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A Hilbert's Type Inequality With Three Parameters

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Abstract. In this paper, by introducing three parameters A, B, α and using the Euler-Maclaurin expansion for the Riemann zeta function, we establish an inequality of a weight coefficient. Using this inequality, we derive generalizations of a Hilbert's type inequality.

1. Introduction

If $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n \geq 0$, $b_n \geq 0$, for $n \geq 1, n \in N$ and $0 < \sum_{n=1}^{\infty} a_n^p < \infty$, $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left\{\sum_{n=1}^{\infty} a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} b_n^q\right\}^{\frac{1}{q}}, \tag{1}$$

and

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}} < pq\left\{\sum_{n=1}^{\infty} a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} b_n^q\right\}^{\frac{1}{q}}, \tag{2}$$

where the constant $\frac{\pi}{\sin\frac{\pi}{p}}$ and pq is best possible for each inequality respectively. Inequality (1) is Hardy-Hilbert's inequality. Inequality (2) is a Hilbert's type inequality [1].

In [4], [8] and [7], Krnić, Pečarić and Yang gave some generalization and reinforcement of inequality (1). In [2], Kuang and Debnath gave a reinforcement of inequality (2):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}} < \left\{\sum_{n=1}^{\infty}[pq - G(p,n)]a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}[pq - G(q,n)]b_n^q\right\}^{\frac{1}{q}} \tag{3}$$

where $G(r,n) = \frac{r + \frac{1}{3r} - \frac{4}{3}}{(2n+1)^{\frac{1}{r}}} > 0$ $(r = p, q)$.

In [6] ,Xi gave a generalization and reinforcement of inequalities (2) and (3):

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$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max(m^{\lambda},n^{\lambda})} < \left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{3qn^{\frac{q+\lambda-2}{q}}}\right]n^{1-\lambda}a_n^p\right\}^{\frac{1}{p}} \times\left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{3pn^{\frac{p+\lambda-2}{p}}}\right]n^{1-\lambda}b_n^q\right\}^{\frac{1}{q}}, \tag{4}$$

where $\kappa(\lambda)=\frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)}>0,\ 2-\min\{p,q\}<\lambda\le 2$.

In[5] and [11], Xi and Zhang gave two generalizations of inequalities (4) :

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^{\lambda}+A,n^{\lambda}+B\}} < \left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{n^{\frac{q+\lambda-2}{q}}}\left(\frac{1}{3q}-\frac{B}{1+B}\right)\right]n^{1-\lambda}a_n^p\right\}^{\frac{1}{p}} \times\left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{n^{\frac{p+\lambda-2}{p}}}\left(\frac{1}{3p}-\frac{A}{1+A}\right)\right]n^{1-\lambda}b_n^q\right\}^{\frac{1}{q}} \tag{5}$$

and

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^{\lambda},n^{\lambda}\}+\alpha} < \left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{n^{\frac{q+\lambda-2}{q}}}\left(\frac{1}{3q}-\frac{\alpha}{1+\alpha}\right)\right]n^{1-\lambda}a_n^p\right\}^{\frac{1}{p}} \times\left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{n^{\frac{p+\lambda-2}{p}}}\left(\frac{1}{3p}-\frac{\alpha}{1+\alpha}\right)\right]n^{1-\lambda}b_n^q\right\}^{\frac{1}{q}}, \tag{6}$$

where $\kappa(\lambda)=\frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)}>0,\ 2-\min\{p,q\}<\lambda\le 2,\ 0\le A\le B\le\min\{\frac{1}{3p-1},\frac{1}{3q-1}\},\ 0\le\alpha\le\min\{\frac{1}{3p-1},\frac{1}{3q-1}\}$.

In this paper, by introducing three parameters A,B,α and using the Euler-Maclaurin expansion for the Riemann zeta function, we establish an inequality for a weight coefficient. Using this inequality, we derive a generalization of inequalities (5) and (6).

2. A Lemma

First, we need the following formula of the Riemann-ζ function (see [3], [10] and [9]):

$$\zeta(\sigma) = \sum_{k=1}^{n}\frac{1}{k^{\sigma}}-\frac{n^{1-\sigma}}{1-\sigma}-\frac{1}{2n^{\sigma}}-\sum_{k=1}^{l-1}\frac{B_{2k}}{2k}\begin{pmatrix}-\sigma\\2k-1\end{pmatrix}\frac{1}{n^{\sigma+2k-1}} -\frac{B_{2l}}{2l}\begin{pmatrix}-\sigma\\2l-1\end{pmatrix}\frac{\varepsilon}{n^{\sigma+2l-1}}, \tag{7}$$

where $\sigma>0$, $\sigma\ne 1$, $n,\ l\ge 1$, $n,\ l\in N$, $0<\varepsilon=\varepsilon(\sigma,l,n)<1$. The numbers $B_1=-1/2$, $B_2=1/6$, $B_3=0$, $B_4=-1/30$, $\cdots$ are Bernoulli numbers. In particular, $\zeta(\sigma)=\sum_{k=1}^{\infty}\frac{1}{k^{\sigma}}$ $(\sigma>1)$.

Since $\zeta(0)=-1/2$, then the formula of the Riemann-ζ function (7) is also true for $\sigma=0$.

Lemma 2.1. *If $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $2 - min\{p, q\} < \lambda \leq 2$, $A, B, \alpha \geq 0$, and $A + \alpha \leq B + \alpha \leq \min\{\frac{1}{3p-1}, \frac{1}{3q-1}\}$, $n \geq 1$ and $n \in N$, then*

$$\begin{aligned} \omega(n, \lambda, p, A, B, \alpha) &= \sum_{k=1}^{\infty} \frac{1}{\max\{k^\lambda + A, n^\lambda + B\} + \alpha} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} \\ &< n^{1-\lambda}\left[\kappa(\lambda) - \frac{1}{n^{\frac{p+\lambda-2}{p}}}\left(\frac{1}{3p} - \frac{A+\alpha}{1+A+\alpha}\right)\right], \end{aligned} \tag{8}$$

and

$$\begin{aligned} \omega(n, \lambda, q, A, B, \alpha) &= \sum_{k=1}^{\infty} \frac{1}{\max\{k^\lambda + B, n^\lambda + A\} + \alpha} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{q}} \\ &< n^{1-\lambda}\left[\kappa(\lambda) - \frac{1}{n^{\frac{q+\lambda-2}{q}}}\left(\frac{1}{3q} - \frac{B+\alpha}{1+B+\alpha}\right)\right], \end{aligned} \tag{9}$$

where $\kappa(\lambda) = \frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)}$. When $\lambda = 1$, we have following the stronger inequality:

$$\begin{aligned} \omega(n, 1, p, A, B, \alpha) &= \sum_{k=1}^{\infty} \frac{1}{\max\{k + A, n + B\}} \left(\frac{n}{k}\right)^{\frac{1}{p}} \\ &< \left[pq - \frac{1}{n^{\frac{1}{q}}}\left(\frac{12q^2 + 3q + 5p}{12pq} - \frac{A+\alpha}{1+A+\alpha}\right)\right], \end{aligned} \tag{10}$$

and

$$\begin{aligned} \omega(n, 1, q, A, B, \alpha) &= \sum_{k=1}^{\infty} \frac{1}{\max\{k + A, n + B\}} \left(\frac{n}{k}\right)^{\frac{1}{q}} \\ &< \left[pq - \frac{1}{n^{\frac{1}{p}}}\left(\frac{12p^2 + 3p + 5q}{12pq} - \frac{B+\alpha}{1+B+\alpha}\right)\right]. \end{aligned} \tag{11}$$

Proof. Equalities (8) and (9) define the weight coefficient. When $2 - min\{p, q\} < \lambda \leq 2$, taking $\sigma = \frac{2-\lambda}{p} \geq 0$, $l = 1$, in (7), we obtain

$$\zeta\left(\frac{2-\lambda}{p}\right) = \sum_{k=1}^{n} \frac{1}{k^{\frac{2-\lambda}{p}}} - \frac{pn^{\frac{p+\lambda-2}{p}}}{p+\lambda-2} - \frac{1}{2n^{\frac{2-\lambda}{p}}} + \frac{2-\lambda}{12pn^{1+\frac{2-\lambda}{p}}}\varepsilon_1, \tag{12}$$

where $0 < \varepsilon_1 < 1$.

Taking $\sigma = \frac{2}{p} + \frac{\lambda}{q}$, $l = 1$, we obtain

$$\zeta\left(\frac{2}{p} + \frac{\lambda}{q}\right) = \sum_{k=1}^{n-1} \frac{1}{k^{\frac{2}{p}+\frac{\lambda}{q}}} + \frac{qn^{-\frac{q+\lambda-2}{q}}}{q+\lambda-2} + \frac{1}{2n^{\frac{2}{p}+\frac{\lambda}{q}}} + \frac{p\lambda + 2q}{12pqn^{1+\frac{2}{p}+\frac{\lambda}{q}}}\varepsilon_2, \tag{13}$$

where $0 < \varepsilon_2 < 1$.

In addition,

$$
\begin{aligned}
\omega(n,\lambda,p,A,B,\alpha) &= \sum_{k=1}^{\infty}\frac{1}{\max\{k^\lambda+A,n^\lambda+B\}+\alpha}\left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}}\\
&= \sum_{k=1}^{n}\frac{1}{\max\{k^\lambda+A,n^\lambda+B\}+\alpha}\left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} - \frac{1}{n^\lambda+A+\alpha}\\
&\quad + \sum_{k=n}^{\infty}\frac{1}{\max\{k^\lambda+A,n^\lambda+B\}+\alpha}\left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}}\\
&= \sum_{k=1}^{n}\frac{1}{n^\lambda+B+\alpha}\left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} - \frac{1}{n^\lambda+A+\alpha} + \sum_{k=n}^{\infty}\frac{1}{k^\lambda+A+\alpha}\left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}}\\
&\le \sum_{k=1}^{n}\frac{1}{n^\lambda}\left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} - \frac{1}{n^\lambda+A+\alpha} + \sum_{k=n}^{\infty}\frac{1}{k^\lambda}\left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}}\\
&= \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}}\sum_{k=1}^{n}\frac{1}{k^{\frac{2-\lambda}{p}}} - \frac{1}{n^\lambda+A+\alpha} + n^{\frac{2-\lambda}{p}}\sum_{k=n}^{\infty}\frac{1}{k^{\frac{2}{p}+\frac{\lambda}{q}}}.
\end{aligned}
$$

By (12) and (13)

$$
\begin{aligned}
\omega(n,\lambda,p,A,B,\alpha) &< \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}}\left[\zeta\left(\frac{2-\lambda}{p}\right)+\frac{pn^{\frac{p+\lambda-2}{p}}}{p+\lambda-2}+\frac{1}{2n^{\frac{2-\lambda}{p}}}\right] - \frac{1}{n^\lambda+A+\alpha}\\
&\quad + n^{\frac{2-\lambda}{p}}\left[\frac{qn^{-\frac{q+\lambda-2}{q}}}{q+\lambda-2}+\frac{1}{2n^{\frac{2}{p}+\frac{\lambda}{q}}}+\frac{p\lambda+2q}{12p\,qn^{1+\frac{2}{p}+\frac{\lambda}{q}}}\right]\\
&= \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}}\zeta\left(\frac{2-\lambda}{p}\right)+\frac{pn^{\,1-\lambda}}{p+\lambda-2}+\frac{1}{2n^\lambda}-\frac{1}{n^\lambda+A+\alpha}+\frac{qn^{\,1-\lambda}}{q+\lambda-2}\\
&\quad +\frac{1}{2n^\lambda}+\frac{p\lambda+2q}{12p\,qn^{1+\lambda}}\\
&= \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}}\zeta\left(\frac{2-\lambda}{p}\right)+\frac{pq\,\lambda\;n^{\,1-\lambda}}{(p+\lambda-2)(q+\lambda-2)}+\frac{p\lambda+2q}{12pqn^{1+\lambda}}+\frac{A+\alpha}{n^\lambda(n^\lambda+A+\alpha)}\\
&= n^{1-\lambda}\left\{\kappa(\lambda)-\frac{1}{n^{\frac{p+\lambda-2}{p}}}\left[-\zeta\left(\frac{2-\lambda}{p}\right)-\frac{p\lambda+2q}{12p\,qn^{\frac{p-\lambda+2}{p}}}-\frac{A+\alpha}{n^{\frac{2-\lambda}{p}}(n^\lambda+A+\alpha)}\right]\right\}.
\end{aligned}
$$

In (12), taking $n=1$, by $2-\min\{p,q\}<\lambda\le 2$, we obtain

$$
\begin{aligned}
\zeta\left(\frac{2-\lambda}{p}\right) &= 1-\frac{p}{p+\lambda-2}-\frac{1}{2}+\frac{(2-\lambda)\varepsilon_1}{12p}\\
&< \frac{1}{2}-\frac{p}{p+\lambda-2}+\frac{2-\lambda}{12p}\\
&= -\frac{(\lambda-2-3p)(\lambda-2-2p)}{12p(p+\lambda-2)}\\
&< 0.
\end{aligned}
$$

So for $n\ge 1$, $n\in N$, $2-\min\{p,q\}<\lambda\le 2$, $A,B,\alpha\ge 0$, and $A+\alpha\le B+\alpha\le\min\{\frac{1}{3p-1},\frac{1}{3q-1}\}$, we have

$$
\begin{aligned}
-\zeta\left(\frac{2-\lambda}{p}\right) &- \frac{p\lambda+2q}{12pqn^{\frac{p-\lambda+2}{p}}} - \frac{A+\alpha}{n^{\frac{2-\lambda}{p}}(n^{\lambda}+A+\alpha)} \\
&> \frac{(\lambda-2-3p)(\lambda-2-2p)}{12p(p+\lambda-2)} - \frac{p\lambda+2q}{12pq} - \frac{\alpha+A}{1+\alpha+A} \\
&= \frac{q(\lambda-2-3p)(\lambda-2-2p)-(p\lambda+2q)(p+\lambda-2)}{12pq(p+\lambda-2)} - \frac{A+\alpha}{1+A+\alpha} \\
&> \frac{-p(p\lambda+2q)+6p^2q}{12pq(p+\lambda-2)} - \frac{A+\alpha}{1+A+\alpha} \\
&\geq \frac{-(2p+2q)+6pq}{12q(p+\lambda-2)} - \frac{A+\alpha}{1+A+\alpha} \\
&> \frac{1}{3(p+\lambda-2)} - \frac{A+\alpha}{1+A+\alpha} \\
&> \frac{1}{3p} - \frac{A+\alpha}{1+A+\alpha} \\
&\geq 0.
\end{aligned}
$$

Using the last result and the inequality for $\omega(n,\lambda,p,A,B,\alpha)$ above, we obtain (8).

When $\lambda = 1$, we have

$$
\begin{aligned}
-\zeta\left(\frac{2-\lambda}{p}\right) &- \frac{p\lambda+2q}{12pqn^{\frac{p-\lambda+2}{p}}} - \frac{A+\alpha}{n^{\frac{2-\lambda}{p}}(n^{\lambda}+A+\alpha)} \\
&> \frac{q(\lambda-2)^2+(p\lambda+5pq+2q)(2-\lambda)-p(p\lambda+2q)+6p^2q}{12pq(p+\lambda-2)} - \frac{A+\alpha}{1+A+\alpha} \\
&= \frac{p+3q-p^2+3pq+6p^2q}{12pq(p-1)} - \frac{A+\alpha}{1+A+\alpha} \\
&= \frac{5p^2+10p+12q}{12pq(p-1)} - \frac{A+\alpha}{1+A+\alpha} \\
&= \frac{(5p^2+10p+12q)(q-1)}{12pq} - \frac{A+\alpha}{1+A+\alpha} \\
&= \frac{12q^2+3q+5p}{12pq} - \frac{A+\alpha}{1+A+\alpha}.
\end{aligned}
$$

Using the last result and the inequality for $\omega(n,\lambda,p,A,B,\alpha)$ above, we obtain (10).

In a similar way, we have

$$
\begin{aligned}
\omega(m,\lambda,q,A,B,\alpha) &= \sum_{k=1}^{\infty} \frac{1}{\max\{m^{\lambda}+A,k^{\lambda}+B\}+\alpha}\left(\frac{m}{k}\right)^{\frac{2-\lambda}{q}} \\
&= \sum_{k=1}^{n} \frac{1}{\max\{m^{\lambda}+A,k^{\lambda}+B\}+\alpha}\left(\frac{m}{k}\right)^{\frac{2-\lambda}{q}} - \frac{1}{m^{\lambda}+B+\alpha} \\
&\quad + \sum_{k=m}^{\infty} \frac{1}{\max\{m^{\lambda}+A,k^{\lambda}+B\}+\alpha}\left(\frac{m}{k}\right)^{\frac{2-\lambda}{q}}
\end{aligned}
$$

$$= \sum_{k=1}^{m} \frac{1}{m^\lambda + A + \alpha}\left(\frac{m}{k}\right)^{\frac{2-\lambda}{q}} - \frac{1}{m^\lambda + B + \alpha} + \sum_{k=m}^{\infty} \frac{1}{k^\lambda + B + \alpha}\left(\frac{m}{k}\right)^{\frac{2-\lambda}{q}}$$
$$\leq \sum_{k=1}^{m} \frac{1}{m^\lambda}\left(\frac{n}{k}\right)^{\frac{2-\lambda}{q}} - \frac{1}{m^\lambda + B + \alpha} + \sum_{k=m}^{\infty} \frac{1}{k^\lambda}\left(\frac{m}{k}\right)^{\frac{2-\lambda}{q}}$$
$$= \frac{1}{m^{\frac{(q+1)\lambda-2}{q}}} \sum_{k=1}^{m} \frac{1}{k^{\frac{2-\lambda}{q}}} - \frac{1}{m^\lambda + B + \alpha} + m^{\frac{2-\lambda}{q}} \sum_{k=m}^{\infty} \frac{1}{k^{\frac{2}{q}+\frac{\lambda}{p}}}.$$

By (12) and (13)

$$\omega(m,\lambda,q,A,B,\alpha) < \frac{1}{m^{\frac{(q+1)\lambda-2}{q}}}\left[\zeta\left(\frac{2-\lambda}{q}\right) + \frac{qm^{\frac{q+\lambda-2}{q}}}{q+\lambda-2} + \frac{1}{2m^{\frac{2-\lambda}{q}}}\right] - \frac{1}{m^\lambda + B + \alpha}$$
$$+ m^{\frac{2-\lambda}{q}}\left[\frac{pm^{-\frac{p+\lambda-2}{p}}}{p+\lambda-2} + \frac{1}{2m^{\frac{2}{q}+\frac{\lambda}{p}}} + \frac{q\lambda+2p}{12qpm^{1+\frac{2}{q}+\frac{\lambda}{p}}}\right]$$
$$= m^{1-\lambda}\left\{\kappa(\lambda) - \frac{1}{m^{\frac{q+\lambda-2}{q}}}\left[-\zeta\left(\frac{2-\lambda}{q}\right) - \frac{q\lambda+2p}{12qpm^{\frac{q-\lambda+2}{q}}} - \frac{B+\alpha}{m^{\frac{2-\lambda}{q}}(m^\lambda + B + \alpha)}\right]\right\}.$$

Since for $m \geq 1$, $m \in N$, $2 - \min\{p,q\} < \lambda \leq 2$, $A, B, \alpha \geq 0$, and $A + \alpha \leq B + \alpha \leq \min\{\frac{1}{3p-1}, \frac{1}{3q-1}\}$, we have

$$-\zeta\left(\frac{2-\lambda}{q}\right) - \frac{q\lambda+2p}{12qpm^{\frac{q-\lambda+2}{q}}} - \frac{B+\alpha}{m^{\frac{2-\lambda}{q}}(m^\lambda + B + \alpha)}$$
$$> \frac{(\lambda-2-3q)(\lambda-2-2q)}{12q(q+\lambda-2)} - \frac{q\lambda+2p}{12pq} - \frac{B+\alpha}{1+B+\alpha}$$
$$> \frac{1}{3q} - \frac{B+\alpha}{1+B+\alpha}$$
$$\geq 0.$$

Using the last result and the inequality for $\omega(m,\lambda,q,A,B,\alpha)$ above, we obtain (9).

When $\lambda = 1$, we have

$$-\zeta\left(\frac{2-\lambda}{q}\right) - \frac{q\lambda+2p}{12qpm^{\frac{q-\lambda+2}{q}}} - \frac{B+\alpha}{m^{\frac{2-\lambda}{q}}(m^\lambda + B + \alpha)}$$
$$> \frac{p(\lambda-2)^2 + (q\lambda + 5pq + 2p)(2-\lambda) - q(q\lambda+2p) + 6q^2p}{12pq(q+\lambda-2)} - \frac{B+\alpha}{1+B+\alpha}$$
$$= \frac{12p^2 + 3p + 5q}{12pq} - \frac{B+\alpha}{1+B+\alpha}.$$

Using the last result and the inequality for $\omega(m,\lambda,q,A,B,\alpha)$ above, we obtain (11). □

3. Main Results

Theorem 3.1. *If* $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $2 - \min\{p, q\} < \lambda \leq 2, A, B, \alpha \geq 0$, *and* $A + \alpha \leq B + \alpha \leq \min\{\frac{1}{3p-1}, \frac{1}{3q-1}\}$, $a_n \geq 0$, $b_n \geq 0$, *for* $n \geq 1, n \in N$ *and* $0 < \sum_{n=1}^{\infty} a_n^p < \infty$, $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, *then*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda + A, n^\lambda + B\} + \alpha} < \left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda) - \frac{1}{n^{\frac{q+\lambda-2}{q}}}\left(\frac{1}{3q} - \frac{B+\alpha}{1+B+\alpha}\right)\right] n^{1-\lambda} a_n^p\right\}^{\frac{1}{p}} \times\left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda) - \frac{1}{n^{\frac{p+\lambda-2}{p}}}\left(\frac{1}{3p} - \frac{A+\alpha}{1+A+\alpha}\right)\right] n^{1-\lambda} b_n^q\right\}^{\frac{1}{q}}, \tag{14}$$

and

$$\sum_{m=1}^{\infty} m^{(p-1)(\lambda-1)}\left(\sum_{n=1}^{\infty}\frac{a_n}{\max\{m^\lambda + A, n^\lambda + B\} + \alpha}\right)^p < \kappa(\lambda)^{p-1}\sum_{n=1}^{\infty}\left[\kappa(\lambda) - \frac{1}{n^{\frac{q+\lambda-2}{q}}}\left(\frac{1}{3q} - \frac{A+\alpha}{1+A+\alpha}\right)\right] n^{1-\lambda} a_n^p, \tag{15}$$

where $\kappa(\lambda) = \frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)} > 0$. *When* $\lambda = 1$, *we have*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m + A, n + B\} + \alpha} < \left\{\sum_{n=1}^{\infty}\left[pq - \frac{1}{n^{\frac{1}{p}}}\left(\frac{12p^2 + 3p + 5q}{12pq} - \frac{B+\alpha}{1+B+\alpha}\right)\right] \times a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty}\left[pq - \frac{1}{n^{\frac{1}{q}}}\left(\frac{12q^2 + 3q + 5p}{12pq} - \frac{A+\alpha}{1+A+\alpha}\right)\right] b_n^q\right\}^{\frac{1}{q}}. \tag{16}$$

Proof. By Hölder's inequality, we have

$$\begin{aligned}
\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda + A, n^\lambda + B\} + \alpha} &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\frac{a_m}{(\max\{m^\lambda + A, n^\lambda + B\} + \alpha)^{\frac{1}{p}}}\left(\frac{m}{n}\right)^{\frac{2-\lambda}{pq}}\right] \\
&\times\left[\frac{b_n}{(\max\{m^\lambda + A, n^\lambda + B\} + \alpha)^{\frac{1}{q}}}\left(\frac{n}{m}\right)^{\frac{2-\lambda}{pq}}\right] \\
&\leq \left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\frac{a_m^p}{\max\{m^\lambda + A, n^\lambda + B\} + \alpha}\left(\frac{m}{n}\right)^{\frac{2-\lambda}{q}}\right]\right\}^{\frac{1}{p}} \\
&\times\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\frac{b_n^q}{\max\{m^\lambda + A, n^\lambda + B\} + \alpha}\left(\frac{n}{m}\right)^{\frac{2-\lambda}{p}}\right]\right\}^{\frac{1}{q}} \\
&= \left\{\sum_{m=1}^{\infty}\omega(m, \lambda, q, A, B, \alpha) a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\omega(n, \lambda, p, A, B, \alpha) b_n^q\right\}^{\frac{1}{q}}.
\end{aligned}$$

By (8), (9), (10) and (11), we obtain (14) and (16).

By Hölder's inequality and Lemma 2.1, we have

$$\sum_{n=1}^{\infty}\frac{a_n}{\max\{m^\lambda+A,n^\lambda+B\}+\alpha}=\sum_{n=1}^{\infty}\left[\frac{1}{(\max\{m^\lambda+A,n^\lambda+B\}+\alpha)^{\frac{1}{p}}}\left(\frac{n}{m}\right)^{\frac{2-\lambda}{pq}}\right.$$
$$\left.\times a_n\frac{1}{(\max\{m^\lambda+A,n^\lambda+B\}+\alpha)^{\frac{1}{q}}}\left(\frac{m}{n}\right)^{\frac{2-\lambda}{pq}}\right]$$
$$\leq\left\{\sum_{n=1}^{\infty}\left[\frac{1}{\max\{m^\lambda+A,n^\lambda+B\}+\alpha}\left(\frac{n}{m}\right)^{\frac{2-\lambda}{q}}a_n^p\right]\right\}^{\frac{1}{p}}$$
$$\times\left\{\sum_{n=1}^{\infty}\left[\frac{1}{\max\{m^\lambda+A,n^\lambda+B\}+\alpha}\left(\frac{m}{n}\right)^{\frac{2-\lambda}{p}}\right]\right\}^{\frac{1}{q}}$$
$$=\left\{\sum_{n=1}^{\infty}\left[\frac{1}{\max\{m^\lambda+A,n^\lambda+B\}+\alpha}\left(\frac{n}{m}\right)^{\frac{2-\lambda}{q}}a_n^p\right]\right\}^{\frac{1}{p}}[\omega(m,\lambda,p,A,B,\alpha)]^{\frac{1}{q}}$$
$$<\left\{\sum_{n=1}^{\infty}\left[\frac{1}{\max\{m^\lambda+A,n^\lambda+B\}+\alpha}\left(\frac{n}{m}\right)^{\frac{2-\lambda}{q}}a_n^p\right]\right\}^{\frac{1}{p}}[m^{1-\lambda}\kappa(\lambda)]^{\frac{1}{q}}.$$

So

$$\sum_{m=1}^{\infty}m^{(p-1)(\lambda-1)}\left(\sum_{n=1}^{\infty}\frac{a_n}{\max\{m^\lambda+A,n^\lambda+B\}+\alpha}\right)^p$$
$$<\kappa(\lambda)^{p-1}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{1}{\max\{m^\lambda+A,n^\lambda+B\}+\alpha}\left(\frac{n}{m}\right)^{\frac{2-\lambda}{q}}a_n^p\right]$$
$$<\kappa(\lambda)^{p-1}\sum_{n=1}^{\infty}\omega(n,\lambda,q,A,B,\alpha)a_n^p.$$

By Lemma 2.1, the proof of the theorem is completed. □

In inequality (16), taking $p=q=2$, we have:

Corollary 3.2. *Let $a_n\geq 0$, $b_n\geq 0$, $A,B,\alpha\geq 0$, $A+\alpha\leq B+\alpha\leq\frac{1}{5}$, and $0<\sum_{n=1}^{\infty}a_n^2<\infty$, $0<\sum_{n=1}^{\infty}b_n^2<\infty$, then*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_mb_n}{\max\{m+A,n+B\}+\alpha}<4\left\{\sum_{n=1}^{\infty}\left[1-\frac{1}{3\sqrt{n}}\left(1-\frac{3B+3\alpha}{4+4B+4\alpha}\right)\right]a_n^2\right\}^{\frac{1}{2}}$$
$$\times\left\{\sum_{n=1}^{\infty}\left[1-\frac{1}{3\sqrt{n}}\left(1-\frac{3A+3\alpha}{4+4A+4\alpha}\right)\right]b_n^2\right\}^{\frac{1}{2}}. \quad (17)$$

In inequality (14), taking $\alpha=0$, we obtain:

Corollary 3.3. *If $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, $2-\min\{p,q\}<\lambda\leq 2$, and $0\leq A\leq B\leq\min\{\frac{1}{3p-1},\frac{1}{3q-1}\}$, $a_n\geq 0$, $b_n\geq 0$, for $n\geq 1, n\in N$ and $0<\sum_{n=1}^{\infty}a_n^p<\infty$, $0<\sum_{n=1}^{\infty}b_n^q<\infty$, then*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_mb_n}{\max\{m^\lambda+A,n^\lambda+B\}}<\left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{n^{\frac{q+\lambda-2}{q}}}\left(\frac{1}{3q}-\frac{B}{1+B}\right)\right]n^{1-\lambda}a_n^p\right\}^{\frac{1}{p}}$$
$$\times\left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{n^{\frac{p+\lambda-2}{p}}}\left(\frac{1}{3p}-\frac{A}{1+A}\right)\right]n^{1-\lambda}b_n^q\right\}^{\frac{1}{q}}, \quad (18)$$

In inequality (14), taking $A = 0, B = 0$, we obtain:

Corollary 3.4. *If* $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $2 - \min\{p, q\} < \lambda \leq 2$, $0 \leq \alpha \leq \min\{\frac{1}{3p-1}, \frac{1}{3q-1}\}$, $a_n \geq 0$, $b_n \geq 0$, *for* $n \geq 1, n \in N$ *and* $0 < \sum\limits_{n=1}^{\infty} a_n^p < \infty$, $0 < \sum\limits_{n=1}^{\infty} b_n^q < \infty$, *then*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda, n^\lambda\} + \alpha} < \left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda) - \frac{1}{n^{\frac{q+\lambda-2}{q}}}\left(\frac{1}{3q} - \frac{\alpha}{1+\alpha}\right)\right] n^{1-\lambda} a_n^p\right\}^{\frac{1}{p}} \times\left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda) - \frac{1}{n^{\frac{p+\lambda-2}{p}}}\left(\frac{1}{3p} - \frac{\alpha}{1+\alpha}\right)\right] n^{1-\lambda} b_n^q\right\}^{\frac{1}{q}}. \quad (19)$$

Apparently, inequality (18) is inequality (5). inequality (19) is inequality (6). So, inequality (14) is a generalization of inequalities (5) and (6).

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