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Existence and Asymptotic Behavior of Intermediate Type of Positive Solutions of Fourth-Order Nonlinear Differential Equations

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Abstract. Under the assumptions that p and q are regularly varying functions satisfying conditions

$$\int_a^\infty \frac{t}{p(t)^{\frac{1}{\alpha}}}\,dt < \infty \quad \text{and} \quad \int_a^\infty \left(\frac{t}{p(t)}\right)^{\frac{1}{\alpha}} dt = \infty$$

existence and asymptotic form of regularly varying intermediate solutions are studied for a fourth-order quasilinear differential equation

$$\left(p(t)|x''(t)|^{\alpha-1}\,x''(t)\right)'' + q(t)|x(t)|^{\beta-1}\,x(t) = 0, \quad \alpha > \beta > 0.$$

It is shown that under certain integral conditions there exist two types of intermediate solutions which according to their asymptotic behavior is to be divided into six mutual distinctive classes, while asymptotic behavior of each member of any of these classes is governed by a unique explicit law.

1. Introduction

This paper is concerned with positive solutions of fourth-order quasilinear differential equations of the form

$$\text{(E)} \qquad (p(t)\Phi_\alpha(x''(t)))'' + q(t)\Phi_\beta(x(t)) = 0, \;\; t \ge a,$$

where $\Phi_\gamma(x) = |x|^\gamma \operatorname{sgn} x$, with $\gamma > 0$, α, β are positive constants such that $\alpha > \beta$ and p, q are positive continuous functions on $[a, \infty)$, $a > 0$ satisfying

$$(P_1) \quad \int_a^\infty \frac{t}{p(t)^{\frac{1}{\alpha}}}\,dt < \infty, \qquad (P_2) \quad \int_a^\infty \left(\frac{t}{p(t)}\right)^{\frac{1}{\alpha}} dt = \infty. \tag{1.1}$$

We emphasize that if (1.1) holds, then $0 < \alpha < 1$.

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By a solution of (E) we mean a twice continuously differentiable function x on $[T, \infty)$, $T \geq a$, such that $p\Phi_\alpha(x'')$ is twice continuously differentiable on $[T, \infty)$ and satisfies the equation (E) at every point of $[T, \infty)$. A solution x of (E) is said to be *nonoscillatory* if $x(t) \neq 0$ for all large t. In other words, a solution x of (E) is nonoscollatory if it is eventually positive or eventually negative. If x is a solution of (E), then so does $-x$ and thus, there is no loss of generality in assuming that a nonoscillatory solution of (E) is eventually positive.

Throughout this paper use is made of the symbol $\sim$ to denote the asymptotic equivalence of two positive functions, i.e.,

$$f(t) \sim g(t), \quad t \to \infty \qquad \Longleftrightarrow \qquad \lim_{t\to\infty} \frac{g(t)}{f(t)} = 1,$$

and the symbol $\prec$ to denote the dominance relation between two positive functions in the sense that

$$f(t) \prec g(t), \quad t \to \infty \qquad \Longleftrightarrow \qquad \lim_{t\to\infty} \frac{g(t)}{f(t)} = \infty.$$

The oscillatory and asymptotic behavior of nonoscillatory solutions of (E) were considered by Kamo and Usami [5], Kusano and Tanigawa [12], Kusano, Manojlović and Tanigawa [10], Manojlović and Milošević [15], Naito and Wu [16, 17], Wu [21, 22]. The qualitative behavior of solutions of (E) are described by means of the integrals

$$P_1 = \int_a^\infty \frac{t}{p(t)^{\frac{1}{\alpha}}}\, dt, \quad P_2 = \int_a^\infty \Big(\frac{t}{p(t)}\Big)^{\frac{1}{\alpha}} dt, \quad P_3 = \int_a^\infty t\Big(\frac{t}{p(t)}\Big)^{\frac{1}{\alpha}} dt\,.$$

In [16, 17, 21, 22] it is assumed that p satisfies $P_1 = \infty$, $P_2 = \infty$, while in [5] it is assumed that p satisfies $P_1 < \infty$, $P_2 = \infty$, and in [15] it is assumed that p satisfies $P_1 < \infty$, $P_2 < \infty$. On the other hand, Kusano, Manojlović and Tanigawa [10, 12] have considered the case $P_3 < \infty$.

The equation (E) under conditions (P_1) and (P_2) has been already considered in [5], where the main objective was to establish necessary and sufficient conditions for oscillation of all solutions. For that cause, necessary and sufficient conditions for the existence of positive solutions satisfying

$$(S_1) \qquad x(t) \sim c\,\varphi(t), \ \ t \to \infty, \quad \text{with } c > 0, \ \varphi(t) = \int_t^\infty \frac{s-t}{p(s)^{\frac{1}{\alpha}}}\, ds;$$

$$(S_4) \qquad x(t) \sim c\,\psi(t), \ \ t \to \infty, \quad \text{with } c > 0, \ \psi(t) = \int_a^t (t-s)\Big(\frac{s}{p(s)}\Big)^{\frac{1}{\alpha}} ds$$

were given.

Theorem 1.1. *(i)* [5, Theorem 4.7] *Equation* (*E*) *has an eventually positive solution x satisfying* (S_1) *if and only if*

$$J_1 = \int_a^\infty tq(t)\varphi(t)^\beta\, dt < \infty. \tag{1.2}$$

(ii) [5, Theorem 4.8] *Equation* (*E*) *has an eventually positive solution x satisfying* (S_4) *if and only if*

$$J_4 = \int_a^\infty q(t)\psi(t)^\beta dt < \infty. \tag{1.3}$$

The aim of this paper is to proceed further and to obtain a more detailed information on the asymptotic behavior of positive solutions of the equation (E) under conditions (1.1). First, detailed analysis is made on the structure of positive solutions of (E) by showing that besides solutions with asymptotic behavior described by (S_1) and (S_4), there exist two other types of positive solutions

$$(S_2) \quad x(t) \sim c, \ \ t \to \infty; \qquad\qquad (S_3) \quad x(t) \sim c\,t, \ \ t \to \infty\,.$$

Necessary and sufficient conditions for the existence of these two types of solutions will be established in Section 2. Thus, in the classification of solutions of (E) under the condition (1.1), a crucial role is played by the four functions

$$\varphi(t) = \int_t^\infty \frac{s-t}{p(s)^{\frac{1}{\alpha}}}\, ds, \quad 1, \quad t, \quad \psi(t) = \int_a^t (t-s)\Big(\frac{s}{p(s)}\Big)^{\frac{1}{\alpha}} ds, \tag{1.4}$$

which are particular solutions of the unperturbed differential equation $(p(t)\Phi_\alpha(x''(t)))'' = 0$ and satisfy the dominance relation $\varphi(t) \prec 1 \prec t \prec \psi_(t), \quad t \to \infty$. It is therefore expected that the equation (E) also possesses intermediate type of solutions x satisfying either

$$(I_1) \quad \varphi(t) \prec x(t) \prec 1, \;\; t \to \infty, \quad \text{or} \quad (I_2) \quad t \prec x(t) \prec \psi(t), \;\; t \to \infty,$$

and the accuracy of this assertion will be also shown in Section 2.

Afterwards, the main goal is to establish the precise asymptotic formula for these two types of intermediate solutions in the framework of regular variation. Asymptotic analysis of differential equations by the means of regularly varying functions was initiated by the monograph of Marić [4]. Its recent development has shown that with the help of theory of regular variation, introduced by Karamata in 1930., it is possible to get a complete asymptotic analysis of nonlinear differential equations with regularly varying coefficients or generalized regularly varying coefficients, introduced by Jaroš and Kusano in [2]; see [3, 6–9, 11, 12, 14, 15, 18, 20]. We consider the equation (E) with generalized regularly varying p and q, showing that each of two classes of its intermediate generalized regularly varying solutions of type (I_1) and (I_2) can be divided into three disjoint subclasses according to their asymptotic behavior at infinity. Necessary and sufficient conditions for the existence of solutions belonging to each of these six types of solution will be established in Section 3. Moreover, the asymptotic behavior of solutions contained in each of the six subclasses will be delivered explicitly and precisely in Section 3. In Section 4 it is shown that our main results, when specialized to the case where p and q are regularly varying functions in the sense of Karamata, provide thorough information about the existence and asymptotic behavior of regularly varying solutions in the sense of Karamata.

2. Asymptotic analysis of solution of (E) with continuous coefficients

We begin by classification of the set of all possible positive solutions of (E) in terms of signs of their derivatives. Let x be a positive solution of (E) and for any such solution x, denote by

$$x^{[3]}(t) = (p(t)\Phi_\alpha(x''(t)))', \qquad x^{[2]}(t) = p(t)\Phi_\alpha(x''(t))\,.$$

It is known (see [5]) that for a positive solution x one of the following three cases holds:

$$x'(t) > 0,\; x''(t) > 0,\; x^{[3]}(t) > 0, \quad t \ge t_0; \tag{2.1}$$

$$x'(t) > 0,\; x''(t) < 0,\; x^{[3]}(t) > 0, \quad t \ge t_0; \tag{2.2}$$

$$x'(t) < 0,\; x''(t) > 0,\; x^{[3]}(t) > 0, \quad t \ge t_0, \tag{2.3}$$

for sufficiently large $t_0 \ge a$.

Next, we give necessary and sufficient conditions for the existence of positive solutions of type (S_2) and (S_3), while otherwise the existence of intermediate type of positive solutions (I_1) and (I_2) will be characterized by sufficient conditions.

2.1. Existence of positive solutions of type (S_2)

Since a positive solution satisfying (2.2) as well as a positive solution satisfying (2.3) may have the asymptotic behavior of type (S_2), we establish necessary and sufficient conditions for the existence of both types of solutions.

Theorem 2.1. *Necessary and sufficient condition for* (E) *to have a solution* x *which satisfies* (S_2) *and* (2.2) *is*

$$\int_a^\infty tq(t)\,dt < \infty. \tag{2.4}$$

Proof. Necessity: Suppose that there exists a positive solution x which satisfies (S_2) and (2.2). We may suppose that $c/2 \le x(t) \le c$ for $t \ge t_0$ and some positive constant c. Multiplying (E) by t and integrating the obtained equation on $[t_0, t]$, we get

$$\left(\frac{c}{2}\right)^\beta \int_{t_0}^t sq(s)\,ds \le \int_{t_0}^t sq(s)x(s)^\beta\,ds = C - t\Big(p(t)\Phi_\alpha(x''(t))\Big)' - p(t)(-x''(t))^\alpha \le C,\ \ t \ge t_0,$$

with $C = t_0\,x^{[3]}(t_0) - x^{[2]}(t_0)$. Thus, letting $t \to \infty$, we obtain (2.4).

Sufficiency: We assume that (2.4) holds. Let a constant $c > 0$ be fixed arbitrarily. Choose $t_0 \ge a$ such that

$$\int_{t_0}^\infty \frac{t}{p(t)^{\frac{1}{\alpha}}}\,dt < 1 \quad \text{and} \quad 2^\alpha \int_{t_0}^\infty tq(t)\,dt < c^{\alpha-\beta}(1 - 2^\alpha). \tag{2.5}$$

Let $C[t_0, \infty)$ be the set of all continuous functions defined on $[t_0, \infty)$ with the topology of uniform convergence on compact subintervals of $[t_0, \infty)$. We defined the subset Ω_1 of $C[t_0, \infty)$ by

$$\Omega_1 = \left\{x \in C[t_0, \infty) : \frac{c}{2} \le x(t) \le c,\ t \ge t_0\right\}.$$

Clearly, Ω_1 is a closed convex subset of the space $C[t_0, \infty)$. Let us define the mapping $\mathcal{F}_1 : \Omega \to C[t_0, \infty)$ by

$$(\mathcal{F}_1 x)(t) = c - \int_t^\infty \frac{s-t}{p(s)^{\frac{1}{\alpha}}}\left(c^\alpha + \int_s^\infty (r-s)q(r)x(r)^\beta\,dr\right)^{\frac{1}{\alpha}} ds, \qquad t \ge t_0. \tag{2.6}$$

Because of (2.5) $\mathcal{F}_1$ maps Ω_1 into itself. It can be shown that $\mathcal{F}_1$ is a continuous mapping by means of the Lebesgue dominated convergence theorem and that the set $\mathcal{F}_1(\Omega_1)$ is relatively compact subset of $C[t_0, \infty)$, with the help of the Ascoli-Arzela theorem. Applying the Schauder-Tychonoff fixed point theorem, there exists a solution $x \in \Omega_1$ of the integral equation $x(t) = (\mathcal{F}_1 x)(t)$, $t \ge t_0$. Then it is easily verifed that $x = x(t)$ is a positive solution of (E) satisfying (S_2) and (2.2). □

Theorem 2.2. *Necessary and sufficient condition for* (E) *to have a solution* x *which satisfies* (S_2) *and* (2.3) *is*

$$J_2 = \int_a^\infty \frac{t}{p(t)^{\frac{1}{\alpha}}}\left(\int_a^t \int_s^\infty q(r)\,dr\,ds\right)^{\frac{1}{\alpha}} dt < \infty. \tag{2.7}$$

Proof. Necessity: Suppose that there exists a positive solution x which satisfies (S_2) and (2.3). If $\lim_{t\to\infty} x(t) = c > 0$, then there exists $t_0 \ge a$ such that $c \le x(t) \le 2c$, for $t \ge t_0$. By [5, Lemma 3.1], for a solution of type (2.3) we have $\lim_{t\to\infty} x^{[3]}(t) = 0$. Moreover, since x' is negative and increasing there exists $\lim_{t\to\infty} x'(t) = \omega_1 \in (-\infty, 0]$. We claim that $\omega_1 = 0$. If we assume that $\omega_1 < 0$, then $x'(t) \le \omega_1$ for $t \ge t_1$ implying that $x(t) \le x(t_1) + \omega_1(t - t_1)$, $t \ge t_1$ and letting $t \to \infty$ leads to the contradiction with positivity of x.

Therefore, integration of (E) first on $[t, \infty)$, then on $[t_0, t]$ and afterwards on $[t, \infty)$ gives

$$\begin{aligned} -x'(t) &= \int_t^\infty \frac{1}{p(s)^{\frac{1}{\alpha}}}\left(\omega_2 + \int_{t_0}^s \int_r^\infty q(u)x(u)^\beta\,du\,dr\right)^{\frac{1}{\alpha}} ds, \\ &\ge c^{\frac{\beta}{\alpha}} \int_t^\infty \frac{1}{p(s)^{\frac{1}{\alpha}}}\left(\int_{t_0}^s \int_r^\infty q(u)\,du\,dr\right)^{\frac{1}{\alpha}} ds, \qquad t \ge t_0, \end{aligned} \tag{2.8}$$

where $\omega_2 = p(t_0)\Phi_\alpha(x''(t_0)) > 0$. Integrating the last inequality from t_0 to ∞ we find

$$x(t_0) - c \geq c^{\frac{\beta}{\alpha}} \int_{t_0}^{\infty} \frac{s - t_0}{p(s)^{\frac{1}{\alpha}}} \left(\int_{t_0}^{s} \int_{r}^{\infty} q(u)\, du\, dr \right)^{\frac{1}{\alpha}} ds, \qquad t \geq t_0. \tag{2.9}$$

Since, (2.8) implies

$$\int_{t_0}^{\infty} \frac{1}{p(s)^{\frac{1}{\alpha}}} \left(\int_{t_0}^{s} \int_{r}^{\infty} q(u)\, du\, dr \right)^{\frac{1}{\alpha}} ds < \infty\,,$$

(2.7) follows from (2.9).

Sufficiency: We assume that (2.7) holds. Let a constant $c > 0$ be fixed arbitrarily and choose $t_0 \geq a$ such that

$$(2c)^{\frac{\beta}{\alpha}} \int_{t_0}^{\infty} \frac{t}{p(t)^{\frac{1}{\alpha}}} \left(\int_{t_0}^{t} \int_{s}^{\infty} q(r)\, dr\, ds \right)^{\frac{1}{\alpha}} dt \leq c.$$

Consider the subset $\Omega_2 = \{x \in C[t_0, \infty) : c \leq x(t) \leq 2c,\ t \geq t_0\}$ of $C[t_0, \infty)$ and define the mapping $\mathcal{F}_2 : \Omega_2 \to C[t_0, \infty)$ by

$$(\mathcal{F}_2 x)(t) = c + \int_{t}^{\infty} \frac{s - t}{p(s)^{\frac{1}{\alpha}}} \left(\int_{t_0}^{s} \int_{r}^{\infty} q(v) x(v)^{\beta}\, dv\, dr \right)^{\frac{1}{\alpha}} ds, \qquad t \geq t_0.$$

Then, by the Schauder-Tychonoff fixed point theorem, $\mathcal{F}_2$ has a fixed element x_2 in the set Ω_2. Since, the fixed element $x_2 = x_2(t)$ satisfies the integral equation $x(t) = (\mathcal{F}_2 x)(t)$, $t \geq t_0$, it provides a positive solution of (E) satisfying (2.3) and (S_2). $\square$

Noting that (2.4) $\Rightarrow$ (2.7), we have the following theorem.

Theorem 2.3. *Equation* (E) *has a solution* x *satisfying* (S_2) *if and only if* (2.7) *holds.*

2.2. *Existence of positive solutions of type* (S_3)

Since solutions with asymptotic behavior of type (S_3) may satisfy condition (2.1) or (2.2), we consider both cases.

Theorem 2.4. *Necessary and sufficient condition for* (E) *to have a solution* x *which satisfies* (S_3) *and* (2.1) *is*

$$J_3 = \int_{a}^{\infty} \frac{1}{p(t)^{\frac{1}{\alpha}}} \left(\int_{a}^{t} \int_{s}^{\infty} q(r) r^{\beta}\, dr\, ds \right)^{\frac{1}{\alpha}} dt < \infty. \tag{2.10}$$

Proof. See the proof of Theorem 3.3. in [16]. $\square$

Theorem 2.5. *Necessary and sufficient condition for* (E) *to have a solution* x *which satisfies* (S_3) *and* (2.2) *is*

$$\int_{a}^{\infty} t^{\beta+1} q(t) dt < \infty. \tag{2.11}$$

Proof. See the proof of Theorem 3.4. in [16]. $\square$

Noting that (2.11) $\Rightarrow$ (2.10), we have the following theorem.

Theorem 2.6. *Equation* (E) *has a solution* x *satisfying* (S_3) *if and only if* (2.10) *holds.*

2.3. Existence of positive solutions of type (I_1)

Theorem 2.7. *If* (2.7) *holds and if* $J_1 = \infty$, *then the equation* (E) *has a positive solution which satisfies* (I_1).

Proof. Choose $t_0 \geq a$ such that

$$\int_{t_0}^{\infty} \frac{t}{p(t)^{\frac{1}{\alpha}}} \left(\int_{t_0}^{t} \int_{s}^{\infty} q(r)\, dr\, ds \right)^{\frac{1}{\alpha}} dt \leq \frac{1}{2^{\frac{1}{\alpha}}} \quad \text{and} \quad \int_{t_0}^{\infty} \frac{t}{p(t)^{\frac{1}{\alpha}}}\, dt \leq \frac{1}{2^{\frac{1}{\alpha}}}, \tag{2.12}$$

for $t \geq t_0$. Define the set $Q_3 = \{x \in C[t_0, \infty) : \varphi(t) \leq x(t) \leq 1,\ t \geq t_0\}$, and the operator $\mathcal{F}_3 : Q_3 \to C[t_0, \infty)$

$$(\mathcal{F}_3 x)(t) = \int_{t}^{\infty} \frac{s-t}{p(s)^{\frac{1}{\alpha}}} \left(1 + \int_{t_0}^{s} \int_{r}^{\infty} q(v) x(v)^{\beta}\, dv\, dr \right)^{\frac{1}{\alpha}} ds\,, \quad t \geq t_0. \tag{2.13}$$

It is clear that Q_3 is a closed convex subset of the locally convex space $C[t_0, \infty)$ equipped with the topology of uniform convergence on compact subintervals of $[t_0, \infty)$. Using inequality

$$(X + Y)^{\frac{1}{\alpha}} \leq 2^{\frac{1-\alpha}{\alpha}} \left(X^{\frac{1}{\alpha}} + Y^{\frac{1}{\alpha}} \right),$$

holding for $\alpha \in (0, 1)$ and (2.12), (2.13), $x \in Q_3$ implies

$$\begin{aligned} \varphi(t) \leq (\mathcal{F}_3 x)(t)(t) &\leq 2^{\frac{1-\alpha}{\alpha}} \int_{t}^{\infty} \frac{s-t}{p(s)^{\frac{1}{\alpha}}} \left(1 + \left(\int_{t_0}^{s} \int_{r}^{\infty} q(v)\, dv\, dr \right)^{\frac{1}{\alpha}} \right) \\ &\leq 2^{\frac{1-\alpha}{\alpha}} \left(\int_{t_0}^{\infty} \frac{s}{p(s)^{\frac{1}{\alpha}}}\, ds + \int_{t_0}^{\infty} \frac{s}{p(s)^{\frac{1}{\alpha}}} \left(\int_{t_0}^{s} \int_{r}^{\infty} q(v)\, dv\, dr \right)^{\frac{1}{\alpha}} ds \right) \leq 1, \end{aligned}$$

for all $t \geq t_0$. This means that $\mathcal{F}_3$ maps Q_3 into itself. Furthermore, it can be shown that $\mathcal{F}_3$ is a continuous map such that $\mathcal{F}_3(Q_3)$ is relatively compact in $C[t_0, \infty)$. Therefore, by the Schauder-Tychonoff fixed point theorem there exists a function $x_3 \in Q_3$ satisfying the integral equation $x(t) = (\mathcal{F}_3 x)(t)(t)$ for $t \geq t_0$. It follows that x_3 is a solution of (E) on $[t_0, \infty)$. It is easy to see that x_3 has the following asymptotic properties: $\lim_{t\to\infty} x_3(t) = 0$ and

$$\lim_{t\to\infty} \frac{x_3(t)}{\varphi(t)} = \lim_{t\to\infty} \left(1 + \int_{t_0}^{t} \int_{s}^{\infty} q(r) x_3(r)^{\beta}\, dr\, ds \right)^{\frac{1}{\alpha}} \geq \left(\lim_{t\to\infty} \int_{t_0}^{t} \int_{s}^{\infty} q(r) \varphi(r)^{\beta}\, dr\, ds \right)^{\frac{1}{\alpha}} = \infty,$$

as a consequence of $J_1 = \infty$, which means that x_3 satisfies $\varphi(t) \prec x_3(t) \prec 1,\ t \to \infty$. $\square$

2.4. Existence of positive solutions of type (I_2)

Theorem 2.8. *If* (1.3) *holds and if* $J_3 = \infty$, *then the equation* (E) *has a positive solution which satisfies* (I_2).

Proof. Choose $t_0 \geq \max\{\frac{1}{2}, a\}$ such that

$$2^{\beta} \int_{t_0}^{\infty} q(t) \psi(t)^{\beta}\, dt \leq 1 \qquad \text{and} \qquad t \leq \psi(t),\ t \geq t_0. \tag{2.14}$$

Define the set $Q_4 = \{x \in C[t_0, \infty) : t \leq x(t) \leq 2\psi(t),\ t \geq t_0\}$, and the integral operator $\mathcal{F}_4 : Q_4 \to C[t_0, \infty)$

$$(\mathcal{F}_4 x)(t) = t + \int_{t_0}^{t} \frac{t-s}{p(s)^{\frac{1}{\alpha}}} \left(\int_{t_0}^{s} \int_{r}^{\infty} q(v) x(v)^{\beta}\, dv\, dr \right)^{\frac{1}{\alpha}} ds, \quad t \geq t_0. \tag{2.15}$$

It is clear that Q_4 is a closed convex subset of the locally convex space $C[t_0,\infty)$ equipped with the topology of uniform convergence on compact subintervals of $[t_0,\infty)$. Using (2.14), (2.15), we see that $x \in Q_4$ implies

$$t \le (\mathcal{F}_4 x)(t) \le t + 2^{\beta/\alpha}\int_{t_0}^{t} \frac{t-s}{p(s)^{\frac{1}{\alpha}}}\left(\int_{t_0}^{s}\int_{r}^{\infty} q(v)\psi(v)^{\beta}\,dv\,dr\right)^{\frac{1}{\alpha}} ds \le t+\psi(t) \le 2\psi(t)\,,$$

for all $t \ge t_0$. This means that $\mathcal{F}_4$ maps Q_4 into itself. Furthermore, it can be shown that $\mathcal{F}_4$ is a continuous map such that $\mathcal{F}_4(Q_4)$ is relatively compact in $C[t_0,\infty)$. Therefore, by the Schauder-Tychonoff fixed point theorem there exists a function $x_4 \in Q_4$ satisfying the integral equation $x(t) = (\mathcal{F}_4 x)(t)$ for $t \ge t_0$. It follows that x_4 is a solution of (E) on $[t_0,\infty)$. It is easy to see that x_4 has the following asymptotic properties:

$$\lim_{t\to\infty}\frac{x_4(t)}{t} \ge \lim_{t\to\infty}\int_{t_0}^{t}\frac{1}{p(s)^{\frac{1}{\alpha}}}\left(\int_a^t\int_s^{\infty} q(r)r^{\beta}\,dr\,ds\right)^{\frac{1}{\alpha}} ds = \infty,$$

and

$$\lim_{t\to\infty}\frac{x_4(t)}{\psi(t)} = \left(\lim_{t\to\infty}\frac{\int_{t_0}^{t}\int_s^{\infty} q(r)x(r)^{\beta}\,dr\,ds}{t}\right)^{\frac{1}{\alpha}} = \left(\lim_{t\to\infty}\int_t^{\infty} q(s)x(s)^{\beta}\,ds\right)^{\frac{1}{\alpha}} \le \left(\lim_{t\to\infty}2^{\beta}\int_t^{\infty} q(s)\psi(s)^{\beta}\,ds\right)^{\frac{1}{\alpha}} = 0,$$

which means that x_4 satisfies $t \prec x_4(t) \prec \psi(t),\ t\to\infty$. □

3. Asymptotic behavior of intermediate solutions of (E) with generalized regularly varying coefficients

In what follows it is always assumed that functions p and q are generalized regularly varying of index η and σ with respect to R, which is defined with

$$R(t) = \int_a^t \left(\frac{s}{p(s)}\right)^{\frac{1}{\alpha}} ds,\ \ t \ge a\,. \tag{3.1}$$

We express functions p and q with

$$p(t) = R(t)^{\eta} l_p(t),\ q(t) = R(t)^{\sigma} l_q(t),\quad l_p,\ l_q \in \mathcal{SV}_R. \tag{3.2}$$

We begin with determining index of regularity of functions in (1.4) what is a fundamental part in the asymptotic analysis. From (3.1) and (3.2) we have that

$$t^{\frac{1}{\alpha}} = R'(t)R(t)^{\frac{\eta}{\alpha}}\, l_p(t)^{\frac{1}{\alpha}}, \tag{3.3}$$

which by integration from a to t implies

$$t^{(\alpha+1)/\alpha} \sim \frac{\alpha+1}{\alpha}\int_a^t R'(s)R(s)^{\eta/\alpha} l_p(s)^{\frac{1}{\alpha}}\,ds,\ \ t\to\infty\,. \tag{3.4}$$

Thus, we must have that $\eta+\alpha \ge 0$, but in what follows we limit ourselves to the case where $\eta+\alpha > 0$. Application of generalized Karamata integration theorem (Proposition 5.4) then gives

$$t \sim \left(\frac{\alpha+1}{\eta+\alpha}\right)^{\frac{\alpha}{\alpha+1}} R(t)^{\frac{\alpha+\eta}{\alpha+1}} l_p(t)^{\frac{1}{\alpha+1}},\quad t\to\infty, \tag{3.5}$$

implying that $\psi_1 \in \mathcal{RV}_R\left(\frac{\alpha+\eta}{\alpha+1}\right)$, with $\psi_1(t) = t$. It is clear that $\varphi_1 \in \mathcal{SV}_R$, with $\varphi_1(t) = 1$. From (3.3) and (3.5) we have

$$R'(t) \sim \left(\frac{\alpha+1}{\eta+\alpha}\right)^{\frac{1}{\alpha+1}} R(t)^{\frac{1-\eta}{\alpha+1}} l_p(t)^{-\frac{1}{\alpha+1}},\quad t\to\infty,$$

which can be rewritten in the form

$$1 \sim R'(t)\left(\frac{\eta+\alpha}{\alpha+1}\right)^{\frac{1}{\alpha+1}} R(t)^{\frac{\eta-1}{\alpha+1}} l_p(t)^{\frac{1}{\alpha+1}}, \quad t \to \infty. \tag{3.6}$$

For furher discussion, to simplify formulation of our main results, we introduce the notation:

$$m_1(\alpha,\eta) = \frac{2\alpha^2-\eta+\alpha\eta}{\alpha(\alpha+1)}, \;\; m_2(\alpha,\eta) = \frac{\alpha+\eta}{\alpha+1} \;\text{ and }\; m_3(\alpha,\eta) = \frac{2\alpha+\eta+1}{\alpha+1}, \tag{3.7}$$

and frequently use the abbreviated notation m_i for $m_i(\alpha,\eta)$, $i = 1,2,3$. In view of (3.6), we may state the next lemma following directly from the generalized Karamata integration theorem.

Lemma 3.1. *Let $f(t) = R(t)^{\mu}\, l_f(t)$, $l_f \in \mathcal{SV}_R$, $\mu \in \mathbb{R}$. Then,*

(i) If $\mu + m_2 > 0$,

$$\int_a^t f(s)\,ds \sim \frac{m_2^{\frac{1}{\alpha+1}}}{\mu+m_2} R(t)^{\mu+m_2} l_f(t) l_p(t)^{\frac{1}{\alpha+1}}, \quad t\to\infty.$$

(ii) If $\mu + m_2 < 0$,

$$\int_t^\infty f(s)\,ds \sim -\frac{m_2^{\frac{1}{\alpha+1}}}{\mu+m_2} R(t)^{\mu+m_2} l_f(t) l_p(t)^{\frac{1}{\alpha+1}}, \quad t\to\infty.$$

(iii) If $\mu + m_2 = 0$, then

$$\int_a^t f(s)\,ds \sim m_2^{\frac{1}{\alpha+1}} \int_a^t R'(s)R(s)^{-1} l_f(s) l_p(s)^{\frac{1}{\alpha+1}}\,ds \in \mathcal{SV}_R, \quad t\to\infty;$$

$$\int_t^\infty f(s)\,ds \sim m_2^{\frac{1}{\alpha+1}} \int_t^\infty R'(s)R(s)^{-1} l_f(s) l_p(s)^{\frac{1}{\alpha+1}}\,ds \in \mathcal{SV}_R, \quad t\to\infty.$$

Before we proceed further in obtaining index of regularity of φ and ψ we give an interpretation of our assumptions (1.1) in terms of index of regularity of coefficients and parameters α, β. Using (3.2) and (3.5) we have

$$\int_t^\infty \frac{s}{p(s)^{\frac{1}{\alpha}}}\,ds \sim m_2^{-\frac{\alpha}{\alpha+1}} \int_t^\infty R(s)^{m_2-\frac{\eta}{\alpha}} l_p(s)^{-\frac{1}{\alpha(\alpha+1)}}\,ds, \;\; t\to\infty,$$

which in view of (P_1), by Lemma 3.1 implies $2m_2 - \eta/\alpha \le 0$, but in what follows we only assume that strict inequality holds. Thus, if (P_1) holds, the following inequalities hold

$$2m_2 - \frac{\eta}{\alpha} = m_1 < 0 < m_2 < m_3 = m_2 + 1, \tag{3.8}$$

what will be used later.

Applying Lemma 3.1 twice, which is possible in view of (3.8), we get

$$\varphi(t) = \int_t^\infty \int_s^\infty \frac{1}{p(r)^{\frac{1}{\alpha}}}\,dr\,ds \sim \frac{m_2^{\frac{2}{\alpha+1}}}{m_1(m_1-m_2)} R(t)^{m_1} l_p(t)^{\frac{\alpha-1}{\alpha(\alpha+1)}}, \quad t\to\infty, \tag{3.9}$$

and

$$\psi(t) = \int_a^t R(s)\,ds \sim \frac{m_2^{\frac{1}{\alpha+1}}}{m_3} R(t)^{m_3} l_p(t)^{\frac{1}{\alpha+1}}, \;\; t\to\infty\,, \tag{3.10}$$

which shows that $\varphi \in \mathcal{RV}_R(m_1)$ and $\psi \in \mathcal{RV}_R(m_3)$.

3.1. Regularly varying solutions of type (I_1)

The first subsection is devoted to the study of the existence and asymptotic behavior of generalized regularly varying solutions of type (I_1) of the equation (E) with p and q satisfying (3.2). We seek such solutions x of (E) expressed in the form

$$x(t) = R(t)^{\rho} l_x(t), \quad l_x \in \mathcal{SV}_R. \tag{3.11}$$

Since $\varphi(t) \prec x(t) \prec 1,\ t \to \infty$, the regularity index ρ of x must satisfy $m_1 \le \rho \le 0$. If $\rho = 0$, then since $x(t) = l_x(t) \to \infty,\ t \to \infty$, x is a member of $ntr - \mathcal{SV}_R$, while if $\rho = m_1$, then since $x(t)/\varphi(t) \sim l_x(t) l_p(t)^{\frac{1-\alpha}{\alpha(\alpha+1)}} \to \infty,\ t \to \infty$, x is a member of $\mathcal{RV}(m_1)$, with $x/\varphi \in ntr - \mathcal{SV}_R$. Thus, the set of all generalized regularly varying solutions of type (I_1) is naturally divided into the three disjoint classes

$$\mathcal{RV}_R\,(m_1) \quad \text{or} \quad \mathcal{RV}_R(\rho) \ \text{ with } \ \rho \in (m_1, 0) \quad \text{or} \quad ntr - \mathcal{SV}_R.$$

Our aim is to establish necessary and sufficient conditions for each of the above classes to be nonempty and to show that the asymptotic behavior of all members of each class is governed by a unique explicit formula.

3.1.1. Main results

Theorem 3.2. *Let $p \in \mathcal{RV}_R(\eta),\ q \in \mathcal{RV}_R(\sigma)$. Equation* (E) *has intermediate solutions $x \in \mathcal{RV}_R(\rho)$ with $\rho \in (m_1, 0)$ if and only if*

$$m_1(\alpha - \beta) - \eta - 2\alpha < \sigma < -\eta - 2\alpha. \tag{3.12}$$

in which case ρ is given by

$$\rho = \frac{\sigma + 2\alpha + \eta}{\alpha - \beta} \tag{3.13}$$

and the asymptotic behavior of any such solution x is governed by the unique formula

$$x(t) \sim X_1(t) = \left(\left(\frac{m_2}{\alpha} \right)^2 \frac{R(t)^{2\alpha} p(t) q(t)}{(\rho(\rho - m_2))^{\alpha} (\rho - m_1)(m_3 - \rho)} \right)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty. \tag{3.14}$$

Theorem 3.3. *Let $p \in \mathcal{RV}_R(\eta),\ q \in \mathcal{RV}_R(\sigma)$. Equation* (E) *has intermediate solutions $x \in \mathcal{RV}_R(m_1)$ satisfying (I_1) if and only if*

$$J_1 = \infty \quad \text{and} \quad \sigma = m_1(\alpha - \beta) - \eta - 2\alpha. \tag{3.15}$$

The asymptotic behavior of any such solution x is governed by the unique formula

$$x(t) \sim X_2(t) = \varphi(t) \left(\frac{\alpha - \beta}{\alpha} \int_a^t s q(s) \varphi(s)^{\beta}\, ds \right)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty. \tag{3.16}$$

Theorem 3.4. *Let $p \in \mathcal{RV}_R(\eta),\ q \in \mathcal{RV}_R(\sigma)$. Equation* (E) *has intermediate solutions $x \in ntr - \mathcal{SV}_R$ satisfying (I_1) if and only if*

$$J_2 < \infty \quad \text{and} \quad \sigma = -\eta - 2\alpha. \tag{3.17}$$

The asymptotic behavior of any such solution x is governed by the unique formula

$$x(t) \sim X_3(t) = \left(\frac{\alpha - \beta}{\alpha} \int_t^{\infty} \frac{s}{p(s)^{\frac{1}{\alpha}}} \left(\int_a^s \int_r^{\infty} q(u)\, du\, dr \right)^{\frac{1}{\alpha}} ds \right)^{\frac{\alpha}{\alpha-\beta}}, \quad t \to \infty. \tag{3.18}$$

3.1.2. Preparatory results

Let x be a solution of (E) on $[t_0, \infty)$ such that $\varphi(t) \prec x(t) \prec 1$ as $t \to \infty$. For such a solution (2.3) holds and

$$\lim_{t\to\infty} x^{[3]}(t) = 0, \;\; \lim_{t\to\infty} x^{[2]}(t) = \infty, \;\; \lim_{t\to\infty} x'(t) = \lim_{t\to\infty} x(t) = 0. \tag{3.19}$$

The three types of intermediate solutions of type (I_1) in the above theorems will be constructed by solving the integral equation

$$x(t) = \int_t^\infty \frac{s-t}{p(s)^{\frac{1}{\alpha}}} \left(c + \int_{T_0}^s \int_r^\infty q(r)x(r)^\beta \, dr \right)^{\frac{1}{\alpha}} ds, \qquad t \ge T_0, \tag{3.20}$$

for some constants $T_0 \ge a$ and $c > 0$. To proceed this, Schauder-Tychonoff fixed point theorem is used as our main tool. Denoting by $\mathcal{G}x(t)$ the right-hand side of (3.20), in order to find a fixed point of $\mathcal{G}$, it is important to choose an appropriate closed convex subset $\mathcal{X} \subset C[t_0, \infty)$ on which $\mathcal{G}$ is a self-map. It will be shown that such a choice of $\mathcal{X}$ is possible by solving the integral asymptotic relation

$$x(t) \sim \int_t^\infty \frac{s-t}{p(s)^{\frac{1}{\alpha}}} \left(\int_a^s \int_r^\infty q(u)x(u)^\beta \, du \, dr \right)^{\frac{1}{\alpha}} ds, \;\; t \to \infty\,, \tag{3.21}$$

which can be considered as an approximation (at infinity) of (3.20) in the sense that it is satisfied by all possible solutions of type (I_1) of (E).

As a preparatory steps toward the proofs of Theorems 3.2-3.4 we show that the generalized regularly varying functions X_i, $i = 1, 2, 3$ defined in (3.14), (3.16), (3.18) satisfy the asymptotic relation (3.21). To simplify the notation, we put

$$J(t; a, X) = \int_t^\infty \frac{s-t}{p(s)^{\frac{1}{\alpha}}} \left(\int_a^s \int_r^\infty q(u)X(u)^\beta \, du \, dr \right)^{\frac{1}{\alpha}} ds, \;\; t \ge a.$$

Lemma 3.5. *Suppose that* (3.12) *holds and let ρ be defined by* (3.13). *Then, X_1 given in* (3.14) *satisfies the asymptotic relation* (3.21) *and $X_1 \in \mathcal{RV}_R(\rho)$.*

Proof. Let (3.12) holds. Using (3.2) and (3.13), function X_1 given in (3.14) can be expressed in the form $X_1(t) = R(t)^\rho L_1(t)$, where

$$L_1(t) = \left(\left(\frac{m_2}{\alpha}\right)^2 \frac{l_p(t) l_q(t)}{(\rho(\rho - m_2))^\alpha (\rho - m_1)(m_3 - \rho)} \right)^{\frac{1}{\alpha-\beta}}, \;\; L_1 \in \mathcal{SV}_R, \tag{3.22}$$

implying that $X_1 \in \mathcal{RV}_R(\rho)$.

By assumption (3.12), from (3.13) we have that $m_1 < \rho < 0$ and using (3.7) and (3.8), we get

$$\sigma + \rho\beta + m_2 = -\alpha(m_3 - \rho) < 0, \quad \sigma + \rho\beta + 2m_2 = \alpha(\rho - m_1) > 0. \tag{3.23}$$

Thus, application of Lemma 3.1 gives

$$\frac{1}{p(t)^{\frac{1}{\alpha}}} \left(\int_a^t \int_s^\infty q(r)X_1(r)^\beta \, dr \, ds \right)^{\frac{1}{\alpha}} \sim \frac{m_2^{\frac{2}{\alpha(\alpha+1)}}}{(\alpha^2(m_3 - \rho)(\rho - m_1))^{\frac{1}{\alpha}}} R(t)^{\rho - m_1 - \frac{\eta}{\alpha}} \left(l_q(t) l_p(t)^{\frac{1-\alpha}{1+\alpha}} \right)^{\frac{1}{\alpha}} L_1(t)^{\frac{\beta}{\alpha}}, \;\; t \to \infty. \tag{3.24}$$

Using that $\rho - m_1 - \frac{\eta}{\alpha} + m_2 = \rho - m_2 < 0$, integration of (3.24) twice over $[t, \infty)$, by Lemma 3.1, gives

$$\int_t^\infty \frac{1}{p(s)^{\frac{1}{\alpha}}} \left(\int_a^s \int_r^\infty q(u)X_1(u)^\beta \, du \, dr \right)^{\frac{1}{\alpha}} ds \sim$$
$$\frac{m_2^{\frac{2+\alpha}{\alpha(\alpha+1)}}}{(\alpha^2(\rho - m_1)(m_3 - \rho))^{\frac{1}{\alpha}} (m_2 - \rho)} R(t)^{\rho - m_2} \left(l_q(t) l_p(t)^{\frac{1}{1+\alpha}} \right)^{\frac{1}{\alpha}} L_1(t)^{\frac{\beta}{\alpha}}, \;\; t \to \infty,$$

and

$$J(t;a,X_1)=\int_t^\infty \frac{s-t}{p(s)^{\frac{1}{\alpha}}}\left(\int_a^s\int_r^\infty q(u)X_1(u)^\beta\,du\,dr\right)^{\frac{1}{\alpha}}ds\sim R(t)^\rho L_1(t)=X_1(t),\ \ t\to\infty.$$

□

Lemma 3.6. *Suppose that* (3.15) *holds. Then,* X_2 *given in* (3.16) *satisfies the asymptotic relation* (3.21) *and* $X_2\in\mathcal{RV}_R(m_1)$, *such that* $X_2/\varphi\in ntr-\mathcal{SV}_R$.

Proof. Let (3.15) holds. From (3.9) we have

$$\varphi(t)\sim R(t)^{m_1}l_\varphi(t),\quad t\to\infty,\quad\text{where}\ \ l_\varphi(t)=\frac{m_2^{\frac{2}{\alpha+1}}}{m_1\,(m_1-m_2)}l_p(t)^{\frac{\alpha-1}{\alpha(\alpha+1)}},\quad l_\varphi\in\mathcal{SV}_R\,, \tag{3.25}$$

implying that $X_2(t)\sim R(t)^{m_1}l_\varphi(t)L_2(t),\ t\to\infty$, with

$$L_2(t)=\left(\frac{\alpha-\beta}{\alpha}\frac{m_2^{\frac{2\beta-\alpha}{\alpha+1}}}{(m_1(m_1-m_2))^\beta}\int_a^t R(s)^{-m_2}l_q(s)l_p(s)^{\frac{\alpha+\beta(\alpha-1)}{\alpha(\alpha+1)}}ds\right)^{\frac{1}{\alpha-\beta}},\ \ L_2\in\mathcal{SV}_R.$$

Consequently, $X_2\in\mathcal{RV}_R(m_1)$ and first assumption of (3.15) implies $X_2/\varphi\in ntr-\mathcal{SV}_R$. Since the second assumption of (3.15) combined with (3.8) implies $\sigma+m_1\beta+m_2=-m_2<0$, application of Lemma 3.1 gives

$$\left(\frac{1}{p(t)}\int_a^t\int_s^\infty q(r)X_2(r)^\beta\,dr\,ds\right)^{\frac{1}{\alpha}}\sim m_2^{-\frac{1}{\alpha+1}}R(t)^{-\frac{\eta}{\alpha}}l_p(t)^{-\frac{1}{\alpha}}W(t)^{\frac{1}{\alpha}},\ \ t\to\infty,$$

where

$$W(t)=\int_a^t R(s)^{-m_2}l_q(s)l_p(s)^{\frac{1}{\alpha+1}}\left(l_\varphi(s)L_2(s)\right)^\beta ds,\ \ W\in\mathcal{SV}_R. \tag{3.26}$$

Integrating the relation above twice over $[t,\infty)$, repeated application of Lemma 3.1, with use of (3.25) and (3.26), yields

$$J(t;a,X_2)\sim\frac{m_2^{\frac{2\beta-\alpha}{\alpha(\alpha+1)}}}{(m_1(m_1-m_2))^{\frac{\beta}{\alpha}}}\varphi(t)\left(\int_a^t R(s)^{-m_2}l_q(s)l_p(s)^{\frac{\alpha+\beta(\alpha-1)}{\alpha(\alpha+1)}}L_2(s)^\beta\,ds\right)^{\frac{1}{\alpha}},\ \ t\to\infty.$$

Integration by substitution in the last integral gives the asymptotic relation (3.21). □

Lemma 3.7. *Suppose that* (3.17) *holds. Then, function* X_3 *given by* (3.18) *satisfies the asymptotic relation* (3.21) *and* $X_3\in ntr-\mathcal{SV}_R$.

Proof. Let (3.17) holds, implying that

$$\sigma+m_2=-m_3\alpha,\qquad\sigma+2m_2=-m_1\alpha. \tag{3.27}$$

From (3.18), using (3.2), (3.5), (3.8), equalities (3.27), with repeated application of Lemma 3.1, we obtain

$$\int_t^\infty\frac{s}{p(s)^{\frac{1}{\alpha}}}\left(\int_a^s\int_r^\infty q(u)\,du\,dr\right)^{\frac{1}{\alpha}}ds\sim\frac{m_2^{\frac{2-\alpha^2}{\alpha(\alpha+1)}}}{(-\alpha^2m_1m_3)^{\frac{1}{\alpha}}}\int_t^\infty R(s)^{-m_2}\left(l_q(s)l_p(s)^{\frac{1}{\alpha+1}}\right)^{\frac{1}{\alpha}}ds\,,$$

so that

$$X_3(t) \sim \left(\frac{\alpha-\beta}{\alpha} \frac{m_2^{\frac{2-\alpha^2}{\alpha(\alpha+1)}}}{(-\alpha^2 m_1 m_3)^{\frac{1}{\alpha}}} \int_t^\infty R(s)^{-m_2} \left(l_q(s) l_p(s)^{\frac{1}{\alpha+1}} \right)^{\frac{1}{\alpha}} ds \right)^{\frac{\alpha}{\alpha-\beta}} = Y_3(t), \tag{3.28}$$

as $t \to \infty$. Therefore, by Lemma 3.1 we conclude that $X_3 \in \mathcal{SV}_R$. Due to $J_2 < \infty$, we have that $X_3(t) \to 0$ as $t \to \infty$, which gives that $X_3 \in ntr - \mathcal{SV}_R$.

To prove that X_3 satisfies the desired asymptotic relation, we apply Lemma 3.1 and use (3.27) to obtain

$$\frac{1}{p(t)^{\frac{1}{\alpha}}} \left(\int_a^t \int_s^\infty q(r) X_3(r)^\beta \, dr \, ds \right)^{\frac{1}{\alpha}} \sim \frac{m_2^{\frac{2}{\alpha(\alpha+1)}}}{(-\alpha^2 m_1 m_3)^{\frac{1}{\alpha}}} R(t)^{-m_1 - \frac{\eta}{\alpha}} \left(l_q(t) l_p(t)^{\frac{1-\alpha}{\alpha+1}} \right)^{\frac{1}{\alpha}} X_3(t)^{\frac{\beta}{\alpha}},$$

as $t \to \infty$, implying with (3.28) that

$$J(t; a, X_3) \sim \frac{m_2^{\frac{2-\alpha^2}{\alpha(\alpha+1)}}}{(-\alpha^2 m_1 m_3)^{\frac{1}{\alpha}}} \int_t^\infty R(s)^{-m_2} \left(l_q(s) l_p(s)^{\frac{1}{\alpha+1}} \right)^{\frac{1}{\alpha}} Y_3(s)^{\frac{\beta}{\alpha}} ds, \ \ t \to \infty.$$

Integration by substitution $u = Y_3(s)^{\frac{\alpha-\beta}{\alpha}}$ in the last integral gives the asymptotic relation (3.21). □

3.1.3. Proofs of main results

Suppose that (E) has (I_1)-type of intermediate solution $x \in \mathcal{RV}_R(\rho)$ on $[t_0, \infty)$. Clearly, $\rho \in [m_1, 0]$. Using (3.2), (3.11) and (3.19), we obtain from (E)

$$(p(t)\Phi_\alpha(x''(t)))' = \int_t^\infty q(s) x(s)^\beta ds = \int_t^\infty R(s)^{\sigma+\rho\beta} l_q(s) l_x(s)^\beta ds, \ \ t \geq t_0. \tag{3.29}$$

The integrability of $x^{[3]}$ on $[t_0, \infty)$ implies that one of the following two cases can be valid

$$(a) \ \ \sigma + \rho\beta + m_2 < 0, \quad (b) \ \ \sigma + \rho\beta + m_2 = 0.$$

Assume that (b) holds. Then,

$$(p(t)\Phi_\alpha(x''(t)))' = \int_t^\infty R(s)^{-m_2} l_q(s) l_x(s)^\beta ds \in \mathcal{SV}_R, \tag{3.30}$$

and integrating (3.30) on $[t_0, t]$, we find via Lemma 3.1 that

$$x''(t) \sim m_2^{-\frac{1}{\alpha+1}} R(t)^{\frac{m_2-\eta}{\alpha}} l_p(t)^{-\frac{1}{\alpha+1}} \left(\int_t^\infty R(s)^{-m_2} l_q(s) l_x(s)^\beta ds \right)^{\frac{1}{\alpha}}, \ \ t \to \infty. \tag{3.31}$$

Integration of (3.31) on $[t, \infty)$, due to (3.19), yileds

$$-x'(t) \sim \int_t^\infty m_2^{-\frac{1}{\alpha+1}} R(s)^{\frac{m_2-\eta}{\alpha}} l_p(s)^{-\frac{1}{\alpha+1}} \left(\int_s^\infty R(r)^{-m_2} l_q(r) l_x(r)^\beta dr \right)^{\frac{1}{\alpha}} ds, \ \ t \to \infty. \tag{3.32}$$

Notice that convergence of the integral in (3.32) implies the contradiction

$$\frac{m_2 - \eta}{\alpha} + m_2 \leq 0 \ \text{ with } \ 1 = \frac{m_2 - \eta}{\alpha} + m_2,$$

concluding that (b) can not be valid. Proceeding further, under the only possible case (a), application of Lemma 3.1 in (3.30) gives

$$(p(t)\Phi_\alpha(x''(t)))' \sim -\frac{m_2^{\frac{1}{\alpha+1}}}{\sigma+\rho\beta+m_2} R(t)^{\sigma+\rho\beta+m_2} l_q(t) l_p(t)^{\frac{1}{\alpha+1}} l_x(t)^\beta,\ \ t\to\infty\,. \tag{3.33}$$

We integrate (3.33) on $[t_0,t]$ to obtain

$$p(t)\Phi_\alpha(x''(t)) \sim -\frac{m_2^{\frac{1}{\alpha+1}}}{\sigma+\rho\beta+m_2}\int_{t_0}^t R(s)^{\sigma+\rho\beta+m_2} l_q(s) l_p(s)^{\frac{1}{\alpha+1}} l_x(s)^\beta\, ds,\ \ t\to\infty. \tag{3.34}$$

Since $\lim_{t\to\infty} p(t)\Phi_\alpha(x''(t)) = \infty$, divergence of the integral in (3.34) implies two possiblities

$$(a.1)\ \ \sigma+\rho\beta+2m_2 > 0, \quad (a.2)\ \ \sigma+\rho\beta+2m_2 = 0.$$

Assume that ($a.2$) holds. Then, (3.34) with (3.2) gives

$$x''(t) \sim m_2^{-\frac{1}{\alpha+1}} R(t)^{-\frac{\eta}{\alpha}} l_p(t)^{-\frac{1}{\alpha}} \left(\int_{t_0}^t R(s)^{-m_2} l_q(s) l_p(s)^{\frac{1}{\alpha+1}} l_x(s)^\beta\, ds\right)^{\frac{1}{\alpha}},\ \ t\to\infty. \tag{3.35}$$

Integration of (3.35) twice on $[t_0,t]$ and application of Lemma 3.1 shows that

$$x(t) \sim \frac{m_2^{\frac{1}{\alpha+1}}}{m_1(m_1-m_2)} R(t)^{m_1} l_p(t)^{\frac{\alpha-1}{\alpha(\alpha+1)}} \left(\int_{t_0}^t R(s)^{-m_2} l_q(s) l_p(s)^{\frac{1}{\alpha+1}} l_x(s)^\beta\, ds\right)^{\frac{1}{\alpha}}, \tag{3.36}$$

as $t\to\infty$, which yileds $x\in\mathcal{RV}_R(m_1)$.

Assume that ($a.1$) holds. Then, application of Lemma 3.1 in (3.34), with (3.2), gives

$$x''(t) \sim M_1(\rho)^{\frac{1}{\alpha}} R(t)^{\frac{k_1(\rho)-\eta}{\alpha}} \left(l_q(t) l_p(t)^{\frac{1-\alpha}{\alpha+1}} l_x(t)^\beta\right)^{\frac{1}{\alpha}},\ \ t\to\infty. \tag{3.37}$$

where

$$M_1(\rho) = \frac{m_2^{\frac{2}{\alpha+1}}}{-(k_1(\rho)-m_2)k_1(\rho)} \ \text{ and }\ k_1(\rho) = \sigma+\rho\beta+2m_2\,. \tag{3.38}$$

The integrability of x'' on $[t,\infty)$ implies

$$-x'(t) \sim M_1(\rho)^{\frac{1}{\alpha}} \int_t^\infty R(s)^{\frac{k_1(\rho)-\eta}{\alpha}} \left(l_q(s) l_p(s)^{\frac{1-\alpha}{\alpha+1}} l_x(s)^\beta\right)^{\frac{1}{\alpha}} ds,\ \ t\to\infty, \tag{3.39}$$

and $k_1(\rho)-\eta+m_2\alpha \le 0$. But the equality is not allowed. In fact, if $k_1(\rho)-\eta+m_2\alpha = 0$, by Lemma 3.1 it follows that $-x'\in\mathcal{SV}_R$, which is impossible because integration on $[t_0,t]$ and Lemma 3.1 would imply that $x\in\mathcal{RV}_R(m_2)$, contradicting assumption that $\rho\in[m_1,0]$. Therefore, $k_1(\rho)-\eta+m_2\alpha < 0$ and by Lemma 3.1 from (3.39) we obtain

$$-x'(t) \sim M_1(\rho)^{\frac{1}{\alpha}} m_2^{\frac{1}{\alpha+1}} \alpha \frac{R(t)^{\frac{k_1(\rho)-\eta+m_2\alpha}{\alpha}}}{-(k_1(\rho)-\eta+m_2\alpha)} \left(l_q(t) l_p(t)^{\frac{1}{\alpha+1}} l_x(t)^\beta\right)^{\frac{1}{\alpha}},\ \ t\to\infty. \tag{3.40}$$

Since, $-x'$ is integrable on $[t,\infty)$, it follows that $k_1(\rho)-\eta+2m_2\alpha \le 0$ and integration of (3.40) on $[t,\infty)$ leads to

$$x(t) \sim M_1(\rho)^{\frac{1}{\alpha}} m_2^{\frac{1}{\alpha+1}} \alpha \int_t^\infty \frac{R(s)^{\frac{k_1(\rho)-\eta+m_2\alpha}{\alpha}}}{-(k_1(\rho)-\eta+m_2\alpha)} \left(l_q(s) l_p(s)^{\frac{1}{\alpha+1}} l_x(s)^\beta\right)^{\frac{1}{\alpha}} ds, \tag{3.41}$$

as $t \to \infty$. We distinguish the two cases:

$$(a.1.1)\;\; k_1(\rho) - \eta + 2m_2\alpha < 0, \quad (a.1.2)\;\; k_1(\rho) - \eta + 2m_2\alpha = 0.$$

If we assume $(a.1.2)$, (3.41) shows that $x \in \mathcal{SV}_R$. On the other hand, if we assume $(a.1.1)$, applying Lemma 3.1 to the integral in (3.41), we get

$$x(t) \sim H_1(\rho) R(t)^{\frac{k_1(\rho)-\eta+2m_2\alpha}{\alpha}} \left(l_q(t) l_p(t) l_x(t)^\beta\right)^{\frac{1}{\alpha}}, \;\; t \to \infty, \tag{3.42}$$

with

$$H_1(\rho) = M_1(\rho)^{\frac{1}{\alpha}} \frac{m_2^{\frac{2}{\alpha+1}} \alpha^2}{(k_1(\rho) - \eta + m_2\alpha)(k_1(\rho) - \eta + 2m_2\alpha)}. \tag{3.43}$$

This means that

$$x \in \mathcal{RV}_R\left(\frac{k_1(\rho) - \eta + 2m_2\alpha}{\alpha}\right), \;\text{ with }\;\; m_1 = 2m_2 - \frac{\eta}{\alpha} < \frac{k_1(\rho) - \eta + 2m_2\alpha}{\alpha} < 0\,.$$

Proof of the "only if" part of Theorem 3.2: Suppose that x is a solution of (E) belonging to $\mathcal{RV}_R(\rho), \rho \in (m_1, 0)$. This is possible only when $(a.1.1)$ holds, in which case x must satisfy the asymptotic relation (3.42). Therefore,

$$\frac{k_1(\rho) - \eta + 2m_2\alpha}{\alpha} = \rho \;\; \Leftrightarrow \;\; \rho = \frac{\sigma + 2\alpha + \eta}{\alpha - \beta}, \tag{3.44}$$

which justifies (3.13). The assumption $\rho \in (m_1, 0)$ determines the range (3.12) of σ. Moreover, using (3.23), we rewrite (3.42) as

$$x(t) = R(t)^\rho l_x(t) \sim \left(\frac{m_2^2}{(\rho - m_1)(m_3 - \rho)\alpha^2}\right)^{\frac{1}{\alpha}} \frac{R(t)^\rho \left(l_q(t) l_p(t) l_x(t)^\beta\right)^{\frac{1}{\alpha}}}{\rho(\rho - m_2)}, \;\; t \to \infty, \tag{3.45}$$

showing that for a slowly varying part of x we have the asymptotic relation

$$l_x(t) \sim \left(\left(\frac{m_2}{\alpha}\right)^2 \frac{l_p(t) l_q(t)}{(\rho(\rho - m_2))^\alpha (\rho - m_1)(m_3 - \rho)}\right)^{\frac{1}{\alpha-\beta}}, \;\; t \to \infty.$$

Thus, we conclude that x enjoys the asymptotic behavior (3.14). This proves the "only if" part of the Theorem 3.2.

Proof of the "only if" part of Theorem 3.3: Suppose that x is a (I_1)−type of intermediate solution of (E) belonging to $\mathcal{RV}_R(m_1)$. Then, the case $(a.2)$ is the only possibility for x and (3.36) is satisfied by x. In view of $\rho = m_1$ this means that $\sigma + m_1\beta + 2m_2 = 0$, i.e. $\sigma = m_1(\alpha - \beta) - \eta - 2\alpha$. Using $x(t) = R(t)^{m_1} l_x(t)$ and (3.9), from (3.36) we get

$$\begin{aligned}\left(\frac{x(t)}{\varphi(t)}\right)^\alpha \;\; &\sim \;\; m_2^{-\frac{\alpha}{\alpha+1}} \int_{t_0}^t R(s)^{-m_2} l_q(s) l_p(s)^{\frac{1}{\alpha+1}} l_x(s)^\beta\, ds \sim \frac{m_2^{\frac{2\beta-\alpha}{\alpha+1}}}{(m_1(m_1 - m_2))^\beta} \int_{t_0}^t \left(\frac{x(s)}{\varphi(s)}\right)^\beta R(s)^{-m_2} l_p(s)^{\frac{\alpha+\beta(\alpha-1)}{\alpha(\alpha+1)}} l_q(s)\, ds \\ &= \;\; \int_{t_0}^t \left(\frac{x(s)}{\varphi(s)}\right)^\beta s q(s) \varphi(s)^\beta\, ds = v(t), \;\; t \to \infty.\end{aligned}$$

Next, for v we obtain the following differential asymptotic relation $v(t)^{-\frac{\beta}{\alpha}} v'(t) \sim t q(t) \varphi(t)^\beta, \;\; t \to \infty$ and by integration on $[t_0, t]$, we find that

$$\frac{x(t)}{\varphi(t)} \sim v(t)^{\frac{1}{\alpha}} \sim \left(\frac{\alpha - \beta}{\alpha} \int_{t_0}^t s q(s) \varphi(s)^\beta\, ds\right)^{\frac{1}{\alpha-\beta}}, \;\; t \to \infty, \tag{3.46}$$

implying the asymptotic relation $x(t) \sim X_2(t),\ t \to \infty$. Moreover, since $x(t)/\varphi(t) \to \infty$ as $t \to \infty$, (3.46) implies that $J_1 = \infty$. This proves the "only if" part of the proof of Theorem 3.3.

Proof of the "only if" part of Theorem 3.4: Let us now suppose that x is a type-(I_1) solution of (E) belonging to $ntr - \mathcal{SV}_R$. From the above observations this is possible only when the case $(a.1.2)$ holds, in which case $\rho = 0$ and x must satisfy the asymptotic behavior (3.41). In view of $\rho = 0$, $(a.1.2)$ gives $\sigma = -\eta - 2\alpha$. Using (3.38) with $\rho = 0$, asymptotic relation (3.41) becomes

$$x(t) \equiv l_x(t) \sim Q\int_t^\infty R(s)^{-m_2}\left(l_q(s)l_p(s)^{\frac{1}{\alpha+1}}l_x(s)^\beta\right)^{\frac{1}{\alpha}}\,ds = \mu(t),\ \ t\to\infty,\ \text{ with }\ Q = m_2^{\frac{2-\alpha^2}{\alpha(\alpha+1)}}\left(\alpha^2(-m_1)m_3\right)^{-\frac{1}{\alpha}}.$$

Noting that

$$-\mu'(t) = Q\,R(t)^{-m_2}\left(l_q(t)l_p(t)^{\frac{1}{\alpha+1}}l_x(t)^\beta\right)^{\frac{1}{\alpha}} \sim Q\,R(t)^{-m_2}\left(l_q(t)l_p(t)^{\frac{1}{\alpha+1}}\mu(t)^\beta\right)^{\frac{1}{\alpha}},\ \ t\to\infty,$$

we obtain the differential asymptotic relation

$$-\mu(t)^{-\frac{\beta}{\alpha}}\mu'(t) \sim QR(t)^{-m_2}\left(l_q(t)l_p(t)^{\frac{1}{\alpha+1}}\right)^{\frac{1}{\alpha}},\ \ t\to\infty.$$

Since the left-hand side of the previous relation is integrable on $[t_0,\infty)$ (note that $x(t)\to 0$ as $t\to\infty$ and so $\mu(t)\to 0$ as $t\to\infty$), in view of (3.28) we conclude that $J_2 < \infty$ and obtain the desired asymptotic relation for x

$$x(t) \sim \mu(t) \sim \left(\frac{\alpha-\beta}{\alpha}Q\int_t^\infty R(s)^{-m_2}\left(l_q(s)l_p(s)^{\frac{1}{\alpha+1}}\right)^{\frac{1}{\alpha}}\,ds\right)^{\frac{\alpha}{\alpha-\beta}} = Y_3(t) \sim X_3(t),\ \ t\to\infty.$$

This proves the "only if" part of Theorem 3.4.

Proof of the "if" part of theorems 3.2, 3.3 and 3.4: Suppose that (3.12) or (3.15) or (3.17) holds. From Lemma 3.5, 3.6 and 3.7 it is known that X_i, $i = 1,2,3$, defined by (3.14), (3.16) and (3.18) satisfy the asymptotic relation (3.21). Let $c > 0$ be arbitrary fixed. Then, since

$$J(t;a,X) \sim \int_t^\infty \frac{s-t}{p(s)^{\frac{1}{\alpha}}}\left(c + \int_a^s\int_r^\infty q(u)X(u)^\beta\,du\,dr\right)^{\frac{1}{\alpha}} ds,\ \ t\to\infty,$$

we have that these functions also satisfy the asymptotic relation

$$X(t) \sim \int_t^\infty \frac{s-t}{p(s)^{\frac{1}{\alpha}}}\left(c + \int_a^s\int_r^\infty q(u)X(u)^\beta\,du\,dr\right)^{\frac{1}{\alpha}} ds,\ \ t \ge a. \tag{3.47}$$

We perform the simultaneous proof for X_i, $i = 1,2,3$ so the subscripts $i = 1,2,3$ will be omitted in the rest of the proof. By (3.47) there exists $T_1 > T_0 > a$ such that

$$\int_t^\infty \frac{s-t}{p(s)^{\frac{1}{\alpha}}}\left(c + \int_{T_0}^s\int_r^\infty q(u)X(u)^\beta\,du\,dr\right)^{\frac{1}{\alpha}} ds \le 2X(t),\ \ t \ge T_0, \tag{3.48}$$

and

$$\int_t^\infty \frac{s-t}{p(s)^{\frac{1}{\alpha}}}\left(c + \int_{T_0}^s\int_r^\infty q(u)X(u)^\beta\,du\,dr\right)^{\frac{1}{\alpha}} ds \ge \frac{X(t)}{2},\ \ t \ge T_1. \tag{3.49}$$

It is possible to choose positive constants m and M so that

$$m \le \frac{\varphi(t)}{X(t)} \le M,\ \ T_0 \le t \le T_1. \tag{3.50}$$

Let k and K be positive constants such that

$$k \le \min\left\{c^{\frac{1}{\alpha}} m, 2^{\frac{\alpha}{\beta-\alpha}}\right\} \quad \text{and} \quad K \ge 2^{\frac{\alpha}{\alpha-\beta}}. \tag{3.51}$$

Define the set

$$\mathcal{X} = \{x \in C[T_0,\infty) : kX(t) \le x(t) \le KX(t),\ \ t \ge T_0\}, \tag{3.52}$$

which is a closed, convex subset of the locally convex space $C[T_0,\infty)$ equipped with the topology of uniform convergence on compact subintervals of $[T_0,\infty)$ and define the integral operator $\mathcal{G} : \mathcal{X} \to C[T_0,\infty)$ by

$$(\mathcal{G}x)(t) = \int_t^\infty \frac{s-t}{p(s)^{\frac{1}{\alpha}}} \left(c + \int_{T_0}^s \int_r^\infty q(v)x(v)^\beta \, dv \, dr \right)^{\frac{1}{\alpha}} ds, \qquad t \ge T_0. \tag{3.53}$$

We claim the existence of a solution $x \in \mathcal{X}$ of the integral equation $x(t) = (\mathcal{G}x)(t)$, $t \ge T_0$ by the Schauder-Tychonoff fixed point theorem. For that cause we show that $\mathcal{G}$ is a self-map on $\mathcal{X}$ and it sends $\mathcal{X}$ continuously to a relatively compact subset of $C[T_0,\infty)$.

(i) $\mathcal{G}(\mathcal{X}) \subset \mathcal{X}$. If $x \in \mathcal{X}$, using (3.48), (3.51) and (3.52) we get

$$\begin{aligned}(\mathcal{G}x)(t) &\le K^{\frac{\beta}{\alpha}} \int_t^\infty \frac{s-t}{p(s)^{\frac{1}{\alpha}}} \left(\frac{c}{K^\beta} + \int_{T_0}^s \int_r^\infty q(v)X(v)^\beta \, dv \, dr \right)^{\frac{1}{\alpha}} ds \\ &\le K^{\frac{\beta}{\alpha}} \int_t^\infty \frac{s-t}{p(s)^{\frac{1}{\alpha}}} \left(c + \int_{T_0}^s \int_r^\infty q(v)X(v)^\beta \, dv \, dr \right)^{\frac{1}{\alpha}} ds \le K^{\frac{\beta}{\alpha}} \cdot 2X(t) \le KX(t),\ \ t \ge T_0.\end{aligned}$$

On the other hand, using (3.49), (3.50), (3.51) and (3.52) we have

$$(\mathcal{G}x)(t) \ge c^{\frac{1}{\alpha}} \int_t^\infty \frac{s-t}{p(s)^{\frac{1}{\alpha}}} \, ds \ge mc^{\frac{1}{\alpha}} X(t) \ge kX(t),\ \ T_0 \le t \le T_1,$$

and

$$\begin{aligned}(\mathcal{G}x)(t) &\ge k^{\frac{\beta}{\alpha}} \int_t^\infty \frac{s-t}{p(s)^{\frac{1}{\alpha}}} \left(\frac{c}{k^\beta} + \int_{T_0}^s \int_r^\infty q(v)X(v)^\beta \, dv \, dr \right)^{\frac{1}{\alpha}} ds \\ &\ge k^{\frac{\beta}{\alpha}} \int_t^\infty \frac{s-t}{p(s)^{\frac{1}{\alpha}}} \left(c + \int_{T_0}^s \int_r^\infty q(v)X(v)^\beta \, dv \, dr \right)^{\frac{1}{\alpha}} ds \ge k^{\frac{\beta}{\alpha}} \frac{X(t)}{2} \ge kX(t),\end{aligned}$$

for all $t \ge T_1$. This shows that $\mathcal{G}x \in \mathcal{X}$, that is, $\mathcal{G}$ maps $\mathcal{X}$ into itself.

(ii) $\mathcal{G}(\mathcal{X})$ *is relatively compact.* The inclusion $\mathcal{G}(\mathcal{X}) \subset \mathcal{X}$ ensures that $\mathcal{G}(\mathcal{X})$ is locally uniformly bounded on $[T_0,\infty)$. From the inequality

$$0 \ge (\mathcal{G}x)'(t) \ge -K^{\frac{\beta}{\alpha}} \int_t^\infty \frac{1}{p(s)^{\frac{1}{\alpha}}} \left(c + \int_{T_0}^s \int_r^\infty q(v)X(v)^\beta \, dv \, dr \right)^{\frac{1}{\alpha}} ds,\ \ t \ge T_0\,,$$

holding for all $x \in \mathcal{X}$ it follows that $\mathcal{G}(\mathcal{X})$ is locally equicontinuous on $[T_0,\infty)$. The relative compactness of $\mathcal{G}(\mathcal{X})$ then follows from the Arzela-Ascoli theorem.

(iii) $\mathcal{G}$ *is continuous.* Let $\{x_n\}$ be a sequence in $\mathcal{X}$ converging to $x \in \mathcal{X}$ uniformly on compact subintervals of $[T_0,\infty)$. Then, by (3.53) we have

$$|(\mathcal{G}x_n)(t) - (\mathcal{G}x)(t)| \le \int_t^\infty \frac{s-t}{p(s)^{\frac{1}{\alpha}}} F_n(s) \, ds \tag{3.54}$$

where

$$F_n(s) = \left| \left(c + \int_{T_0}^{s} \int_{r}^{\infty} q(v) x_n(v)^\beta \, dv \, dr \right)^{\frac{1}{\alpha}} - \left(c + \int_{T_0}^{s} \int_{r}^{\infty} q(v) x(v)^\beta \, dv \, dr \right)^{\frac{1}{\alpha}} \right|, \ s \in [t, \infty).$$

Using the mean value theorem we get

$$F_n(t) \le \theta \int_{T_0}^{t} \int_{s}^{\infty} q(r) \big| x_n(r)^\beta - x(r)^\beta \big| drds, \quad t \ge T_0, \quad \text{since} \quad 0 < \alpha < 1\,, \tag{3.55}$$

where

$$\theta = \frac{1}{\alpha} \left(c + K^\beta \int_{T_0}^{t} \int_{s}^{\infty} q(r) X(r)^\beta drds \right)^{\frac{1-\alpha}{\alpha}}.$$

Thus, using that $q(t)\big|x_n(t)^\beta - x(t)^\beta| \to 0$ as $n \to \infty$ at each point $t \in [T_0, \infty)$ and $q(t)\big|x_n(t)^\beta - x(t)^\beta| \le (2K)^\beta q(t) X(t)^\beta$ for $t \ge T_0$, while $q(t)X(t)^\beta$ is integrable on $[T_0, \infty)$, the uniform convergence $F_n(t) \rightrightarrows 0$ as $n \to \infty$ on compact subinterval of $[T_0, \infty)$ follows by the application of the Lebesgue dominated convergence theorem. We conclude that

$$\lim_{n\to\infty} \big| (\mathcal{G} x_n)(t) - (\mathcal{G} x)(t) \big| = 0,$$

uniformly on any compact subinterval of $[T_0, \infty)$, which proves the continuity of $\mathcal{G}$.

Thus, all the hypotheses of the Schauder-Tychonoff fixed point theorem are fulfilled and so there exists a fixed point $x \in \mathcal{X}$ of $\mathcal{G}$, which satisfies integral equation $x(t) = (\mathcal{G}x)(t)$, $t \ge T_0$. Differentiating this equation four times shows that x is a solution of (E) on $[T_0, \infty)$, which due to (3.52) is an intermediate solution of type (I_1). Therefore, the proof of our main results will be completed with the verification that the intermediate solutions of (E) constructed above are actually regularly varying functions with respect to R. We put

$$l = \liminf_{t\to\infty} \frac{x(t)}{J(t; T_0, X)}, \quad L = \limsup_{t\to\infty} \frac{x(t)}{J(t; T_0, X)}.$$

By Lemmas 3.5, 3.6 and 3.7 we have $X(t) \sim J(t; T_0, X)$, $t \to \infty$. Since, $x \in \mathcal{X}$, it is clear that $0 < l \le L < \infty$. Applying generalized L'Hospital's rule four times, we obtain

$$\begin{aligned}
L &= \limsup_{t\to\infty} \frac{x(t)}{J(t; T_0, X)} \le \limsup_{t\to\infty} \frac{x'(t)}{J'(t; T_0, X)} \le \limsup_{t\to\infty} \frac{x''(t)}{J''(t; T_0, X)} \\
&= \limsup_{t\to\infty} \left(\frac{c + \int_{T_0}^{t} \int_{r}^{\infty} q(s) x(s)^\beta \, ds \, dr}{\int_{T_0}^{t} \int_{r}^{\infty} q(s) X(s)^\beta \, ds \, dr} \right)^{\frac{1}{\alpha}} \le \left(\limsup_{t\to\infty} \frac{\int_{T_0}^{t} \int_{r}^{\infty} q(s) x(s)^\beta \, ds \, dr}{\int_{T_0}^{t} \int_{r}^{\infty} q(s) X(s)^\beta \, ds \, dr} \right)^{\frac{1}{\alpha}} \\
&\le \left(\limsup_{t\to\infty} \frac{\int_{t}^{\infty} q(s) x(s)^\beta \, ds}{\int_{t}^{\infty} q(s) X(s)^\beta \, ds} \right)^{\frac{1}{\alpha}} \le \left(\limsup_{t\to\infty} \frac{q(t) x(t)^\beta}{q(t) X(t)^\beta} \right)^{\frac{1}{\alpha}} = \left(\limsup_{t\to\infty} \frac{x(t)}{X(t)} \right)^{\frac{\beta}{\alpha}} = L^{\frac{\beta}{\alpha}}.
\end{aligned}$$

Since $\beta/\alpha < 1$, the inequality $L \le L^{\frac{\beta}{\alpha}}$ implies that $L \le 1$. Similarly, repeated application of generalized L'Hospital's rule to l leads to $l \ge 1$, from which it follows that $L = l = 1$, that is, $x(t) \sim J(t; T_0, X) \sim X(t)$, $t \to \infty$. Therefore it is concluded that if $p \in \mathcal{RV}_R(\eta)$ and $q \in \mathcal{RV}_R(\sigma)$, then solution x of the type (I_1) is a member of $\mathcal{RV}_R(\rho)$, where

$$\rho = \frac{\sigma + 2\alpha + \eta}{\alpha - \beta} \in (m_1, 0) \quad \text{or} \quad \rho = m_1 \quad \text{or} \quad \rho = 0,$$

according to whether the pair (η, σ) satisfies (3.12), (3.15) or (3.17), respectively. Any such solution $x \in \mathcal{RV}_R(\rho)$ enjoys one and the same asymptotic behavior (3.14), (3.16) or (3.18) according as $\rho \in (m_1, 0)$, $\rho = m_1$ or $\rho = 0$. This completes the "if" parts of Theorems 3.2, 3.3 and 3.4.

3.2. Regularly varying solutions of type (I_2)

Let us turn our attention to the study of intermediate solutions of type (I_2) of the equation (E) with regularly varying coefficients satisfying (3.2), i.e. solutions x such that $t \prec x(t) \prec \psi(t)$ as $t \to \infty$. Since $\psi_1 \in \mathcal{RV}_R(m_2)$, $\psi_1(t) = t$ and $\psi \in \mathcal{RV}_R(m_3)$ (see (3.5) and (3.10)), the regularity index ρ of x must satisfy $m_2 \le \rho \le m_3$. If $\rho = m_2$, then since $x(t)/t \to \infty,\ t \to \infty$, x is a member of $\mathcal{RV}_R(m_2)$ with $x/\psi_1 \in ntr-\mathcal{SV}_R$, while if $\rho = m_3$, then since $x(t)/\psi(t) \to 0,\ t \to \infty$, x is a member of $\mathcal{RV}_R(m_3)$ with $x/\psi \in ntr-\mathcal{SV}_R$. Therefore, the totality of type-(I_2) intermediate solutions of (E) is divideed into the following three classes

$$\mathcal{RV}_R(m_2) \quad \text{or} \quad \mathcal{RV}_R(\rho),\ \rho \in (m_2, m_3) \quad \text{or} \quad \mathcal{RV}_R(m_3). \tag{3.56}$$

Our purpose is to show that, for each of the above classes, necessary and sufficient conditions for the membership are established and that the asymptotic behavior at infinity of all members of each class is determined precisely by a unique explicit formula.

3.2.1. Main results

Theorem 3.8. *Let $p \in \mathcal{RV}_R(\eta)$, $q \in \mathcal{RV}_R(\sigma)$. Equation* (E) *has intermediate solutions $x \in \mathcal{RV}_R(\rho)$ with $\rho \in (m_2, m_3)$ if and only if*

$$m_2(\alpha-\beta) - \eta - 2\alpha < \sigma < m_3(\alpha-\beta) - \eta - 2\alpha. \tag{3.57}$$

in which case ρ is given by (3.13) *and the asymptotic behavior of any such solution x is governed by the unique formula* (3.14).

Theorem 3.9. *Let $p \in \mathcal{RV}_R(\eta)$, $q \in \mathcal{RV}_R(\sigma)$. Equation* (E) *has intermediate solutions $x \in \mathcal{RV}_R(m_2)$ satisfying (I_2) if and only if*

$$\sigma = m_2(\alpha-\beta) - \eta - 2\alpha \ \text{and} \ J_3 = \infty. \tag{3.58}$$

The asymptotic behavior of any such solution x is governed by the unique formula

$$x(t) \sim Y_2(t) = t\left(\frac{\alpha-\beta}{\alpha}\int_a^t \frac{1}{p(s)^{\frac{1}{\alpha}}}\left(\int_a^s\int_r^\infty q(u)u^\beta\,du\,dr\right)^{\frac{1}{\alpha}} ds\right)^{\frac{\alpha}{\alpha-\beta}},\ \ t \to \infty. \tag{3.59}$$

Theorem 3.10. *Let $p \in \mathcal{RV}_R(\eta)$, $q \in \mathcal{RV}_R(\sigma)$. Equation* (E) *has intermediate solutions $x \in \mathcal{RV}_R(m_3)$ satisfying (I_2) if and only if*

$$\sigma = m_3(\alpha-\beta) - \eta - 2\alpha \ \ \text{and} \ \ J_4 < \infty. \tag{3.60}$$

The asymptotic behavior of any such solution x is governed by the unique formula

$$x(t) \sim Y_3(t) = \psi(t)\left(\frac{\alpha-\beta}{\alpha}\int_t^\infty q(s)\psi(s)^\beta\,ds\right)^{\frac{1}{\alpha-\beta}},\ \ t \to \infty. \tag{3.61}$$

3.2.2. Preparatory results

Let x be a type-(I_2) intermediate solution of (E) defined on $[t_0, \infty)$. It is known that for such a solution (2.1) holds and

$$\lim_{t\to\infty} x(t) = \infty,\ \lim_{t\to\infty} x'(t) = \infty,\ \lim_{t\to\infty} x^{[2]}(t) = \infty,\ \lim_{t\to\infty} x^{[3]}(t) = 0. \tag{3.62}$$

The three types of generalized $\mathcal{RV}-$intermediate solutions of type (I_2) will be constructed in what follows by solving the integral equation

$$x(t) = c + \int_{t_0}^t \frac{t-s}{p(s)^{\frac{1}{\alpha}}}\left(\int_{t_0}^s\int_r^\infty q(u)x(u)^\beta\,du\,dr\right)^{\frac{1}{\alpha}} ds, \qquad t \ge t_0, \tag{3.63}$$

for some constants $t_0 \geq a$ and $c > 0$. From (3.62) and (3.63) we easily see that all possible solutions x of type (I_1) of (E) satisfy the integral asymptotic relation

$$x(t) \sim \int_{t_0}^{t} \frac{t-s}{p(s)^{\frac{1}{\alpha}}} \left(\int_{t_0}^{s} \int_{r}^{\infty} q(u)x(u)^{\beta}\, du\, dr \right)^{\frac{1}{\alpha}} ds, \; t \to \infty. \tag{3.64}$$

First, we show that the generalized regularly varying functions X_1, Y_2, Y_3 defined in (3.14), (3.59), (3.61) satisfy the asymptotic relation (3.64). To simplify notation we put

$$I(t; a, X) = \int_{a}^{t} \frac{t-s}{p(s)^{\frac{1}{\alpha}}} \left(\int_{a}^{s} \int_{r}^{\infty} q(u)X(u)^{\beta}\, du\, dr \right)^{\frac{1}{\alpha}} ds, \;\; t \geq a.$$

Lemma 3.11. *Suppose that* (3.57) *holds and let* ρ *be defined by* (3.13). *Then,* X_1 *given in* (3.14) *satisfies the asymptotic relation* (3.64) *and* $X_1 \in \mathcal{RV}_R(\rho)$, *where* $m_2 < \rho < m_3$.

Proof. Let (3.57) holds. Using (3.2) and (3.13), we express $X_1(t)$ in the form $X_1(t) = R(t)^{\rho} L_1(t)$, where L_1 is defined in (3.22) and $L_1 \in \mathcal{SV}_R$. With the help of (3.57), we see that ρ defined by (3.13) satisfies $m_2 < \rho < m_3$ and so inequalities (3.23) hold. Thus, for X_1 we obtain asymptotic relation (3.24). Since

$$\rho - m_1 - \frac{\eta}{\alpha} + m_2 = \rho - m_2 > 0,$$

we integrate (3.24) on $[a, t]$ and obtain with the application of Lemma 3.1

$$\int_{a}^{t} \frac{1}{p(s)^{\frac{1}{\alpha}}} \left(\int_{a}^{s} \int_{r}^{\infty} q(u)X_1(u)^{\beta}\, du\, dr \right)^{\frac{1}{\alpha}} ds \sim$$

$$\frac{m_2^{\frac{2+\alpha}{\alpha(\alpha+1)}}}{\left(\alpha^2(\rho - m_1)(m_3 - \rho)\right)^{\frac{1}{\alpha}} (\rho - m_2)} R(t)^{\rho - m_2} \left(l_q(t) l_p(t)^{\frac{1}{1+\alpha}} \right)^{\frac{1}{\alpha}} L_1(t)^{\frac{\beta}{\alpha}}, \;\; t \to \infty,$$

while integration again over $[a, t]$ gives the desired asymptotic relation (3.64) for X_1. □

Lemma 3.12. *Suppose that* (3.58) *holds. Then,* Y_2 *given in* (3.59) *satisfies the asymptotic relation* (3.64) *and* $Y_2 \in \mathcal{RV}_R(m_2)$.

Proof. From (3.7) and (3.58) we have that

$$\sigma + m_2\beta + m_2 = -\alpha, \qquad \sigma + m_2\beta + 2m_2 = \alpha(m_2 - m_1)\,. \tag{3.65}$$

Thus, applying Lemma 3.1, with the help of (3.5) and (3.65) we get

$$\left(\frac{1}{p(t)} \int_{a}^{t} \int_{s}^{\infty} q(r) r^{\beta}\, dr\, ds \right)^{\frac{1}{\alpha}} \sim \left(\frac{m_2^{\frac{2-\alpha\beta}{\alpha+1}}}{\alpha^2(m_2 - m_1)} \right)^{\frac{1}{\alpha}} R(t)^{-m_2} \left(l_p(t)^{\frac{1-\alpha+\beta}{\alpha+1}} l_q(t) \right)^{\frac{1}{\alpha}} \tag{3.66}$$

which by integration over $[a, t]$ and combined with (3.5) gives the following expression for Y_2

$$Y_2(t) \sim t \left(\frac{\alpha - \beta}{\alpha} \left(\frac{m_2^{\frac{2-\alpha\beta}{\alpha+1}}}{\alpha^2(m_2 - m_1)} \right)^{\frac{1}{\alpha}} \int_{a}^{t} R(s)^{-m_2} \left(l_p(s)^{\frac{1-\alpha+\beta}{\alpha+1}} l_q(s) \right)^{\frac{1}{\alpha}} ds \right)^{\frac{\alpha}{\alpha-\beta}} \sim R(t)^{m_2} l_2(t) L_2(t) \tag{3.67}$$

where $l_2(t) = m_2^{-\frac{\alpha}{\alpha+1}} l_p(t)^{\frac{1}{\alpha+1}}$, $l_2 \in \mathcal{SV}_R$, and

$$L_2(t) = \left(\frac{\alpha - \beta}{\alpha} \left(\frac{m_2^{\frac{2-\alpha\beta}{\alpha+1}}}{\alpha^2(m_2 - m_1)} \right)^{\frac{1}{\alpha}} \int_{a}^{t} R(s)^{-m_2} \left(l_p(s)^{\frac{1-\alpha+\beta}{\alpha+1}} l_q(s) \right)^{\frac{1}{\alpha}} ds \right)^{\frac{\alpha}{\alpha-\beta}}, \quad L_2 \in \mathcal{SV}_R,$$

implying that $Y_2 \in \mathcal{RV}_R(m_2)$. Next, using (3.65) and the asymptotic relation (3.67), applying Lemma 3.1 we obtain

$$\left(\frac{1}{p(t)}\int_a^t\int_s^\infty q(r)Y_2(r)^\beta\,dr\,ds\right)^{\frac{1}{\alpha}} \sim \left(\frac{m_2^{\frac{2}{\alpha+1}}}{\alpha^2(m_2-m_1)}\right)^{\frac{1}{\alpha}} R(t)^{-m_2}\left(l_p(t)^{\frac{1-\alpha}{\alpha+1}}l_q(t)l_2(t)^\beta L_2(t)^\beta\right)^{\frac{1}{\alpha}}, \quad t\to\infty.$$

Integration of the previous relation on $[a,t]$ gives

$$I(t;a,Y_2) \sim R(t)^{m_2}l_2(t)\left(\frac{m_2^{\frac{2-\alpha\beta}{\alpha+1}}}{\alpha^2(m_2-m_1)}\right)^{\frac{1}{\alpha}}\int_a^t R(s)^{-m_2}\left(l_p(s)^{\frac{1-\alpha+\beta}{\alpha+1}}l_q(s)L_2(s)^\beta\right)^{\frac{1}{\alpha}}ds, \quad t\to\infty.$$

Integration by substitution $u=L_2(s)^{\frac{\alpha-\beta}{\alpha}}$ in the last integral gives the asymptotic relation $I(t;a,Y_2)\sim Y_2(t), t\to\infty$. □

Lemma 3.13. *Suppose that* (3.60) *holds. Then,* Y_3 *given in* (3.61) *satisfies the asymptotic relation* (3.64) *and* $Y_3 \in \mathcal{RV}_R(m_3)$.

Proof. From (3.10) we may express ψ with

$$\psi(t)\sim R(t)^{m_3}l_\psi(t), \quad t\to\infty, \ \text{ where } \ l_\psi(t)=\frac{m_2^{\frac{1}{\alpha+1}}}{m_3}l_p(t)^{\frac{1}{\alpha+1}}, \ \ l_\psi\in\mathcal{SV}_R\,. \tag{3.68}$$

Combining (3.68) with (3.61), we obtain the following asymptotic representation for Y_3:

$$Y_3(t)\sim R(t)^{m_3}l_\psi(t)L_3(t), \ \ t\to\infty, \tag{3.69}$$

where

$$L_3(t)=\left(\frac{\alpha-\beta}{\alpha}\int_t^\infty R(s)^{-m_2}l_q(s)l_\psi(s)^\beta\,ds\right)^{\frac{1}{\alpha-\beta}}, \ \ L_3\in\mathcal{SV}_R.$$

Therefore, $Y_3\in\mathcal{RV}_R(m_3)$. Using (3.60) and (3.69), by Lemma 3.1 we obtain

$$\left(\frac{1}{p(t)}\int_a^t\int_s^\infty q(r)Y_3(r)^\beta\,dr\,ds\right)^{\frac{1}{\alpha}} \sim m_2^{-\frac{1}{\alpha+1}}R(t)^{\frac{m_2-\eta}{\alpha}}l_p(t)^{-\frac{1}{\alpha+1}}W(t)^{\frac{1}{\alpha}}, \ \ t\to\infty, \tag{3.70}$$

where

$$W(t)=\int_t^\infty R(s)^{-m_2}l_q(s)\left(l_\psi(s)L_3(s)\right)^\beta ds, \ \ W\in\mathcal{SV}_R. \tag{3.71}$$

Integrating (3.70) twice on $[a,t]$ and using (3.68) and (3.71), we have

$$I(t;a,Y_3)\sim\frac{m_2^{\frac{1}{\alpha+1}}}{m_3}R(t)^{m_3}l_p(t)^{\frac{1}{\alpha+1}}W(t)^{\frac{1}{\alpha}}\sim\psi(t)\left(\int_t^\infty R(s)^{-m_2}l_q(s)\left(l_\psi(s)L_3(s)\right)^\beta ds\right)^{\frac{1}{\alpha}}, \ \ t\to\infty,$$

leading to the desired conclusion that $Y_3(t)$ satisfies the integral asymptotic relation (3.64), with integration by substitution $u=L_3(s)^{\alpha-\beta}$ in the last integral. □

3.2.3. Proof of main results

Suppose that the equation (E) has a type-(I_2) intermediate solution $x \in \mathcal{RV}_R(\rho)$ with $\rho \in [m_2, m_3]$ which is defined on $[t_0, \infty)$. In view of (3.62), the integrability of $x^{[3]}$ on $[t_0, \infty)$ implies (3.29) and that one of the following two cases can be valid

$$(a)\ \sigma + \rho\beta + m_2 < 0, \quad (b)\ \sigma + \rho\beta + m_2 = 0.$$

Assume that (b) holds. Then, as previously we obtain (3.30) and (3.31). Using that $(m_2 - \eta)/\alpha + m_2 = 1$, integration of (3.31) twice on $[t_0, t]$, with Lemma 3.1 gives

$$x'(t) \sim R(t) \left(\int_t^\infty R(s)^{-m_2} l_q(s) l_x(s)^\beta \, ds \right)^{\frac{1}{\alpha}}, \ \ t \to \infty, \tag{3.72}$$

and

$$x(t) \sim \frac{m_2^{\frac{1}{\alpha+1}}}{m_3} R(t)^{m_3} l_p(t)^{\frac{1}{\alpha+1}} \left(\int_t^\infty R(s)^{-m_2} l_q(s) l_x(s)^\beta \, ds \right)^{\frac{1}{\alpha}}, \ \ t \to \infty. \tag{3.73}$$

Thus, in this case, $x \in \mathcal{RV}_R(m_3)$.

Assume that (a) holds and as previously we obtain (3.33) and (3.34). As a consequence of the divergence of the integral in (3.34), we further consider the following two cases separately:

$$(a.1)\ \sigma + \rho\beta + 2m_2 > 0, \quad (a.2)\ \sigma + \rho\beta + 2m_2 = 0.$$

We show that $(a.2)$ can not hold, because otherwise from the asymptotic relation (3.34) we obtain (3.35), which by integration over $[t_0, t]$ yields

$$x'(t) \sim m_2^{-\frac{1}{\alpha+1}} \int_{t_0}^t R(s)^{-\frac{\eta}{\alpha}} l_p(s)^{-\frac{1}{\alpha}} \left(\int_{t_0}^s R(u)^{-m_2} l_q(u) l_p(u)^{\frac{1}{\alpha+1}} l_x(u)^\beta \, du \right)^{\frac{1}{\alpha}} ds, \ \ t \to \infty.$$

But, the last integral is divergent, because $x'(t) \to \infty$ as $t \to \infty$, leading to the contradiction that $m_2 - \eta/\alpha > 0$.

Therefore only $(a.1)$ can be valid and from (3.34) we obtain (3.37), which integrated over $[t_0, t]$, implies

$$x'(t) \sim M_1(\rho)^{\frac{1}{\alpha}} \int_{t_0}^t R(s)^{\frac{k_1(\rho) - \eta}{\alpha}} \left(l_q(s) l_p(s)^{\frac{1-\alpha}{\alpha+1}} l_x(s)^\beta \right)^{\frac{1}{\alpha}} ds, \ \ t \to \infty, \tag{3.74}$$

where $M_1(\rho)$ and $k_1(\rho)$ are defined in (3.38). To proceed further, due to divergence of the integral in (3.74), we consider cases

$$(a.1.1)\ k_1(\rho) - \eta + m_2\alpha > 0, \quad (a.1.2)\ k_1(\rho) - \eta + m_2\alpha = 0.$$

Suppose that $(a.1.2)$ holds. Integration of (3.74) on $[t_0, t]$ and application of Lemma 3.1 gives

$$x(t) \sim M_1(\rho)^{\frac{1}{\alpha}} m_2^{-\frac{\alpha}{\alpha+1}} R(t)^{m_2} l_p(t)^{\frac{1}{\alpha+1}} \int_{t_0}^t R(s)^{-m_2} \left(l_q(s) l_p(s)^{\frac{1-\alpha}{\alpha+1}} l_x(s)^\beta \right)^{\frac{1}{\alpha}} ds, \ \ t \to \infty, \tag{3.75}$$

which means that $x \in \mathcal{RV}_R(m_2)$.

Suppose that $(a.1.1)$ holds. Then, application of Lemma 3.1 in (3.74) gives

$$x'(t) \sim M_1(\rho)^{\frac{1}{\alpha}} m_2^{\frac{1}{\alpha+1}} \alpha \frac{R(t)^{\frac{k_1(\rho) - \eta + m_2\alpha}{\alpha}}}{k_1(\rho) - \eta + m_2\alpha} \left(l_q(t) l_p(t)^{\frac{1}{\alpha+1}} l_x(t)^\beta \right)^{\frac{1}{\alpha}}, \ \ t \to \infty, \tag{3.76}$$

which by integration on $[t_0, t]$ implies the asymptotic relation (3.42), where $H_1(\rho)$ is given in (3.43). This means that

$$x \in \mathcal{RV}_R\left(\frac{k_1(\rho) - \eta + 2m_2\alpha}{\alpha} \right), \ \text{with} \ \ m_2 < \frac{k_1(\rho) - \eta + 2m_2\alpha}{\alpha} < \frac{m_2 - \eta}{\alpha} + 2m_2 = m_3\,.$$

Proof of the "only if" part of theorem 3.8: Let x be an intermediate solution of (E) belonging to $\mathcal{RV}_R(\rho)$ for some $\rho \in (m_2, m_3)$. Clearly, the only case when this is possible is (a.1.1) and x must satisfy the asymptotic relation (3.42). Therefore, it holds (3.44), verifying that the regularity index ρ is given by (3.13). From the requirement $m_2 < \rho < m_3$ it follows that the range of σ is given by (3.57). Since

$$-(\sigma + \beta\rho + m_2) = \alpha(m_3 - \rho), \quad \sigma + \beta\rho + 2m_2 = \alpha(\rho - m_1),$$

the relation (3.42) can be rewritten as (3.45) from which it follows that x enjoys the asymptotic behavior (3.14). This proves the "only if" part of the Theorem 3.8.

Proof of the "only if" part of theorem 3.9: Now, let x be a type-(I_2) intermediate solution of (E) belonging to $\mathcal{RV}_R(m_2)$. Then, from the above observations it is clear that only the case (a.1.2) is admissible, so that (3.75) holds and $\rho = m_2$ and $k_1(m_2) - \eta + m_2\alpha = 0$ implying that $\sigma = m_2(\alpha - \beta) - \eta - 2\alpha$, using (3.7). From (3.5) and (3.75) we obtain

$$\frac{x(t)}{t} \sim M_1(m_2)^{\frac{1}{\alpha}} m_2^{-\frac{\beta}{\alpha+1}} \int_{t_0}^{t} \left(\frac{x(s)}{s}\right)^{\frac{\beta}{\alpha}} R(s)^{-m_2} \left(l_q(s) l_p(s)^{\frac{1-\alpha+\beta}{\alpha+1}}\right)^{\frac{1}{\alpha}} ds = \mu(t), \;\; t \to \infty.$$

Using (3.66) we obtain

$$\mu'(t) \sim \left(\frac{x(t)}{t}\right)^{\frac{\beta}{\alpha}} \left(\frac{1}{p(t)} \int_{t_0}^{t} \int_{s}^{\infty} q(r) r^{\beta}\, dr\, ds\right)^{\frac{1}{\alpha}} \sim \mu(t)^{\frac{\beta}{\alpha}} \left(\frac{1}{p(t)} \int_{t_0}^{t} \int_{s}^{\infty} q(r) r^{\beta}\, dr\, ds\right)^{\frac{1}{\alpha}}, \;\; t \to \infty,$$

which gives us the differential asymptotic relation for $\mu(t)$:

$$\mu(t)^{-\frac{\beta}{\alpha}} \mu'(t) \sim \left(\frac{1}{p(t)} \int_{t_0}^{t} \int_{s}^{\infty} q(r) r^{\beta}\, dr\, ds\right)^{\frac{1}{\alpha}}, \;\; t \to \infty. \tag{3.77}$$

Integration of (3.77) on $[t_0, t]$ gives

$$\frac{x(t)}{t} \sim \mu(t) \sim \left(\frac{\alpha - \beta}{\alpha} \int_{t_0}^{t} \left(\frac{1}{p(s)} \int_{t_0}^{s} \int_{r}^{\infty} q(v) v^{\beta}\, dv\, dr\right)^{\frac{1}{\alpha}} ds\right)^{\frac{\alpha}{\alpha-\beta}}, \;\; t \to \infty, \tag{3.78}$$

implying the desired asymptotic relation $x(t) \sim Y_2(t)$, $t \to \infty$. Since $x(t)/t \to \infty$ as $t \to \infty$, (3.78) also implies that $J_3 = \infty$. This completes the "only if" part of the Theorem 3.9.

Proof of the "only if" part of theorem 3.10: Let x is a type-(I_2) intermediate solution of (E) belonging to $\mathcal{RV}_R(m_3)$. Since only the case (b) is possible for x, it satisfies (3.73), which implies $\rho = m_3$ and $\sigma = -m_2 - \beta m_3$. From (3.73), using (3.10) we get

$$\begin{aligned}\left(\frac{x(t)}{\psi(t)}\right)^{\alpha} &\sim \int_{t}^{\infty} R(s)^{-m_2} l_q(s) l_x(s)^{\beta}\, ds \sim \left(\frac{m_2^{\frac{1}{\alpha+1}}}{m_3}\right)^{\beta} \int_{t}^{\infty} \left(\frac{x(s)}{\psi(s)}\right)^{\beta} R(s)^{-m_2} l_p(s)^{\frac{\beta}{\alpha+1}} l_q(s)\, ds \\ &\sim \int_{t}^{\infty} \left(\frac{x(s)}{\psi(s)}\right)^{\beta} q(s)\psi(s)^{\beta}\, ds = \nu(t), \;\; t \to \infty.\end{aligned}$$

Using the relation

$$\nu'(t) = -\left(\frac{x(t)}{\psi(t)}\right)^{\beta} q(t)\psi(t)^{\beta} \sim -\nu(t)^{\frac{\beta}{\alpha}} q(t)\psi(t)^{\beta}, \;\; t \to \infty,$$

we obtain the differential asymptotic relation

$$-\nu(t)^{-\frac{\beta}{\alpha}} \nu'(t) \sim q(t)\psi(t)^{\beta}, \;\; t \to \infty. \tag{3.79}$$

Since the left-hand side of (3.79) is integrable on $[t_0, \infty)$ (note that $\lim_{t\to\infty} x(t)/\psi(t) = 0$ and so $\lim_{t\to\infty} \nu(t) = 0$), so is the right-hand side, that is, $J_4 < \infty$. Integrating (3.79) over $[t, \infty)$, then yields

$$\frac{x(t)}{\psi(t)} \sim \nu(t)^{\frac{1}{\alpha}} \sim \left(\frac{\alpha-\beta}{\alpha}\int_t^\infty q(s)\psi(s)^\beta\, ds\right)^{\frac{1}{\alpha-\beta}}, \ t \to \infty, \tag{3.80}$$

which determines the precise asymptotic behavior of x as $x(t) \sim Y_3(t),\ t \to \infty$. Thus the "only if" part of the Theorem 3.10 has been proved.

Proof of the "if" part of theorems 3.8, 3.9 and 3.10: Suppose that (3.57) or (3.58) or (3.60) holds. From Lemmas 3.11, 3.12 and 3.13 it is known that X_1, Y_2, Y_3 defined by (3.14), (3.59) and (3.61) satisfy the asymptotic relation (3.64). We perform the simultaneous proof for X_1, Y_2, Y_3 so in the rest of the proof we will denote them by Y. By (3.64) there exist $T_1 \geq T_0 > a$ such that

$$I(t; T_0, Y) \leq 2Y(t),\ \ t \geq T_0 \ \ \text{and} \ \ \frac{Y(t)}{2} \leq I(t; T_0, Y),\ \ t \geq T_1.$$

Moreover, since $Y \in \mathcal{RV}_R(\rho)$, with $\rho > 0$, this function is almost increasing, that is there exist constant $A > 1$ such that $Y(x) \leq AY(y)$ for each $y \geq x > a$. Choose positive constants k and K such that

$$k \leq \min\left\{2^{\frac{\alpha}{\beta-\alpha}}, \frac{K\,Y(T_0)}{2A^2 Y(T_1)}\right\}, \quad K \geq 4^{\frac{\alpha}{\alpha-\beta}} .$$

Considering the integral operator

$$(\mathcal{H}x)(t) = c + \int_{T_0}^t \frac{t-s}{p(s)^{\frac{1}{\alpha}}}\left(\int_{T_0}^s \int_r^\infty q(v)x(v)^\beta\, dv\, dr\right)^{\frac{1}{\alpha}} ds, \qquad t \geq T_0,$$

where $c > 0$ is a constant such that $AkY(T_1) \leq c \leq \dfrac{K}{2A}Y(T_0)$, we may verify that $\mathcal{H}$ is continuous self-map on the set $\mathcal{Y} = \{x \in C[T_0, \infty) : kY(t) \leq x(t) \leq KY(t),\ \ t \geq T_0\}$ and that $\mathcal{H}$ sends $\mathcal{Y}$ into relatively compact subset of $C[T_0, \infty)$. Thus, $\mathcal{H}$ has a fixed point $x \in \mathcal{Y}$, which generates a solution of equation (E) of type (I_2) satisfying above inequalities and thus yields that

$$0 < \liminf_{t\to\infty} \frac{x(t)}{Y(t)} \leq \limsup_{t\to\infty} \frac{x(t)}{Y(t)} < \infty.$$

We put

$$l = \liminf_{t\to\infty} \frac{x(t)}{I(t; T_0, Y)}, \quad L = \limsup_{t\to\infty} \frac{x(t)}{I(t; T_0, Y)}.$$

By Lemmas 3.11, 3.12 and 3.13 we have $Y(t) \sim I(t; T_0, Y),\ t \to \infty$. Since, $x \in \mathcal{Y}$, it is clear that $0 < l \leq L < \infty$. Then, proceeding exactly as in the proof of the "if" part of Theorems 3.2, 3.3, 3.4, with application of generalized L'Hospital's rule, we conclude that $x(t) \sim I(t; T_0, Y) \sim Y(t)$, $t \to \infty$. Therefore, x is a generalized regularly varying solution of (E) with requested regularity index and the asymptotic behavior (3.14), (3.59), (3.61) depending on if (σ, η) satisfies, respectively, (3.57) or (3.58) or (3.60). Thus, the "if part" of Theorems 3.8, 3.9 and 3.10 has been proved.

4. Corollaries

The final section is concerned with equation (E) whose coefficients p and q are regularly varying functions in the sense of Karamata. Our purpose here is to show that this new problem can be embedded in the framework of generalized regularly varying functions, so that the results of the preceding section provide full information about the existence and the precise asymptotic behavior of regularly varying solutions of (E).

We assume that p and q are regularly varying functions of indices η and σ, respectively, i.e.,

$$p(t) = t^{\eta} l_p(t), \qquad q(t) = t^{\sigma} l_q(t), \qquad l_p, l_q \in \mathcal{SV},$$

and consider regularly varying solutions x of (E) expressed in the from $x(t) = t^{\rho} l_x(t)$, $l_x \in \mathcal{SV}$. Conditions (1.1) are satisfied if and only if $2\alpha \le \eta \le \alpha + 1$. In what follows we assume that $\eta < \alpha + 1$, because if $\eta = \alpha + 1$, that is $R \in \mathcal{SV}$, the assumption $p \in \mathcal{RV}_R(\eta^*)$, for some $\eta^* \in \mathbb{R}$ would imply the contradiction that $p \in \mathcal{SV}$. Then, it is easy to see that

$$R(t) = \int_a^t s^{\frac{1-\eta}{\alpha}} l_p(s)^{-\frac{1}{\alpha}}\, ds \sim \frac{\alpha}{1+\alpha-\eta} t^{\frac{1+\alpha-\eta}{\alpha}} l_p(t)^{-\frac{1}{\alpha}},$$

which means that

$$R \in \mathcal{RV}\left(\frac{\alpha+1-\eta}{\alpha}\right) \quad \text{and} \quad R^{-1} \in \mathcal{RV}\left(\frac{\alpha}{\alpha+1-\eta}\right).$$

Therefore, any regularly varying function $f \in \mathcal{RV}(\lambda)$ is considered as a generalized regularly varying function of index $\alpha\lambda/(\alpha + 1 - \eta)$ with respect to R, and conversely any generalized regularly varying function $f \in \mathcal{RV}_R(\lambda^*)$ is regarded as an regularly varying function of index $\lambda = \lambda^*(\alpha + 1 - \eta)/\alpha$. It follows that

$$p \in \mathcal{RV}_R\left(\frac{\eta\alpha}{1+\alpha-\eta}\right), \ q \in \mathcal{RV}_R\left(\frac{\sigma\alpha}{1+\alpha-\eta}\right), \ x \in \mathcal{RV}_R\left(\frac{\rho\alpha}{1+\alpha-\eta}\right).$$

Put

$$\eta^* = \frac{\eta\alpha}{1+\alpha-\eta}, \ \sigma^* = \frac{\sigma\alpha}{1+\alpha-\eta}, \ \rho^* = \frac{\rho\alpha}{1+\alpha-\eta}.$$

Three positive constants given by (3.7) are reduced to

$$m_1(\alpha,\eta^*) = \frac{2\alpha-\eta}{1+\alpha-\eta}, \ m_2(\alpha,\eta^*) = \frac{\alpha}{1+\alpha-\eta}, \ m_3(\alpha,\eta^*) = \frac{2\alpha-\eta-1}{1+\alpha-\eta}.$$

Based on the above observations we are able to apply the theory of generalized regularly varying functions built in Section 3 to establish necessary and sufficient conditions for the existence of intermediate regularly varying solutions of (E) and to determine the asymptotic behavior of all such solutions explicitly and accurately. First, we state the results on type-(I_1) intermediate solutions that can be derived as corollaries of Theorems 3.2, 3.3 and 3.4.

Theorem 4.1. *Assume that $p \in \mathcal{RV}(\eta)$ and $q \in \mathcal{RV}(\sigma)$. Equation* (E) *possess (I_1)–type intermediate intermediate regularly varying solutions belonging to $\mathcal{RV}(\rho)$ with $\rho \in \left(2 - \frac{\eta}{\alpha}, 0\right)$, if and only if*

$$\eta\frac{\beta}{\alpha} - 2\beta - 2 < \sigma < \eta - 2\alpha - 2.$$

in which case ρ is given by

$$\rho = \frac{\sigma + 2\alpha + 2 - \eta}{\alpha - \beta} \tag{4.1}$$

and any such solution x enjoys one and the same asymptotic behavior

$$x(t) \sim \left(\frac{t^{2\alpha+2} p(t)^{-1} q(t)}{(\rho(\rho-1))^{\alpha}(\alpha\rho - 2\alpha + \eta)(2\alpha - \eta + 1 - \alpha\rho)}\right)^{\frac{1}{\alpha-\beta}}, \ t \to \infty. \tag{4.2}$$

Theorem 4.2. *Assume that $p \in \mathcal{RV}(\eta)$ and $q \in \mathcal{RV}(\sigma)$. Equation* (E) *possess $(I_1)-$type intermediate solutions belonging to $\mathcal{RV}\left(2 - \frac{\eta}{\alpha}\right)$ if and only if*

$$\sigma = \eta\frac{\beta}{\alpha} - 2\beta - 2 \quad and \quad J_1 = \infty.$$

Any such solution x enjoys one and the same asymptotic behavior $x(t) \sim X_2(t)$, $t \to \infty$, where X_2 is given by (3.16).

Theorem 4.3. *Assume that $p \in \mathcal{RV}(\eta)$ and $q \in \mathcal{RV}(\sigma)$. Equation* (E) *possess $(I_1)-$type intermediate nontrivial slowly varying solutions if and only if*

$$\sigma = \eta - 2\alpha - 2 \quad and \quad J_2 < \infty.$$

Any such solution x enjoys one and the same asymptotic behavior $x(t) \sim X_3(t)$, $t \to \infty$, where X_3 is given by (3.18).

Similarly, from Theorems 3.8, 3.9 and 3.10, we are able to completely charaterize existence and asymptotic behavior of type-(I_2) intermediate regularly varying solutions of (E).

Theorem 4.4. *Assume that $p \in \mathcal{RV}(\eta)$ and $q \in \mathcal{RV}(\sigma)$. Equation* (E) *possess $(I_2)-$type intermediate regularly varying solutions belonging to $\mathcal{RV}(\rho)$ with $\rho \in \left(1, 2 - \frac{\eta-1}{\alpha}\right)$ if and only if*

$$-\alpha - \beta + \eta - 2 < \sigma < \frac{\beta}{\alpha}(\eta - 1) - 2\beta - 1.$$

in which case ρ is given by (4.1) *and the asymptotic behavior of any such solution x is governed by the unique formula* (4.2).

Theorem 4.5. *Assume that $p \in \mathcal{RV}(\eta)$ and $q \in \mathcal{RV}(\sigma)$. Equation* (E) *possess $(I_2)-$type intermediate regularly varying solutions belonging to $\mathcal{RV}(1)$ if and only if*

$$\sigma = -\alpha - \beta + \eta - 2 \quad and \quad J_3 = \infty.$$

The asymptotic behavior of any such solution $x(t)$ is governed by the unique formula $x(t) \sim Y_2(t)$, $t \to \infty$, where Y_2 is given by (3.59).

Theorem 4.6. *Assume that $p \in \mathcal{RV}(\eta)$ and $q \in \mathcal{RV}(\sigma)$. Equation* (E) *possess $(I_2)-$type intermediate regularly varying solutions belonging to $\mathcal{RV}\left(2 - \frac{\eta-1}{\alpha}\right)$ if and only if*

$$\sigma = \frac{\beta}{\alpha}(\eta - 1) - 2\beta - 1 \quad and \quad J_4 < \infty.$$

The asymptotic behavior of any such solution x is governed by the unique formula $x(t) \sim Y_3(t)$, $t \to \infty$, where Y_3 is given by (3.61).

5. Basic properties of regularly varying functions

We recall that the set of regularly varying functions of index $\rho \in \mathbb{R}$ is introduced by the following definition.

Definition 5.1. *A measurable function $f : (a, \infty) \to (0, \infty)$ for some $a > 0$ is said to be regularly varying at infinity of index $\rho \in \mathbb{R}$ if*

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^\rho \quad for\ all\ \lambda > 0.$$

The totality of all regularly varying functions of index ρ is denoted by $\mathcal{RV}(\rho)$. In the special case when $\rho = 0$, we use the notation $\mathcal{SV}$ instead of $\mathcal{RV}(0)$ and refer to members of $\mathcal{SV}$ as *slowly varying functions*.

The reader is referred to N.H. Bingham et al. [1] and E. Seneta [19] for the most complete exposition of theory of regular variation and its application to various branches of mathematical analysis.

Jaroš and Kusano introduced in [2] the class of generalized Karamata functions with the following definition.

Definition 5.2. *Let R be a positive function which is continuously differentiable on (a,∞) and satisfies $R'(t) > 0$, $t > a$ and $\lim_{t\to\infty} R(t) = \infty$. A measurable function $f : (a,\infty) \to (0,\infty)$ for some $a > 0$ is said to be regularly varying of index $\rho \in \mathbb{R}$ with respect to R if $f \circ R^{-1}$ is defined for all large t and is regularly varying function of index ρ in the sense of Karamata, where R^{-1} denotes the inverse function of R.*

The symbol $\mathcal{RV}_R(\rho)$ is used to denote the totality of regularly varying functions of index $\rho \in \mathbb{R}$ with respect to $R(t)$. The symbol $\mathcal{SV}_R$ is often used for $\mathcal{RV}_R(0)$.

We emphasize that there exists a function which is regularly varying in generalized sense, but is not regularly varying in the sense of Karamata, so that, roughly speaking, the class of generalized Karamata functions is larger than that of classical Karamata functions.

The following proposition summarizes selected properties of generalized regularly varying functions.

Proposition 5.3.

(i) *$f \in \mathcal{RV}_R(\sigma)$ if and only if $f(t) = R(t)^\sigma\, \ell(t)$, $\ell \in \mathcal{SV}_R$.*

(ii) *If $g_1 \in \mathcal{RV}_R(\sigma_1)$, then $(g_1)^\alpha \in \mathcal{RV}_R(\alpha\sigma_1)$ for any $\alpha \in \mathbb{R}$.*

(iii) *If $g_i \in \mathcal{RV}_R(\sigma_i)$, $i = 1,2$, then $g_1 + g_2 \in \mathcal{RV}_R(\sigma)$, $\sigma = \max(\sigma_1,\sigma_2)$.*

(iv) *If $g_i \in \mathcal{RV}_R(\sigma_i)$, $i = 1,2$, then $g_1 \cdot g_2 \in \mathcal{RV}_R(\sigma_1 + \sigma_2)$.*

(v) *If $g_i \in \mathcal{RV}_R(\sigma_i)$, $i = 1,2$ and $g_2(t) \to \infty$ as $t \to \infty$, then $g_1 \circ g_2 \in \mathcal{RV}_R(\sigma_1\sigma_2)$.*

(vi) *If $f \in \mathcal{RV}_R(\sigma)$ and $f(t) \sim g(t)$ as $t \to \infty$, then $g \in \mathcal{RV}_R(\sigma)$.*

(vii) *If $\ell \in \mathcal{SV}_R$, then for any $\varepsilon > 0$, $\lim_{t\to\infty} R(t)^\varepsilon \ell(t) = \infty$, and $\lim_{t\to\infty} R(t)^{-\varepsilon} \ell(t) = 0$.*

In view of Proposition 5.3-(i), if

$$\lim_{t\to\infty} \frac{f(t)}{R(t)^\rho} = \lim_{t\to\infty} \ell(t) = \text{const} > 0$$

then f is said to be a *trivial* regularly varying function of index ρ with respect to R and it is denoted by $f \in tr - \mathcal{RV}_R(\rho)$. Otherwise, f is said to be a *nontrivial* regularly varying function of index ρ with respect to R and it is denoted by $f \in ntr - \mathcal{RV}_R(\rho)$.

Next, we present a fundamental result (see [2]), called *Generalized Karamata integration theorem*, which played a central role in establishing our main results.

Proposition 5.4. (Generalized Karamata integration theorem) *Let $\ell \in \mathcal{SV}_R$. Then,*

(i) *If $\alpha > -1$,*

$$\int_a^t R'(s) R(s)^\alpha \ell(s)\, ds \sim \frac{R(t)^{\alpha+1}\, \ell(t)}{\alpha+1}, \quad t \to \infty;$$

(ii) *If $\alpha < -1$,*

$$\int_t^\infty R'(s)\, R(s)^\alpha\, \ell(s)\, ds \sim -\frac{R(t)^{\alpha+1}\, \ell(t)}{\alpha+1}, \quad t \to \infty;$$

(iii) *If $\alpha = -1$,*

$$\int_a^t R'(s)\, R(s)^{-1}\, \ell(s)\, ds \in \mathcal{SV}_R \quad \text{and} \quad \int_t^\infty R'(s)\, R(s)^{-1}\, \ell(s)\, ds \in \mathcal{SV}_R.$$

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