned} f\left(\sqrt{ab}\right) g\left(\sqrt{ab}\right) &\leq g\left(\sqrt{ab}\right) \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \\ &\leq \frac{1}{\ln b/a} \int_a^b \frac{\breve{G}_f(x)\, g(x)}{x} dx \\ &\leq \frac{g(a) + g(b)}{2} \frac{1}{\ln b/a} \int_a^b \frac{f(x)}{x} dx \\ &\leq \frac{f(a) + f(b)}{2} \frac{g(a) + g(b)}{2}, \end{aligned} \tag{30}$$

Proof. If $f, g : [a, b] \subseteq (0, \infty) \to \mathbb{R}$ are symmetrized GA-convex, then by (12) we have

$$0 \leq f\left(\sqrt{ab}\right) \leq \breve{G}_f(x) \leq \frac{f(a) + f(b)}{2} \tag{31}$$

and

$$0 \leq g\left(\sqrt{ab}\right) \leq \breve{G}_g(x) \leq \frac{g(a) + g(b)}{2}, \tag{32}$$

for any $x \in [a, b]$.

If we multiply (31) by $\breve{G}_g(x)$, then we get

$$0 \leq f\left(\sqrt{ab}\right) \breve{G}_g(x) \leq \breve{G}_f(x)\, \breve{G}_g(x) \leq \frac{f(a) + f(b)}{2} \breve{G}_g(x)$$

for any $x \in [a, b]$. If we multiply the last inequality by $1/x$ and integrate over x on $[a, b]$ and divide by $\ln b/a$, then we get

$$0 \leq f\left(\sqrt{ab}\right) \frac{1}{\ln b/a} \int_a^b \frac{\breve{G}_g(x)}{x} dx \leq \frac{1}{\ln b/a} \int_a^b \frac{\breve{G}_f(x)\, \breve{G}_g(x)}{x} dx \leq \frac{f(a) + f(b)}{2} \frac{1}{\ln b/a} \int_a^b \frac{\breve{G}_g(x)}{x} dx,$$

namely

$$0 \leq f\left(\sqrt{ab}\right) \frac{1}{\ln b/a} \int_a^b \frac{g(x)}{x} dx \leq \frac{1}{\ln b/a} \int_a^b \frac{\breve{G}_f(x)\, g(x)}{x} dx \leq \frac{f(a) + f(b)}{2} \frac{1}{\ln b/a} \int_a^b \frac{g(x)}{x} dx, \tag{33}$$

Since, by (10) we have

$$g\left(\sqrt{ab}\right) \leq \frac{1}{\ln b - \ln a} \int_a^b \frac{g(x)}{x} dx \leq \frac{g(a) + g(b)}{2},$$

then

$$f\left(\sqrt{ab}\right)g\left(\sqrt{ab}\right) \leq f\left(\sqrt{ab}\right)\frac{1}{\ln b - \ln a}\int_a^b \frac{g(x)}{x}dx, \tag{34}$$

and

$$\frac{f(a)+f(b)}{2}\frac{1}{\ln b/a}\int_a^b \frac{g(x)}{x}dx \leq \frac{f(a)+f(b)}{2}\frac{g(a)+g(b)}{2}. \tag{35}$$

By utilising (33)-(35) we get (29).

The inequality (30) follows in a similar way by making use of (32) and a similar procedure. The details are omitted. □

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