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On (L, M)-Fuzzy Convex Structures

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Abstract. This paper defines a new class of L-fuzzy sets called r-L-fuzzy biconvex sets in (L, M)-fuzzy convex structures (X, C), where C is an (L, M)-fuzzy convexity on X, and some of their properties were studied. In addition, we introduce (L, M)-fuzzy topological convexity space and study some of its properties. Finally, we introduce locally (L, M)-fuzzy topology (L, M)-fuzzy convexity space and study some of its properties.

1. Introduction and Preliminaries

Abstract convexity theory in [26] plays an important role in various branches of mathematics. It deals with set-theoretic structures which satisfies axioms similar to that usual convex sets fulfill. Here, by "usual convex sets", we mean convex sets in real linear spaces. Also, abstract convexity theory has been applied to many different mathematical research fields, such as topological spaces, lattices, metric spaces and graphs (see, for example, [7, 11, 12, 24, 27, 29, 35]). The concept of convex structures as a topology-like structure, it can be also treated as a special kind of spatial structures and some topology-like properties.

For a generalization of a convex structure, Rosa in 1994 introduced the notion of fuzzy convex structure in [20, 21] which is called I-convex structure. Also, he studied a fuzzy topology together with a fuzzy convexity on the same underlying set X, and introduced fuzzy topology fuzzy convexity spaces and the notion of fuzzy local convexity. By framework, which proposed in [23], Li [9] presented a categorical approach to enrich (L, M)-fuzzy convex structures, Xiu et al [32] presented a degree approach to study the relationship between (L, M)-fuzzy convex structures and (L, M)-fuzzy closure systems and Wu and Li [31] introduced (L, M)-fuzzy domain finiteness, (L, M)-fuzzy restricted hull spaces and several characterizations of the category (L, M)-CS of (L, M)-fuzzy convex spaces. Recently, there has been significant research on fuzzy convex structures ([8, 13–17, 22, 28, 33, 34]).

The main contributions of the present paper are to give some further investigations on (L, M)-fuzzy convex structures, mainly including fuzzy hull operators and fuzzy topological convexity structures with respect to (L, M)-fuzzy convex structures. The transformation method between L-fuzzy hull operators and (L, M)-fuzzy convex structures were introduced. The continuous image of the locally (L, M)-fuzzy topology (L, M)-fuzzy convexity space was given. A characterization of the product of the L-fuzzy hull operator and the locally fuzzy convex space was obtained.

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Throughout this paper, let X be a non-empty set, both L and M be completely distributive lattices with order reversing involution $'$ where $\perp_M$ ($\perp_L$) and $\top_M(\top_L)$ denote the least and the greatest elements in $M(L)$ respectively, and $M_{\perp_M} = M - \{\perp_M\}(L_{\perp_L} = L - \{\perp_L\})$. An L-fuzzy subset of X is a mapping $\mu : X \longrightarrow L$ and the family L^X denoted the set of all fuzzy subsets of a given X [3]. The least and the greatest elements in L^X are denoted by $\chi_\emptyset$ and χ_X, respectively. For each $\alpha \in L$, let $\underline{\alpha}$ denote the constant L-fuzzy subset of X with the value α. The complementation of a fuzzy subset are defined as $\mu'(x) = (\mu(x))'$ for all $x \in X$, (e.g. $\mu'(x) = 1 - \mu(x)$ in the case of $L = [0, 1]$). Let $X = \prod_{i\in\Gamma} X_i$ and $\mu_i \in L^{X_i}$, then $\mu \in L^X$ denote the product of all $\mu_i \in L^{X_i}$ is defined as follows: $\mu(x) = \wedge_{i\in\Gamma}\mu_i(x_i)$ for all $x \in X$ [25].

Definition 1.1. ([5]) Let $\emptyset \neq Y \subseteq X$ and $\mu \in L^X$; the restriction of μ on Y, is denoted by $\mu|Y$. The extension of $\mu \in L^Y$ on X, denoted by μ_X, is defined by

$$\mu_X(x) = \begin{cases} \mu(x), & \text{if } x \in Y, \\ \perp_L, & \text{if } x \in X - Y. \end{cases}$$

Definition 1.2. ([4, 18]) A fuzzy point x_t for $t \in L_{\perp_L}$ is an element of L^X such that

$$x_t(y) = \begin{cases} t, & \text{if } y = x, \\ \perp_L, & \text{if } y \neq x. \end{cases}$$

The set of all fuzzy points in X is denoted by $P_t(X)$.

Definition 1.3. ([36]) Let $f : X \longrightarrow Y$. Then the image $f^{\rightarrow}(\mu)$ of $\mu \in L^X$ and the preimage $f^{\leftarrow}(\nu)$ of $\nu \in L^Y$ are defined by:

$$f^{\rightarrow}(\mu)(y) = \bigvee\{\mu(x) : x \in X, f(x) = y\} \text{ and } f^{\leftarrow}(\nu) = \nu \circ f, \text{ respectively.}$$

Definition 1.4. ([23]) The pair $(X, \mathcal{C})$ is called an (L, M)-fuzzy convex structure, where $\mathcal{C} : L^X \longrightarrow M$ satisfies the following axioms:

(LMC1) $\mathcal{C}(\chi_\emptyset) = \mathcal{C}(\chi_X) = \top_M$.

(LMC2) If $\{\mu_i : i \in \Gamma\} \subseteq L^X$ is nonempty, then $\mathcal{C}(\bigwedge_{i\in\Gamma} \mu_i) \geq \bigwedge_{i\in\Gamma} \mathcal{C}(\mu_i)$.

(LMC3) If $\{\mu_i : i \in \Gamma\} \subseteq L^X$ is nonempty and totally ordered by inclusion, then $\mathcal{C}(\bigvee_{i\in\Gamma} \mu_i) \geq \bigwedge_{i\in\Gamma} \mathcal{C}(\mu_i)$.

The mapping $\mathcal{C}$ is called an (L, M)-fuzzy convexity on X and $\mathcal{C}(\mu)$ can be regarded as the degree to which μ is an L-convex fuzzy set.

Definition 1.5. ([23]) Let $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ be (L, M)-fuzzy convex structures. A function $f : X \longrightarrow Y$ is called:

(1) An (L, M)-fuzzy convexity preserving function if $\mathcal{C}(f^{\leftarrow}(\mu)) \geq \mathcal{D}(\mu)$ for all $\mu \in L^Y$.

(2) An (L, M)-fuzzy convex-to-convex function if $\mathcal{D}(f^{\rightarrow}(\mu)) \geq \mathcal{C}(\mu)$ for all $\mu \in L^X$.

Theorem 1.6. ([23]) *Let $(X, \mathcal{C})$ be an (L, M)-fuzzy convex structure, $\emptyset \neq Y \subseteq X$. Then $(Y, \mathcal{C}|Y)$ is an (L, M)-fuzzy convex structure on Y, where*

$$(\mathcal{C}|Y)(\mu) = \bigvee\{\mathcal{C}(\nu) : \nu \in L^X, \nu|Y = \mu\},$$

for each $\mu \in L^Y$. The pair $(Y, \mathcal{C}|Y)$ is called an (L, M)-fuzzy convex sub-structure of $(X, \mathcal{C})$.

Definition 1.7. ([23]) Let $\{(X_i, \mathcal{C}_i) : i \in \Gamma\}$ be a set of (L, M)-fuzzy convex structures, X be the product of the sets X_i for $i \in \Gamma$ and $\pi_i : X \longrightarrow X_i$ be the projection for each $i \in \Gamma$. Define a mapping $\varphi : L^X \longrightarrow M$ by

$$\varphi(\mu) = \bigvee_{i\in\Gamma} \bigvee_{\pi_i^{\leftarrow}(\nu)=\mu} \mathcal{C}_i(\nu), \qquad \text{for each } \mu, \nu \in L^X.$$

Then the product convexity $\mathcal{C}$ of X is the one generated by subbase φ. The resulting (L, M)-fuzzy convex structure $(X, \mathcal{C})$ is called the product of $\{(X_i, \mathcal{C}_i) : i \in \Gamma\}$ and is denoted by $\prod_{i\in\Gamma}(X_i, \mathcal{C}_i)$.

Definition 1.8. ([6], [25]) An (L, M)-fuzzy topology on X is a map $\mathcal{T} : L^X \longrightarrow M$ with the following conditions:

(1) $\mathcal{T}(\chi_\emptyset) = \mathcal{T}(\chi_X) = \top_M$.
(2) $\mathcal{T}(\mu \wedge \nu) \geq \mathcal{T}(\mu) \wedge \mathcal{T}(\nu), \quad \forall \mu, \nu \in L^X$
(3) $\mathcal{T}(\bigvee_{i\in\Gamma} \mu_i) \geq \bigwedge_{i\in\Gamma} \mathcal{T}(\mu_i), \quad \forall \mu_i \in L^X, i \in \Gamma$.

The pair $(X, \mathcal{T})$ is called an (L, M)-fuzzy topological space.

Definition 1.9. ([25]) Let $f : (X, \mathcal{T}^1) \longrightarrow (Y, \mathcal{T}^2)$ be a mapping. Then, f is called

(1) An (L, M)-fuzzy continuous if $\mathcal{T}^1(f^{\leftarrow}(\mu)) \geq \mathcal{T}^2(\mu)$ for all $\mu \in L^Y$;
(2) An (L, M)-fuzzy open if $\mathcal{T}^2(f^{\rightarrow}(\mu)) \geq \mathcal{T}^1(\mu)$ for all $\mu \in L^X$.

Proposition 1.10. ([2, 19]) *Let $(X, \mathcal{T})$ be an (L, M)-fuzzy topological space and $A \subseteq X$. Define a mapping $\mathcal{T}_A : L^X \longrightarrow M$ by*

$$\mathcal{T}_A(\mu) = \bigvee \{\mathcal{T}(\nu) : \nu \in L^X, \nu|A = \mu\}.$$

($\bigvee$ being the supremum operation on M). Then $\mathcal{T}_A$ is an (L, M)-fuzzy topology A.

Theorem 1.11. ([1, 30]) *Let $f : X \longrightarrow Y$. Then, for all $\mu, \mu_i \in L^Y$ and $\nu, \nu_i \in L^X$*

(1) $\mu \geq f^{\rightarrow}(f^{\leftarrow}(\mu))$ with equality if f is surjective.
(2) $\nu \leq f^{\leftarrow}(f^{\rightarrow}(\nu))$ with equality if f is injective.
(3) $f^{\leftarrow}(\mu') = (f^{\leftarrow}(\mu))'$.
(4) $f^{\leftarrow}(\bigvee_{i\in\Gamma} \mu_i) = \bigvee_{i\in\Gamma} f^{\leftarrow}(\mu_i)$.
(5) $f^{\leftarrow}(\bigwedge_{i\in\Gamma} \mu_i) = \bigwedge_{i\in\Gamma} f^{\leftarrow}(\mu_i)$.
(6) $f^{\rightarrow}(\bigvee_{i\in\Gamma} \nu_i) = \bigvee_{i\in\Gamma} f^{\rightarrow}(\nu_i)$.
(7) $f^{\rightarrow}(\bigwedge_{i\in\Gamma} \nu_i) \leq \bigwedge_{i\in\Gamma} f^{\rightarrow}(\nu_i)$ with equality if f is injective.

2. *r-L*-Fuzzy Biconvex Sets

Definition 2.1. Let $(X, \mathcal{C})$ be an (L, M)-fuzzy convex structure, $r \in M_{\perp_M}$ and $\mu \in L^X$. Then μ is called *r-L*-fuzzy biconvex set if $\mathcal{C}(\mu) \geq r$ and $\mathcal{C}(\mu') \geq r$.

Note: $\chi_\emptyset$ and χ_X are *r-L*-fuzzy biconvex sets.

Proposition 2.2. *Let $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ be an (L, M)-fuzzy convex structures, $f : X \longrightarrow Y$ be (L, M)-fuzzy convexity preserving function and μ be r-L-fuzzy biconvex set in Y. Then $f^{\leftarrow}(\mu)$ is r-L-fuzzy biconvex set in X.*

Proof. Let μ be *r-L*-fuzzy biconvex set in Y. Then $\mathcal{D}(\mu) \geq r$ and $\mathcal{D}(\mu') \geq r$. Therefore, by assumption we obtain $\mathcal{C}(f^{\leftarrow}(\mu)) \geq r$ and $\mathcal{C}(f^{\leftarrow}(\mu')) \geq r$. By the equality, $f^{\leftarrow}(\mu') = (f^{\leftarrow}(\mu))'$ we have $\mathcal{C}((f^{\leftarrow}(\mu))') \geq r$. So, $f^{\leftarrow}(\mu)$ is *r-L*-fuzzy biconvex set in X is obtained. □

Proposition 2.3. *Let $(X, \mathcal{C})$ be an (L, M)-fuzzy convex structure, $\emptyset \neq Y \subseteq X$ and μ is an r-L-fuzzy biconvex set in $(X, \mathcal{C})$. Then $\mu|Y$ is an r-L-fuzzy biconvex set in $(Y, \mathcal{C}|Y)$.*

Proof. Let μ be an *r-L*-fuzzy biconvex set in $(X, \mathcal{C})$. On one hand, $\mathcal{C}(\mu) \geq r$. Then,

$$(\mathcal{C}|Y)(\mu|Y) = \bigvee \{\mathcal{C}(\nu) : \nu \in L^X, \nu|Y = \mu|Y\}.$$

Put $\nu = \mu$, we obtain $(\mathcal{C}|Y)(\mu|Y) \geq r$. On the other hand $\mathcal{C}(\mu') \geq r$. Hence,

$$\begin{aligned}(\mathcal{C}|Y)((\mu|Y)') &= \bigvee \{\mathcal{C}(\lambda) : \lambda \in L^X, \lambda|Y = (\mu|Y)'\} \\ &= \bigvee \{\mathcal{C}(\lambda) : \lambda \in L^X, \lambda|Y = \mu'|Y\}.\end{aligned}$$

Put $\lambda = \mu'$, we obtain $(\mathcal{C}|Y)((\mu|Y)') \geq r$. Therefor $\mu|Y$ is *r-L*-fuzzy biconvex set in $(Y, \mathcal{C}|Y)$. □

Theorem 2.4. *Let $(X,\mathcal{C})$ be an (L,M)-fuzzy convex structure. For each $\mu \in L^X$ and $r \in M_{\perp_M}$ a mapping $CO : L^X \times M_{\perp_M} \longrightarrow L^X$ is defined as follows:*

$$CO(\mu,r) = \bigwedge\{\nu \in L^X : \mu \leq \nu,\ \mathcal{C}(\nu) \geq r\}.$$

For $\mu,\nu \in L^X$ and $r,s \in M_{\perp_M}$ the operator CO satisfies the following conditions:

(1) $CO(\chi_\emptyset,r) = \chi_\emptyset$.
(2) $\mu \leq CO(\mu,r)$.
(3) If $\mu \leq \nu$, then $CO(\mu,r) \leq CO(\nu,r)$.
(4) if $r \leq s$, then $CO(\mu,r) \leq CO(\mu,s)$.
(5) $CO(CO(\mu,r),r) = CO(\mu,r)$.
(6) For $\{\mu_i : i \in \Gamma\} \subseteq L^X$ is nonempty and totally ordered by inclusion, $CO(\bigvee_{i\in\Gamma}\mu_i,r) = \bigvee_{i\in\Gamma} CO(\mu_i,r)$.
A mapping CO is called an L-fuzzy hull operator.

Proof. (1) For all $r \in M_{\perp_M}$, we have $\mathcal{C}(\chi_\emptyset) \geq r$. So, we obtain $CO(\chi_\emptyset,r) = \chi_\emptyset$.
(2) and (3) are satisfied from the definition of CO.
(4) Suppose that $r \leq s$. Then by (2) we have

$$CO(\mu,r) \leq CO(CO(\mu,s),r).$$

By the definition of CO, we obtain $\mathcal{C}(CO(\mu,s)) \geq r$. So, $CO(CO(\mu,s),r) = CO(\mu,s)$. Hence $CO(\mu,r) \leq CO(\mu,s)$.

(5) It is enough to verify that $CO(CO(\mu,r),r) \leq CO(\mu,r)$. Suppose that there exists $\mu \in L^X, r \in M_{\perp_M}$ and $x \in X$ such that

$$CO(CO(\mu,r),r)(x) > CO(\mu,r)(x).$$

By the definition of $CO(\mu,r)$, there exists $\nu \in L^X$ with $\mu \leq \nu$ and $\mathcal{C}(\nu) \geq r$ such that

$$CO(CO(\mu,r),r)(x) > \nu(x) \geq CO(\mu,r)(x).$$

On the other hand, $CO(\mu,r) \leq \nu$ and $\mathcal{C}(\nu) \geq r$. By the definition of $CO(CO(\mu,r),r)$, we have $CO(CO(\mu,r),r)(x) \leq \nu(x)$. It is a contradiction. Thus, $CO(CO(\mu,r),r) = CO(\mu,r)$.

(6) For $i \in \Gamma$, we have

$\mu_i \leq \bigvee \mu_i$. Therefore by (3) we have $CO(\mu_i,r) \leq CO(\bigvee \mu_i,r)$.

Hence,

$$\bigvee CO(\mu_i,r) \leq CO(\bigvee \mu_i,r). \tag{1}$$

On the other hand, by (2), we have $\bigvee \mu_i \leq \bigvee CO(\mu_i,r)$. Since $CO(\mu_i,r)$ are L-fuzzy convex sets totally ordered by inclusion, $\bigvee CO(\mu_i,r)$ is an r-L-fuzzy convex set containing $\bigvee \mu_i$. So, $CO(\bigvee \mu_i,r)$ is the smallest fuzzy convex set containing $\bigvee \mu_i$ and hence,

$$\bigvee \mu_i \leq CO(\bigvee \mu_i,r) \leq \bigvee CO(\mu_i,r). \tag{2}$$

From equations (1) and (2), we have $CO(\bigvee \mu_i,r) = \bigvee CO(\mu_i,r)$. □

The triple $(X,\mathcal{C}^1,\mathcal{C}^2)$ is called an (L,M)-fuzzy biconvex structure ((L,M)-fbcs, for short) where $\mathcal{C}^1$ and $\mathcal{C}^2$ are (L,M)-fuzzy convexities on X.

Proposition 2.5. *Let $(X,\mathcal{C}^1,\mathcal{C}^2)$ be an (L,M)-fbcs. For each $r \in M_{\perp_M}$ and $\mu \in L^X$, a mapping $CO^{12} : L^X \times M_{\perp_M} \longrightarrow L^X$ is defined as follows:*

$$CO^{12}(\mu,r) = CO^1(\mu,r) \wedge CO^2(\mu,r).$$

Then, CO^{12} is an L-fuzzy hull operator.

Proof. (1) By Theorem 2.4 (1), we have $CO^1(\chi_\emptyset, r) = \chi_\emptyset$ and $CO^2(\chi_\emptyset, r) = \chi_\emptyset$ for all $r \in M_{\perp_M}$. So,

$$\begin{aligned} CO^{12}(\chi_\emptyset, r) &= CO^1(\chi_\emptyset, r) \wedge CO^2(\chi_\emptyset, r) \\ &= \chi_\emptyset \wedge \chi_\emptyset = \chi_\emptyset. \end{aligned}$$

(2) Since, $\mu \leq CO^1(\mu, r)$ and $\mu \leq CO^2(\mu, r)$, we obtain

$$\begin{aligned} \mu = \mu \wedge \mu &\leq CO^1(\mu, r) \wedge CO^2(\mu, r) \\ &= CO^{12}(\mu, r). \end{aligned}$$

(3) Let $\mu \leq \nu$. Then by Theorem 2.4 (3) we obtain

$$CO^1(\mu, r) \leq CO^1(\nu, r) \text{ and } CO^2(\mu, r) \leq CO^2(\nu, r).$$

Therefore,

$$\begin{aligned} CO^{12}(\mu, r) &= CO^1(\mu, r) \wedge CO^2(\mu, r) \\ &\leq CO^1(\nu, r) \wedge CO^2(\nu, r) \\ &= CO^{12}(\nu, r). \end{aligned}$$

(4) Let $r \leq s$. Then we have from Theorem 2.4 (4)

$$CO^1(\mu, r) \leq CO^1(\mu, s) \text{ and } CO^2(\mu, r) \leq CO^2(\mu, s).$$

Therefore,

$$\begin{aligned} CO^{12}(\mu, r) &= CO^1(\mu, r) \wedge CO^2(\mu, r) \\ &\leq CO^1(\mu, s) \wedge CO^2(\mu, s) \\ &= CO^{12}(\mu, s). \end{aligned}$$

(5) For all $\mu \in L^X, r \in M_{\perp_M}$.

$$\begin{aligned} CO^{12}(CO^{12}(\mu, r), r) &= CO^1(CO^{12}(\mu, r), r) \wedge CO^2(CO^{12}(\mu, r), r) \\ &\leq CO^1(CO^1(\mu, r), r) \wedge CO^2(CO^2(\mu, r), r) \\ &= CO^1(\mu, r) \wedge CO^2(\mu, r) = CO^{12}(\mu, r). \end{aligned}$$

(6) Let $\{\mu_i : i \in \Gamma\} \subset L^X$ be nonempty and totally ordered by inclusion. Then, for $r \in M_{\perp_M}$, by applying Theorem 2.4 (6) we have

$$\begin{aligned} CO^{12}(\textstyle\bigvee_{i\in\Gamma} \mu_i, r) &= CO^1(\bigvee_{i\in\Gamma} \mu_i, r) \wedge CO^2(\bigvee_{i\in\Gamma} \mu_i, r) \\ &= \bigvee_{i\in\Gamma} CO^1(\mu_i, r) \wedge \bigvee_{i\in\Gamma} CO^2(\mu_i, r) \\ &= \bigvee_{i\in\Gamma} (CO^1(\mu_i, r) \wedge CO^2(\mu_i, r)) \quad \text{Since } L \text{ is distributive lattices} \\ &= \bigvee_{i\in\Gamma} CO^{12}(\mu_i, r). \end{aligned}$$

So we obtain $CO^{12}(\bigvee_{i\in\Gamma} \mu_i, r) = \bigvee_{i\in\Gamma} CO^{12}(\mu_i, r)$. □

Proposition 2.6. *For an (L, M)-fuzzy hull operator CO^{12}, $\mu \in L^X$ and $r \in M_{\perp_M}$ a mapping $C^{CO^{12}} : L^X \longrightarrow M$ is defined as follows*

$$C^{CO^{12}}(\mu) = \bigvee\{r \in M_{\perp_M} : \mu = CO^{12}(\mu, r)\}.$$

Then:

(1) $C^{CO^{12}}$ is an (L, M)-fuzzy convexity on X.

(2) $(CO^{12})^{C^{CO^{12}}} = CO^{12}$.

Proof. (1) (LMC1) Since for all $r \in M_{\perp_M}$, $CO^{12}(\chi_\emptyset, r) = \chi_\emptyset$ and $\chi_X \leq CO^{12}(\chi_X, r)$ we have $C^{CO^{12}}(\chi_\emptyset) = C^{CO^{12}}(\chi_X) = \top_M$.

(LMC2) Let $\mu = \bigwedge_{i\in\Gamma} \mu_i$ and $C^{CO^{12}}(\bigwedge_{i\in\Gamma} \mu_i) \ngeq \bigwedge_{i\in\Gamma} C^{CO^{12}}(\mu_i)$. Then there exists $r_0 \in M_{\perp_M}$ such that

$$CO^{12}(\mu, r_0) \leq CO^{12}(\mu_i, r_0) \text{ for all } i \in \Gamma.$$

and

$$C^{CO^{12}}(\bigwedge_{i\in\Gamma} \mu_i) < r_0 < \bigwedge_{i\in\Gamma} C^{CO^{12}}(\mu_i).$$

So, $CO^{12}(\mu, r_0) \leq \bigwedge_{i\in\Gamma} CO^{12}(\mu_i, r_0)$. For all $i \in \Gamma$, there exists $r_i \in M_{\perp_M}$ with $CO^{12}(\mu_i, r_i) = \mu_i$ such that $r_0 < r_i \leq C^{CO^{12}}(\mu_i)$. On the other hand,

$$\mu_i \leq CO^{12}(\mu_i, r_0) \leq CO^{12}(\mu_i, r_i) = \mu_i.$$

Implies that $CO^{12}(\mu_i, r_0) = \mu_i$. Therefore,

$$CO^{12}(\mu, r_0) \leq \bigwedge_{i\in\Gamma} CO^{12}(\mu_i, r_0) = \bigwedge_{i\in\Gamma} \mu_i = \mu.$$

Hence $CO^{12}(\mu, r_0) = \mu$. So, $C^{CO^{12}}(\bigwedge_{i\in\Gamma} \mu_i) \geq r_0$. It is a contradiction.

(LMC3) Let $\{\mu_i : i \in \Gamma\} \subseteq L^X$ is nonempty and totally ordered by inclusion and suppose that $C^{CO^{12}}(\bigvee_{i\in\Gamma} \mu_i) \ngeq \bigwedge_{i\in\Gamma} C^{CO^{12}}(\mu_i)$. Then there exists $r_0 \in M_{\perp_M}$ such that

$$C^{CO^{12}}(\bigvee_{i\in\Gamma} \mu_i) < r_0 < \bigwedge_{i\in\Gamma} C^{CO^{12}}(\mu_i).$$

For all $i \in \Gamma$, there exist $r_i \in M_{\perp_M}$ with $CO^{12}(\mu_i, r_i) = \mu_i$ such that $r_0 < r_i \leq C^{CO^{12}}(\mu_i)$. On the other hand,

$$\mu_i \leq CO^{12}(\mu_i, r_0) \leq CO^{12}(\mu_i, r_i) = \mu_i.$$

Implies that $CO^{12}(\mu_i, r_0) = \mu_i$. Since $CO^{12}(\bigvee_{i\in\Gamma} \mu_i, r_0) = \bigvee_{i\in\Gamma} CO^{12}(\mu_i, r_0) = \bigvee_{i\in\Gamma} \mu_i$, then $C^{CO^{12}}(\bigvee_{i\in\Gamma} \mu_i) \geq r_0$. It is a contradiction.

(2) Let $\mu, \nu \in L^X$ and $r \in M_{\perp_M}$. Then,

$$\begin{aligned}
(CO^{12})^{C^{CO^{12}}}(\mu, r) &= (CO)^{C_1^{CO^{12}}}(\mu, r) \wedge (CO)^{C_2^{CO^{12}}}(\mu, r) \\
&= (\wedge\{\nu \in L^X : \mu \leq \nu,\ \ C_1^{CO^{12}}(\nu) \geq r\}) \\
&\quad \bigwedge(\wedge\{\nu \in L^X : \mu \leq \nu,\ \ C_2^{CO^{12}}(\nu) \geq r\}) \\
&= (\wedge\{\nu \in L^X : \mu \leq \nu = CO^{12}(\nu, r)\}) \\
&\quad \bigwedge(\wedge\{\nu \in L^X : \mu \leq \nu = CO^{12}(\nu, r)\}) \\
&= \wedge\{\nu \in L^X : \mu \leq \nu = CO^{12}(\nu, r)\}.
\end{aligned}$$

On one hand, take each $\mu \in L^X$ such that $\mu \leq \nu = CO^{12}(\nu, r)$. Then it follows that

$$CO^{12}(\mu, r) \leq CO^{12}(CO^{12}(\nu, r), r) = CO^{12}(\nu, r) = \nu.$$

This implies

$$CO^{12}(\mu, r) \leq (CO^{12})^{C^{CO^{12}}}(\mu, r). \tag{3}$$

On the other hand, since $\mu \leq CO^{12}(\mu, r) = CO^{12}(CO^{12}(\mu, r), r)$, it follows that

$$(CO^{12})^{C^{CO^{12}}}(\mu, r) \leq CO^{12}(\mu, r). \tag{4}$$

From equations (3) and (4) we have $(CO^{12})^{C^{CO^{12}}}(\mu, r) = CO^{12}(\mu, r)$. □

Corollary 2.7. *For a nonempty set X, there is a one-to-one correspondence between an L-fuzzy hull operators and an (L, M)-fuzzy convex structures.*

Proposition 2.8. *Let $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ be (L, M)-fuzzy convex structures. Then, $f : X \longrightarrow Y$ is*
(1) An (L, M)-fuzzy convexity preserving function if and only if $f^{\rightarrow}(CO_{\mathcal{C}}(\mu, r)) \leq CO_{\mathcal{D}}(f^{\rightarrow}(\mu), r)$ for all $\mu \in L^X$.
(2) An (L, M)-fuzzy convex-to-convex function if and only if $CO_{\mathcal{D}}(f^{\rightarrow}(\mu), r) \leq f^{\rightarrow}(CO_{\mathcal{C}}(\mu, r))$ for all $\mu \in L^X$.

Proof. (1) ($\Longrightarrow$) Suppose there exist $\mu \in L^X$ and $r \in M_{\perp_M}$ such that $f^{\rightarrow}(CO_{\mathcal{C}}(\mu, r)) \nleq CO_{\mathcal{D}}(f^{\rightarrow}(\mu), r)$. There exists $y \in Y$ and $t \in M_{\perp_M}$ such that

$$f^{\rightarrow}(CO_{\mathcal{C}}(\mu, r))(y) > t > CO_{\mathcal{D}}(f^{\rightarrow}(\mu), r)(y).$$

If $f^{\leftarrow}\{y\} = \emptyset$, it is a contradiction because $f(CO_{\mathcal{C}}(\mu, r)) = \perp_M$. If $f^{\leftarrow}\{y\} \neq \emptyset$, there exists $x \in f^{\leftarrow}\{y\}$ such that

$$f^{\rightarrow}(CO_{\mathcal{C}}(\mu, r))(y) > CO_{\mathcal{C}}(\mu, r)(x) > t > CO_{\mathcal{D}}(f^{\rightarrow}(\mu), r)(f^{\rightarrow}(x)). \quad (5)$$

Since $CO_{\mathcal{D}}(f^{\rightarrow}(\mu), r)(f^{\rightarrow}(x)) < t$, there exists $\nu \in L^Y, \mathcal{D}(\nu) \geq r$ with $f^{\rightarrow}(\mu) \leq \nu$ such that $CO_{\mathcal{D}}(f^{\rightarrow}(\mu), r)(f^{\rightarrow}(x)) \leq \nu(f^{\rightarrow}(x)) < t$. Moreover, $f^{\rightarrow}(\mu) \leq \nu$ implies that $\mu \leq f^{\leftarrow}(\nu)$. Since $\mathcal{C}(f^{\leftarrow}(\nu)) \geq r, CO_{\mathcal{C}}(\mu, r)(x) \leq CO_{\mathcal{C}}(f^{\leftarrow}(\nu), r)(x) = f^{\leftarrow}(\nu)(x) = \nu(f(x)) < t$. It is a contradiction for (5).

($\Longleftarrow$) Let $\mu \in L^Y$ such that $\mathcal{D}(\mu) \geq r$. Then

$$f^{\rightarrow}(CO_{\mathcal{C}}(f^{\leftarrow}(\mu), r)) \leq CO_{\mathcal{D}}(f^{\rightarrow}(f^{\leftarrow}(\mu)), r) \leq CO_{\mathcal{D}}(\mu, r) = \mu.$$

Therefore $CO_{\mathcal{C}}(f^{\leftarrow}(\mu), r) \leq f^{\leftarrow}(\mu)$. By Theorem 2.4 (2), we obtain $CO_{\mathcal{C}}(f^{\leftarrow}(\mu), r) = f^{\leftarrow}(\mu)$. Hence $\mathcal{C}(f^{\leftarrow}(\mu)) \geq r$ and f is an (L, M)-fuzzy convexity preserving function.

(2) ($\Longrightarrow$) Let $\mu \in L^X$ and suppose $f : X \longrightarrow Y$ is an (L, M)-fuzzy convex-to-convex function. Then, $\mathcal{C}(CO_{\mathcal{C}}(\mu, r)) \geq r$ and $\mu \leq CO_{\mathcal{C}}(\mu, r)$. Since f is an (L, M)-fuzzy convex-to-convex function, $\mathcal{D}(f^{\rightarrow}(CO_{\mathcal{C}}(\mu, r))) \geq r$ and $f^{\rightarrow}(\mu) \leq f^{\rightarrow}(CO_{\mathcal{C}}(\mu, r))$. Hence

$$CO_{\mathcal{D}}(f^{\rightarrow}(\mu), r) \leq CO_{\mathcal{D}}(f^{\rightarrow}(CO_{\mathcal{C}}(\mu, r))) = f^{\rightarrow}(CO_{\mathcal{C}}(\mu, r)).$$

($\Longleftarrow$) Let $\mu \in L^X$ such that $\mathcal{C}(\mu) \geq r$. Then, $CO_{\mathcal{C}}(\mu, r) = \mu$ and hence $f^{\rightarrow}(CO_{\mathcal{C}}(\mu, r)) = f^{\rightarrow}(\mu)$. Therefore,

$$CO_{\mathcal{D}}(f^{\rightarrow}(\mu), r) \leq f^{\rightarrow}(CO_{\mathcal{C}}(\mu, r)) = f^{\rightarrow}(\mu).$$

By Theorem 2.4 (2), we have $CO_{\mathcal{D}}(f^{\rightarrow}(\mu), r) = f^{\rightarrow}(\mu)$. Hence $\mathcal{D}(f^{\rightarrow}(\mu)) \geq r$ and f is an (L, M)-fuzzy convex-to-convex function. □

3. (L, M)-Fuzzy Topology (L, M)-Fuzzy Convexity Spaces

In this section we introduce the concept of an (L, M)-fuzzy topology (L, M)-fuzzy convexity space, (L, M)-fuzzy topological convexity space and define a locally (L, M)-fuzzy topology (L, M)-fuzzy convex space and their properties were studied. Also, the relationships between these concepts were investigated.

Definition 3.1. A triple $(X, \mathcal{C}, \mathcal{T})$ consisting of a set X, an (L, M)-fuzzy convexity, and an (L, M)-fuzzy topology is called an (L, M)-fuzzy topology (L, M)-fuzzy convexity space ((L, M)-ftfcs for short).

Definition 3.2. Let $(X, \mathcal{C}, \mathcal{T})$ be an (L, M)-ftfcs and $\emptyset \neq Y \subseteq X$. Then, the corresponding triple $(Y, \mathcal{C}|Y, \mathcal{T}_Y)$ is an (L, M)-fuzzy subspace of $(X, \mathcal{C}, \mathcal{T})$ such that $\mathcal{T}_Y$ is an (L, M)-fuzzy topology on Y.

Definition 3.3. Let $\mathcal{C}, \mathcal{T}$ be an (L, M)-fuzzy convexity and an (L, M)-fuzzy topology respectively. Then, $\mathcal{T}$ is said to be compatible with $\mathcal{C}$, if $\mathcal{T}((CO_{\mathcal{C}}(\mu, r))') \geq r$ for each $\mu \in L^X$ and the triple $(X, \mathcal{C}, \mathcal{T})$ is called an (L, M)-fuzzy topological convexity space ((L, M)-ftcs for short).

Remark 3.4. It is obvious that an (L, M)-ftcs is always an (L, M)-ftfcs and the converse is not true.

Example 3.5. Let $L = M = [0,1]$ and μ_i be fuzzy subsets of $X = \{a,b,c\}$ where $i = \{1,2,3\}$ is defined as follows:

$$\mu_1(a) = 1.0, \quad \mu_1(b) = 1.0, \quad \mu_1(c) = 0.0,$$
$$\mu_2(a) = 0.2, \quad \mu_2(b) = 0.2, \quad \mu_2(c) = 1.0,$$
$$\mu_3(a) = 0.0, \quad \mu_3(b) = 0.0, \quad \mu_3(c) = 1.0.$$

Define an (L,M)-fuzzy topology in [[6], [25]] $\mathcal{T}^1, \mathcal{T}^2 : [0,1]^X \longrightarrow [0,1]$ on X as follows:

$$\mathcal{T}^1(\nu) = \begin{cases} 1, & \text{if } \nu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{4}, & \text{if } \nu = \mu_1, \\ \frac{1}{4}, & \text{if } \nu = \mu_2, \\ \frac{1}{4}, & \text{if } \nu = \mu_3, \\ \frac{1}{2}, & \text{if } \nu = \mu_1 \wedge \mu_2, \\ 0, & \text{otherwise.} \end{cases} \qquad \mathcal{T}^2(\nu) = \begin{cases} 1, & \text{if } \nu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{4}, & \text{if } \nu = \mu_2, \\ \frac{1}{2}, & \text{if } \nu = \mu_3, \\ 0, & \text{otherwise.} \end{cases}$$

Define an (L,M)-fuzzy convexity $\mathcal{C} : [0,1]^X \longrightarrow [0,1]$ on X as follows:

$$\mathcal{C}(\nu) = \begin{cases} 1, & \text{if } \nu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{4}, & \text{if } \nu = \mu_1, \\ \frac{1}{4}, & \text{if } \nu = \underline{1} - \mu_2, \\ \frac{1}{3}, & \text{if } \nu = \mu_3, \\ 0, & \text{otherwise.} \end{cases}$$

Then $(X, \mathcal{C}, \mathcal{T}^1)$ is an (L,M)-ftcs. On the other hand, $(X, \mathcal{C}, \mathcal{T}^2)$ is an (L,M)-ftfcs but it is not (L,M)-ftcs because $0 = \mathcal{T}^2(\underline{1} - CO_{\mathcal{C}}(\mu_3, \frac{1}{4})) \not\geq \frac{1}{4}$.

Theorem 3.6. *An (L,M)-fuzzy subspace of (L,M)-ftcs is an (L,M)-ftcs.*

Proof. Let $(X, \mathcal{C}, \mathcal{T})$ be an (L,M)-ftcs and $(Y, \mathcal{C}|Y, \mathcal{T}_Y)$ be an (L,M)-fuzzy subspace of $(X, \mathcal{C}, \mathcal{T})$. Then by Theorem 1.6, $(Y, \mathcal{C}|Y, \mathcal{T}_Y)$ is an (L,M)-ftfcs. To show that it is an (L,M)-ftcs, let $\lambda = CO_{(\mathcal{C}|Y)}(\mu, r)$ for each $\lambda, \mu \in L^Y$. Then, $(\mathcal{C}|Y)(\lambda) \geq r$, $\lambda = \nu|Y$ and $\mathcal{C}(\nu) \geq r$ for each $\nu \in L^X$. Put $\nu = CO_{\mathcal{C}}(\mu, r)$. Since $(X, \mathcal{C}, \mathcal{T})$ is an (L,M)-ftcs, $\mathcal{T}(\nu') \geq r$ and hence $\mathcal{T}_Y(\lambda') \geq r$. Hence, $(Y, \mathcal{C}|Y, \mathcal{T}_Y)$ be an (L,M)-ftcs. □

Remark 3.7. An (L,M)-fuzzy convexity preserving and an (L,M)-fuzzy continuous image of an (L,M)-ftcs need not be an (L,M)-ftcs.

Example 3.8. Let $L = M = [0,1]$ and ν_i be fuzzy subsets of $X = \{a,b,c\}$ where $i = \{1,2,3,4,5\}$ are defined as follows:

$$\nu_1(a) = 1.0, \quad \nu_1(b) = 0.0, \quad \nu_1(c) = 0.0,$$
$$\nu_2(a) = \frac{1}{3}, \quad \nu_2(b) = 0.0, \quad \nu_2(c) = 0.0,$$
$$\nu_3(a) = 0.0, \quad \nu_3(b) = 1.0, \quad \nu_3(c) = 1.0,$$
$$\nu_4(a) = \frac{4}{5}, \quad \nu_4(b) = 0.0, \quad \nu_4(c) = 0.0,$$
$$\nu_5(a) = \frac{1}{5}, \quad \nu_5(b) = 0.0, \quad \nu_5(c) = 0.0.$$

Define an (L, M)-fuzzy topology in [[6], [25]] $\mathcal{T}^1 : I^X \longrightarrow I$ and (L, M)-fuzzy convexity $C : I^X \longrightarrow I$ on X as follows:

$$\mathcal{T}^1(\lambda) = \begin{cases} 1, & \text{if } \nu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{4}, & \text{if } \lambda = \nu_1, \\ \frac{1}{4}, & \text{if } \lambda = \nu_2, \\ \frac{1}{4}, & \text{if } \lambda = \nu_3, \\ \frac{1}{4}, & \text{if } \lambda = \nu_4, \\ \frac{1}{2}, & \text{if } \lambda = \nu_2 \vee \nu_3, \\ \frac{1}{2}, & \text{if } \lambda = \nu_3 \vee \nu_4, \\ 0, & \text{otherwise.} \end{cases} \qquad C(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{6}, & \text{if } \lambda = \nu_1, \\ \frac{1}{5}, & \text{if } \lambda = \nu_3, \\ \frac{1}{5}, & \text{if } \lambda = \nu_5, \\ 0, & \text{otherwise.} \end{cases}$$

Let μ_i be fuzzy subsets of $Y = \{y_1, y_2\}$ where $i = \{1, 2, 3, 4\}$ is defined as follows:

$$\mu_1(y_1) = 0.0, \qquad \mu_1(y_2) = 1.0,$$
$$\mu_2(y_1) = 0.0, \qquad \mu_2(y_2) = \frac{1}{3},$$
$$\mu_3(y_1) = 1.0, \qquad \mu_3(y_2) = 0.0,$$
$$\mu_4(y_1) = 0.0, \qquad \mu_4(y_2) = \frac{1}{5}.$$

Define an (L, M)-fuzzy topology in [[6], [25]] $\mathcal{T}^2 : I^Y \longrightarrow I$ and (L, M)-fuzzy convexity $\mathcal{D} : I^Y \longrightarrow I$ on Y as follows:

$$\mathcal{T}^2(\nu) = \begin{cases} 1, & \text{if } \nu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{4}, & \text{if } \nu = \mu_1, \\ \frac{1}{4}, & \text{if } \nu = \mu_2, \\ \frac{1}{4}, & \text{if } \nu = \mu_3, \\ \frac{1}{2}, & \text{if } \nu = \mu_2 \vee \mu_3, \\ 0, & \text{otherwise.} \end{cases} \qquad \mathcal{D}(\nu) = \begin{cases} 1, & \text{if } \nu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{6}, & \text{if } \nu = \mu_1, \\ \frac{1}{5}, & \text{if } \nu = \mu_3, \\ \frac{1}{5}, & \text{if } \nu = \mu_4, \\ 0, & \text{otherwise.} \end{cases}$$

Let $f : (X, C, \mathcal{T}^1) \longrightarrow (Y, \mathcal{D}, \mathcal{T}^2)$ be defined as follows:

$$f(a) = y_2 \text{ and } f(b) = f(c) = y_1.$$

Then, $\mathcal{T}^1(f^{\leftarrow}(\mu_i)) \geq \mathcal{T}^2(\mu_i)$ for each $\mu_i \in I^Y, i = \{1, 2, 3, 4\}$. Therefore, f is an (L, M)-fuzzy continuous map. Also, $C(f^{\leftarrow}(\mu_i)) \geq \mathcal{D}(\mu_i)$ for each $\mu_i \in I^Y, i = \{1, 3, 4\}$. Therefore, f is an (L, M)-fuzzy convexity preserving map. On the other hand $(X, C, \mathcal{T}^1)$ is an (L, M)-ftcs but $(Y, \mathcal{D}, \mathcal{T}^2)$ is not an (L, M)-ftcs because $\mathcal{T}^2(\underline{1} - CO_{\mathcal{D}}(\mu_4, r)) \ngeq r,\ r \in (0, 1]$.

Definition 3.9. Let x_t be an L-fuzzy point of an (L, M)-ftfcs $(X, C, \mathcal{T})$. Then, $\mu \in L^X$ is called r-fuzzy neighbourhood of x_t if there exists $\nu \in L^X, \mathcal{T}(\nu) \geq r$ such that $x_t \in \nu \leq \mu$.

Definition 3.10. An (L, M)-ftfcs $(X, \mathcal{C}, \mathcal{T})$ is said to be locally fuzzy convex at an L-fuzzy point x_t if for every r-fuzzy neighbourhood μ of x_t there exists some r-convex fuzzy nieghbourhood ν of x_t such that $\nu \leq \mu$.

$(X, \mathcal{C}, \mathcal{T})$ is locally fuzzy convex if it is locally fuzzy convex at each of its L-fuzzy points.

Proposition 3.11. *An (L, M)-fuzzy convex-to-convex, (L, M)-fuzzy open and (L, M)-fuzzy continuous image of a locally (L, M)-ftfcs is a locally (L, M)-ftfcs.*

Proof. Let $f : (X, \mathcal{C}, \mathcal{T}^1) \longrightarrow (Y, \mathcal{D}, \mathcal{T}^2)$ be an (L, M)-fuzzy convex-to-convex, (L, M)-fuzzy open and (L, M)-fuzzy continuous onto map. Let y_s be an L-fuzzy point in Y. Then there exists an L-fuzzy point x_t in X such that $f^{\rightarrow}(x_t) = y_s$. Let μ be r-fuzzy neighbourhood of y_s in Y. Then $f^{\leftarrow}(\mu)$ is r-fuzzy neighbourhood of x_t in X. Since X is a locally (L, M)-ftfcs, there exists r-convex fuzzy neighbourhood ν of x_t in X such that

$$x_t \in \nu \leq f^{\leftarrow}(\mu).$$

Therefore

$$f^{\rightarrow}(x_t) \in f^{\rightarrow}(\nu) \leq \mu, \quad \text{i.e. } y_s \in f^{\rightarrow}(\nu) \leq \mu.$$

Since f is an (L, M)-fuzzy convex-to-convex and (L, M)-fuzzy open onto a map, $f^{\rightarrow}(\nu)$ is r-convex fuzzy neighbourhood of y_s in Y. Hence, Y is a locally (L, M)-ftfcs. □

Proposition 3.12. *An (L, M)-fuzzy convex subspaces of a locally (L, M)-ftfcs is a locally (L, M)-ftfcs.*

Proof. Let $(X, \mathcal{C}, \mathcal{T})$ be a locally (L, M)-ftfcs, $\emptyset \neq Y \subseteq X$ and $(Y, \mathcal{C}|Y, \mathcal{T}_Y)$ be the corresponding an (L, M)-fuzzy subspace of $(X, \mathcal{C}, \mathcal{T})$. Let x_t be an L-fuzzy point in Y and μ be r-fuzzy open neighborhood of x_t in Y, i.e., $x_t \in \mu$ such that $\mathcal{T}_Y(\mu) \geq r$. Since $\mathcal{T}_Y(\mu) \geq r$ we have $\mu = \nu|Y$, where $\mathcal{T}(\nu) \geq r$. Since X is locally fuzzy convex, there exists r- convex fuzzy neighborhood λ of x_t such that $x_t \in \lambda \leq \nu$. So, $x_t \in \lambda|Y \leq \nu|Y$. Since Y is an (L, M)-fuzzy convex, $\lambda|Y$ is r-convex fuzzy neighborhood in $(Y, \mathcal{C}|Y, \mathcal{T}_Y)$ and hence $(Y, \mathcal{C}|Y, \mathcal{T}_Y)$ is a locally (L, M)-ftfcs. □

Proposition 3.13. *Let $(X, \mathcal{C})$ be the product of $\{(X_i, \mathcal{C}_i) : i \in \Gamma\}$. Then for $\pi_i : X \longrightarrow X_i$, $r \in M_{\perp_M}$ and $\mu \in L^X$, a mapping $CO_{\mathcal{C}} : L^X \times M_{\perp_M} \longrightarrow L^X$ is defined as follows:*

$$CO_{\mathcal{C}}(\mu, r) = \prod_{i \in \Gamma} CO_{\mathcal{C}_i}(\pi_i^{\rightarrow}(\mu), r).$$

Then, $CO_{\mathcal{C}}$ is an L-fuzzy hull operator.

Proof. (1) From Theorem 2.4 (1), we have $CO_{\mathcal{C}_i}(\pi_i^{\rightarrow}(\chi_{\emptyset}), r)(x_i) = \perp_M$ for all $x_i \in X_i, i \in \Gamma$ and $r \in M_{\perp_M}$. Hence,

$$\begin{aligned} CO_{\mathcal{C}}(\chi_{\emptyset}, r)(x) &= \prod_{i \in \Gamma} CO_{\mathcal{C}_i}(\pi_i^{\rightarrow}(\chi_{\emptyset}), r)(x_i) \\ &= \perp_M \qquad \text{for all } x \in X. \end{aligned}$$

So, we obtain $CO_{\mathcal{C}}(\chi_{\emptyset}, r) = \chi_{\emptyset}$.

(2) Let $\mu \in L^X$. Then by definition of product fuzzy sets,

$$\mu(x) \leq \mu_i(x_i) \text{ for all } x \in X.$$

Therefore,

$$\mu(x) = \wedge_{i \in \Gamma} \mu(x) \leq \wedge_{i \in \Gamma} \mu_i(x_i) = \prod_{i \in \Gamma} \mu_i(x_i) \text{ for all } x \in X.$$

Put $\mu_i(x_i) = \pi_i^{\rightarrow}(\mu)(x_i)$ for all $x_i \in X_i$ where $\pi_i^{\rightarrow} : L^X \longrightarrow L^{X_i}$ is a projection and

$$\pi_i^{\rightarrow}(\mu)(x_i) = \vee\{\mu(x) : x \in X, \pi_i^{\rightarrow}(x) = x_i\}(\text{ Definition 1.3 [10]}).$$

We have

$$\mu(x) \leq \prod_{i \in \Gamma} \mu_i(x_i) = \prod_{i \in \Gamma} \pi_i^{\rightarrow}(\mu)(x_i). \tag{6}$$

For all $i \in \Gamma$ we have $\pi_i^{\rightarrow}(\mu)(x_i) \leq CO_{C_i}(\pi_i^{\rightarrow}(\mu), r)(x_i)$ for each $\mu \in L^X$. So, by equation (6) we obtain,

$$\begin{aligned} \mu(x) &\leq \prod_{i\in\Gamma} \pi_i^{\rightarrow}(\mu)(x_i) \\ &\leq \prod_{i\in\Gamma} CO_{C_i}(\pi_i^{\rightarrow}(\mu), r)(x_i) \\ &= CO_C(\mu, r)(x) \qquad \text{for all } x \in X. \end{aligned}$$

Hence, $\mu \leq CO_C(\mu, r)$.

(3) Suppose that $\mu(x_i) \leq \nu(x_i)$ for all $x_i \in X_i$. Then by Theorem 2.4 (3) it is obtained that

$$CO_{C_i}(\pi_i^{\rightarrow}(\mu), r)(x_i) \leq CO_{C_i}(\pi_i^{\rightarrow}(\nu), r)(x_i).$$

Therefore,

$$\begin{aligned} CO_C(\mu, r)(x) &= \prod_{i\in\Gamma} CO_{C_i}(\pi_i^{\rightarrow}(\mu), r)(x_i) \\ &\leq \prod_{i\in\Gamma} CO_{C_i}(\pi_i^{\rightarrow}(\nu), r)(x_i) \\ &= CO_C(\nu, r)(x) \qquad \text{for all } x \in X. \end{aligned}$$

(4) Let $r \leq s$. Then from Theorem 2.4 (4), we have

$$CO_{C_i}(\pi_i^{\rightarrow}(\mu), r)(x_i) \leq CO_{C_i}(\pi_i^{\rightarrow}(\mu), s)(x_i) \qquad \text{for all } x_i \in X_i.$$

Therefore,

$$\begin{aligned} CO_C(\mu, r)(x) &= \prod_{i\in\Gamma} CO_{C_i}(\pi_i^{\rightarrow}(\mu), r)(x_i) \\ &\leq \prod_{i\in\Gamma} CO_{C_i}(\pi_i^{\rightarrow}(\mu), s)(x_i) \\ &= CO_C(\mu, s)(x) \qquad \text{for all } x \in X. \end{aligned}$$

(5) It is enough to verify that $CO_C(CO_C(\mu, r), r) \leq CO_C(\mu, r)$. So taking any $\mu \in L^X$ and $r \in M_{\perp_M}$,

$$\begin{aligned} CO_C(CO_C(\mu, r), r) &= \prod_{i\in\Gamma} CO_{C_i}(\pi_i^{\rightarrow}(CO_C(\mu, r)), r) \\ &\leq \prod_{i\in\Gamma} CO_{C_i}(CO_{C_i}(\pi_i^{\rightarrow}(\mu), r), r) \\ &= \prod_{i\in\Gamma} CO_{C_i}(\pi_i^{\rightarrow}(\mu), r) = CO_C(\mu, r). \end{aligned}$$

(6) Let $\{\mu_\alpha : \alpha \in \Delta\} \subset L^X$ be nonempty and totally ordered by inclusion. Then, for $r \in M_{\perp_M}$, we have,

$$\begin{aligned} CO_C(\textstyle\bigvee_{\alpha\in\Delta} \mu_\alpha, r) &= \prod_{i\in\Gamma} CO_{C_i}(\pi_i^{\rightarrow}(\bigvee_{\alpha\in\Delta} \mu_\alpha), r) \\ &= \prod_{i\in\Gamma} CO_{C_i}(\bigvee_{\alpha\in\Delta} \pi_i^{\rightarrow}(\mu_\alpha), r) \\ &= \prod_{i\in\Gamma} \bigvee_{\alpha\in\Delta} CO_{C_i}(\pi_i^{\rightarrow}(\mu_\alpha), r) \quad \text{Theorem 2.4 (6)} \\ &= \bigvee_{\alpha\in\Delta} \prod_{i\in\Gamma} CO_{C_i}(\pi_i^{\rightarrow}(\mu_\alpha), r) \quad \text{Since } L \text{ is distributive lattices} \\ &= \bigvee_{\alpha\in\Delta} CO_C(\mu_\alpha, r). \end{aligned}$$

Hence, $CO_C(\bigvee_{\alpha\in\Delta}\mu_\alpha, r) = \bigvee_{\alpha\in\Delta} CO_C(\mu_\alpha, r)$. □

Theorem 3.14. *Let (X, C) be the product of $\{(X_i, C_i) : i \in \Gamma\}$. Then for each $i \in \Gamma$, $\pi_i : X \longrightarrow X_i$ is an (L, M)-fuzzy convex-to-convex function.*

Proof. By Proposition 3.13, we obtain $\pi_i^{\rightarrow}(\mu) \leq \pi_i^{\rightarrow}(CO_C(\mu, r))$. Therefore,

$$\begin{aligned} CO_{C_i}(\pi_i^{\rightarrow}(\mu), r) &\leq CO_{C_i}(\pi_i^{\rightarrow}(CO_C(\mu, r)), r) \\ &= \pi_i^{\rightarrow}(CO_C(\mu, r)) \quad \text{because } C_i(\pi_i^{\rightarrow}(CO_C(\mu, r))) \geq r. \end{aligned}$$

Hence from Proposition 2.8 (2) we obtain π_i is an (L, M)-fuzzy convex-to-convex function. □

Theorem 3.15. *The product space $\prod_{i\in\Gamma}(X_i, C_i, \mathcal{T}^i)$ is locally fuzzy convex if and only if $(X_i, C_i, \mathcal{T}^i)$ is locally fuzzy convex.*

Proof. Suppose that each X_i is locally fuzzy convex. Let x_t be a fuzzy point in $X = \prod_{i\in\Gamma} X_i$ and $\bigwedge_\alpha \pi_{i_\alpha}^{\leftarrow}(\mu_\alpha)$ be r-fuzzy neighborhood of x_t where $\pi_i : X \longrightarrow X_i$ is the projection map, μ_α is r-fuzzy open neighbourhood of $(x_{i_\alpha})_t$ in X_{i_α} for $\alpha = 1, 2, 3, ..., n$. Since X_{i_α} is locally fuzzy convex, then there exist r-convex fuzzy neighbourhood ν_α of $(x_{i_\alpha})_t$ such that

$$(x_{i_\alpha})_t \in \nu_\alpha \leq \mu_\alpha.$$

Which implies that

$$x_t \in \bigwedge_\alpha \pi_{i_\alpha}^{\leftarrow}(\nu_\alpha) \leq \bigwedge_\alpha \pi_{i_\alpha}^{\leftarrow}(\mu_\alpha).$$

Therefore, $\bigwedge_\alpha \pi_{i_\alpha}^{\leftarrow}(\nu_\alpha)$ is r-convex fuzzy neighbourhood of x_t. Hence X is locally fuzzy convex. On the other hand, let $(x_i)_t$ be a fuzzy point in X_i. Then we can find a fuzzy point $x_t \in X$ such that $\pi_i^{\rightarrow}(x_t) = (x_i)_t$. Let μ_i be r-fuzzy neighbourhood of $(x_i)_t \in X_i$. Then $\pi_i^{\leftarrow}(\mu_i)$ is r-fuzzy neighbourhood of $x_t \in X$. Since, X is locally fuzzy convex, there exists r-convex fuzzy neighourhood ν of x_t such that $\nu \leq \pi_i^{\leftarrow}(\mu_i)$. Since, π_i is an (L, M)-fuzzy convex-to-convex function, $\pi_i^{\rightarrow}(\nu)$ is r-convex fuzzy neighbourhood of $(x_i)_t \in X_i$ such that

$$(x_i)_t \in \pi_i^{\rightarrow}(\nu) \leq \mu_i.$$

Hence, X_i is locally fuzzy convex. □

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