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Characterization of Symmetric Distributions Based on Concomitants of Ordered Variables from FGMs Family of Bivariate Distributions

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Abstract. Several characterization results of a symmetric distribution based on concomitants of order statistics as well as k-records from Farlie-Gumbel-Morgenstern (FGM) family of bivariate distributions are established. These include characterizations of a symmetric distribution on the basis of equality in distribution, moments, Rényi and Tsallis entropies of concomitants of upper and lower order statistics, also in terms of the same properties of concomitants of upper and lower k-records.

1. Introduction and preliminaries

It is well-known that characterization problems in applied probability and mathematical statistics are statements in which the description of possible distributions of random variables follows from properties of some functions in these variables. There are a lot of interesting characterizations of probability distributions based on ordered variables in the literature, which have been established by several authors. See, for example, Arnold et al. (1992, Ch. 6), Arnold et al. (1998, Ch. 4), Ahsanullah (2017) and references cited therein. It is known that the class of symmetric distributions is so broad and includes several well-known distributions such as normal, logistic, Student-t, Cauchy, Laplace, beta (with equal shape parameters) and uniform distributions. Also, the properties and characterization of symmetric distributions are widely used in many applications, see for example, Johnson et al. (1995) for more details. Let $X_1, X_2, ..., X_n$ be independent and identically distributed (iid) random variables with continuous distribution function (cdf) F_X and probability density function (pdf) f_X with support S_X and denote the corresponding order statistics by $X_{1:n}, ..., X_{n:n}$. Characterization of symmetric distributions based on the order statistics has been done by a number of authors. Milošević and Obradović (2016) proved that $|X_{r:n}|$ and $|X_{n-r+1:n}|$, $(r \leq n/2)$, are equally distributed if and only if X has a symmetric distribution with respect to zero. Balakrishnan and Selvitella (2017) proved that $X_{r:n} \stackrel{d}{=} -X_{n-r+1:n}$ for a given n and some fixed $r = 1, ..., n$ if and only if $f_X(x) = f_X(-x)$ for all $x \in \mathbb{R}$, i.e., the population distribution is symmetric, say around 0. Here, $\stackrel{d}{=}$ means that the two random variables have the same distribution. Testing of symmetry based on characterizations have been also considered by several authors, see, for example, Baringhaus and Henze (1992), Nikitin and Ahsanullah (2015), Amiri and Khaledi (2016), Milošević and Obradović (2016) and Božin et al. (2018) and references

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therein. The aim of this paper is to provide some new characterization results for symmetric distributions by using some properties of concomitants of ordered variables from the FGM family of bivariate distributions. This family of distributions has found extensive use in practice and is characterized by the specified marginal cdf $F_X(x)$ and $F_Y(y)$ of random variables X and Y, respectively, and a parameter λ, resulting in the bivariate distribution function given by

$$F_{X,Y}(x,y) = F_X(x)F_Y(y)[1 + \lambda(1 - F_X(x))(1 - F_Y(y))], \tag{1}$$

where $\lambda \in [-1,1]$ is known as the association parameter. Two random variables X and Y are independent whenever $\lambda = 0$, positively associated when $\lambda > 0$ and negatively associated when $\lambda < 0$. Bairamov and Kotz (2002) introduced an extension of FGM family of bivariate distributions. For more details about FGM family see, for example, Drouet Mari and Kotz (2001) and Balakrishnan and Lai (2009). In what follows, based on some properties of concomitants of order statistics as well as concomitants of record values from the FGM family of bivariate distributions, we establish characterizations of symmetric distributions. These include characterizations on the basis of identity in distribution, moments, Rényi entropy, Tsallis entropy and cross entropy of concomitants of upper and lower order statistics, also in terms of same properties of concomitants of upper and lower k-records. With this in mind and for the convenience of the readers, let us first recall some of notions here. Let X be a random variable having an absolutely continuous cdf F_X and pdf f_X with support S_X. Then, the entropy of order α or Rényi entropy is defined as

$$H_\alpha(X) = \frac{1}{1-\alpha}\log\int_{S_X} f_X^\alpha(x)dx, \ \alpha > 0, \ \alpha \neq 1. \tag{2}$$

It should be mentioned that in information theory, the Rényi entropy of general order unifies the well-known Shannon entropy with several other entropy notions, like the min-entropy or collision entropy. We refer the reader to Cover and Thomas (1991) for more details and references. Also, Tsallis entropy was introduced by Tsallis (1988) and it is a generalization of Boltzmann-Gibbs statistics. For a continuous random variable X with pdf f_X, Tsallis entropy of order α is defined by

$$T_\alpha(X) = \frac{1}{\alpha-1}\left(1 - \int_{S_X} f_X^\alpha(x)dx\right), \tag{3}$$

where the entropic index α is any real number. It can be shown that $\lim_{\alpha\to 1} H_\alpha(X) = H(X)$ and also $\lim_{\alpha\to 1} T_\alpha(X) = H(X)$, where

$$H(X) = -\int_{S_X} f_X(x)\log f_X(x)dx, \tag{4}$$

is commonly referred to as the entropy or Shannon information measure of X. Kerridge (1961) introduced the concept of inaccuracy in the context of information theory. Nath (1968) extended Kerridge's inaccuracy to the case of continuous situation and discussed some properties. Let X and Y be two continuous random variables with cdfs F and G, respectively. If F is the actual distribution function corresponding to the observations and G is the distribution assigned by the experimenter and f and g are the corresponding pdfs, then, the inaccuracy measure is defined as

$$K(X,Y) = -\int f(x)\log g(x)dx. \tag{5}$$

The inaccuracy measure in Eq. (5) is also known as the cross entropy between two probability distributions F and G, see for example Ghosh and Kundu (2018). We remind that when $g(x) = f(x)$ for all x, then Eq. (5) becomes the Shannon's entropy as given in (4). There are also characterization results based on some information measures of order statistics and record values in the literature. See, Baratpour et al. (2007, 2008), Ahmadi and Fashandi (2009), Thapliyal et al. (2015) and Kumar (2017). Fashandi and Ahmadi (2012) showed that the equality of entropies (Shannon and Rényi) of upper and lower order statistics as well as upper and lower records is a characteristic property of symmetric distributions. Recently, Ahmadi and Fashandi (2019) established some characterization results of symmetric continuous distributions based on various information measures properties of order statistics. In this work, we intend to obtain some characteristic results for symmetric distributions based on concomitants.

The rest of this paper is organized as follows. Section 2 contains characterization results of symmetric distributions based on concomitants of order statistics from FGM family of bivariate distributions. The results based on concomitants of k-records are given in Section 3.

Throughout the paper, we assume that the set where the pdf is positive is an interval, where it may be $(-\infty, +\infty)$. Also, throughout this paper we assume that the integrals exist.

2. Results based on concomitants of order statistics

Suppose $(X_i, Y_i), i = 1, ..., n$ is a random sample from a bivariate population (X, Y) with joint cdf $F(x, y)$. If we order the sample by the X-variate, and obtain the order statistics, for the X sample, then the Y-variate associated with the r-th order statistic $X_{r:n}$ is called the concomitant of the r-th order statistic, and is denoted by $Y_{[r:n]}$. Then, the pdf of $Y_{[r:n]}$ is given by

$$f_{Y_{[r:n]}}(y) = \int_{S_X} f_{Y|X}(y|x) f_{X_{r:n}}(x)dx, \quad y \in S_Y, \tag{6}$$

where S_Y stands for the support of Y and $f_{X_{r:n}}$ is the pdf of $X_{r:n}$ and is given by

$$f_{X_{r:n}}(x) = \frac{n!}{(r-1)!(n-r)!} F_X^{r-1}(x)(1 - F_X(x))^{n-r} f(x), \quad x \in S_X, \tag{7}$$

see for example, David and Nagaraja (2003). For FGM family, from Eq. (1) the corresponding joint pdf is given by

$$f_{X,Y}(x, y) = f_X(x) f_Y(y)[1 + \lambda(1 - 2F_X(x))(1 - 2F_Y(y))]. \tag{8}$$

Then, from Eqs. (6), (7) and (8), we immediately have the pdf of $Y_{[r:n]}$ to be

$$f_{Y_{[r:n]}}(y) = f_Y(y)[1 + \lambda_{[r,n]}(1 - 2F_Y(y))], \quad y \in S_Y, \tag{9}$$

where $\lambda_{[r,n]} = (1 - \frac{2r}{n+1})\lambda$. For some recent discussion on concomitants of order statistics arising from FGM family, see for example, Veena and Thomas (2017) and references therein.

In this section, we intend to investigate the conditions in which Y has a symmetric distribution based on some properties of concomitants of order statistics from FGM family of bivariate distributions. In all theorems, it is assumed that $(X_1, Y_1), ..., (X_n, Y_n)$ are iid pairs of bivariate random variables from a bivariate FGM distribution function (1).

Theorem 2.1. *The following two statements are equivalent:*
(i) Y has a symmetric distribution about μ;
(ii) $\mu - Y_{[r:n]} \stackrel{d}{=} \mu + Y_{[n-r+1:n]}$ for some fixed positive integers r and n, such that $1 \le r \le n/2$.

Proof. Suppose the marginal distribution of Y is symmetric around μ, then from (9) we have

$$\begin{aligned} f_{Y_{[r:n]}}(\mu - y) &= f_Y(\mu - y)[1 + \lambda_{[r,n]}(1 - 2F_Y(\mu - y))] \\ &= f_Y(\mu + y)[1 + \lambda_{[n-r+1,n]}(1 - 2F_Y(\mu + y))] \\ &= f_{Y_{[n-r+1:n]}}(\mu + y), \quad \text{for all } y \in S_Y, \end{aligned}$$

because of $\lambda_{[n-r+1,n]} = -\lambda_{[r,n]}$. This means that $\mu - Y_{[r:n]} \stackrel{d}{=} \mu + Y_{[n-r+1:n]}$ for all $r \in \{1, ..., n\}$. For proving (ii) $\Rightarrow$ (i), from Eq. (9), we obtain the cdf of $\mu - Y_{[r:n]}$ to be

$$F_{\mu - Y_{[r:n]}}(y) = \frac{1}{2} \int_{-1}^{1-2F_Y(\mu-y)} (1 + \lambda_{[r,n]}u)du. \tag{10}$$

Similarly, from Eq. (9), we have the cdf of $\mu + Y_{[n-r+1:n]}$ to be

$$\begin{aligned} F_{\mu+Y_{[n-r+1:n]}}(y) &= \int_{-\infty}^{y-\mu} f_Y(u)[1 + \lambda_{[n-r+1,n]}(1 - 2F_Y(u))]du \\ &= \frac{1}{2}\int_{-1}^{2F_Y(y-\mu)-1} (1 + \lambda_{[r,n]}u)du. \end{aligned} \tag{11}$$

By (ii) there exists some n and r, $(1 \le r \le n/2)$ such that $\mu - Y_{[r:n]} \overset{d}{=} \mu + Y_{[n-r+1:n]}$, i.e., $F_{\mu-Y_{[r:n]}}(t) = F_{\mu+Y_{[n-r+1:n]}}(t)$, for all $t \in \mathbb{R}$. Then, Eqs. (10) and (11) resulted that $F_Y(\mu - t) = 1 - F_Y(t - \mu)$, for all $t \in \mathbb{R}$. So, the proof is complete. □

Let C_1 be the class of all continuous cdf, F, with $F^{-1}(u) - \mu \ge \mu - F^{-1}(1 - u)$ for all $u \in (0, 1/2)$ or $F^{-1}(u) - \mu \le \mu - F^{-1}(1 - u)$ for all $u \in (0, 1/2)$, where $F^{-1}(u)$ is the quantile function and for $u \in [0, 1]$ is defined as $F^{-1}(u) = \inf\{t, F(t) \ge u\}$ and μ is the median of F (without loss of generality, we assume $\mu = 0$). It is known that the equality in C_1 means that F is symmetric. Using this fact, Amiri and Khaledi (2015) proposed a new test of symmetry against right skewness. It is easy to show that the class C_1 is not empty. For example, let $F(y) = \exp(-e^{-y}), -\infty < y < +\infty$, i.e., Y has Gumbel distribution (extreme value type I distribution for maximums), see, Kotz and Nadarajah (2000). Then, we immediately find $F^{-1}(u) = -\log(-\log u)$ and $\mu = -\log(\log 2)$. It is not difficult to verify that, for Gumbel distribution we have $F^{-1}(u) - \mu \ge \mu - F^{-1}(1 - u)$ for all $u \in (0, 1/2)$, i.e., the Gumbel distribution belongs to C_1. Many characterization results of distribution by recurrence relations of moment of order statistics are known. We have the next result for symmetric distribution based on the moments of concomitants of order statistics from FGM type bivariate distributions.

Theorem 2.2. *Suppose $E(Y^m)$ exists for some positive integer number m. Then, the following two statements are equivalent for any F_Y belongs to C_1:*
(i) Y has a symmetric distribution;
(ii) $E(Y^m_{[r:n]}) = (-1)^m E(Y^m_{[n-r+1:n]})$, for some fixed positive integers r and n, such that $1 \le r \le n/2$.

Proof. The proof for (i) ⇒ (*ii*) is easy. We prove that (*i*) follows by (*ii*). From Eq. (9), it is easy to show that for any positive integer number m,

$$E(Y^m_{[r:n]}) = \int_0^1 \{1 + \lambda_{[r,n]}(1 - 2u)\} [F_Y^{-1}(u)]^m du. \tag{12}$$

It follows that (12) showing that $E(Y^m_{[r:n]})$ exists provided $E(Y^m)$ exists. Similarly $E(Y^m_{[n-r+1:n]})$ is given by

$$E(Y^m_{[n-r+1:n]}) = \int_0^1 \{1 + \lambda_{[r,n]}(1 - 2u)\} [F_Y^{-1}(1 - u)]^m du. \tag{13}$$

By Eqs. (12) and (13), we obtain

$$\begin{aligned} E(Y^m_{[r:n]}) - (-1)^m E(Y^m_{[n-r+1:n]}) &= \int_0^1 \{1 + \lambda_{[r,n]}(1 - 2u)\} \left([F_Y^{-1}(u)]^m - [-F_Y^{-1}(1 - u)]^m\right) du \\ &= 2\lambda_{[r,n]} \int_0^{1/2} (1 - 2u) \left([F_Y^{-1}(u)]^m - [-F_Y^{-1}(1 - u)]^m\right) du. \end{aligned}$$

Now, suppose (*ii*) holds, then

$$\int_0^{1/2} (1 - 2u) \left([F_Y^{-1}(u)]^m - [-F_Y^{-1}(1 - u)]^m\right) du = 0.$$

By assumptions F_Y belongs to C_1, this implies that

$$[F_Y^{-1}(u)]^m - [-F_Y^{-1}(1-u)]^m = 0,$$

for all $u \in (0, 1/2)$. If m is an odd number, then $F_X^{-1}(u) = -F_X^{-1}(1-u)$ and if m is an even number then $F_X^{-1}(u) = \pm F_X^{-1}(1-u)$, the positive sign is not acceptable. These complete the proof. □

Let C_2 be the class of all continuous pdf, f, with connected support and $f(F^{-1}(u)) \geq f(F^{-1}(1-u))$ for all $u \in (0, 1/2)$ or $f(F^{-1}(u)) \leq f(F^{-1}(1-u))$ for all $u \in (0, 1/2)$. The class C_2 also is not empty. For example, if F is the Gumbel distribution, then we have $f(F^{-1}(u)) = -u \log u \geq -(1-u)\log(1-u) = f(F^{-1}(1-u))$ for all $u \in (0, 1/2)$. For FGM family, Fashandi and Ahmadi (2012) proved that (in Theorem 5) if $H(Y_{[r:n]}) = H(Y_{[n-r+1:n]})$, then Y has uniform distribution on its support under some conditions. We have the next result for symmetric distribution based on the entropies of concomitants of order statistics from FGM type bivariate distribution. Let us remind the following lemma that will be used to obtain the new results.

Lemma 2.3. *(Fashandi and Ahmadi, 2012) Let X be a continuous random variable with cdf F_X and pdf f_X with support S_X. Then, the identity*

$$f_X(F_X^{-1}(u)) = f_X(F_X^{-1}(1-u)), \text{ for almost all } u \in (0, 1/2),$$

implies that there exists a constant c such that $F_X(c-x) = 1 - F_X(c+x)$ for all $x \in S_X$.

Theorem 2.4. *The following two statements are equivalent for any f_Y belongs to C_2:*
(i) Y has a symmetric distribution;
(ii) $H(Y_{[r:n]}) = H(Y_{[n-r+1:n]})$, for some fixed positive integers r and n, such that $1 \leq r \leq n/2$.

Proof. It is easy to show that $(i) \Rightarrow (ii)$. We prove that (i) follows from (ii), Fashandi and Ahmadi (2012) proved the following identity

$$H(Y_{[n-r+1:n]}) - H(Y_{[r:n]} \quad = \quad \lambda_{[r:n]} \int_0^1 (1-2u) \log\left(\frac{f_Y(F_Y^{-1}(u))}{f_Y(F_Y^{-1}(1-u))}\right) du. \tag{14}$$

From Eq. (14), $H(Y_{[n-r+1:n]}) - H(Y_{[r:n]}) = 0$ is equivalent to

$$\int_0^1 (1-2u) \log\left(\frac{f_Y(F_Y^{-1}(u))}{f_Y(F_Y^{-1}(1-u))}\right) du = 0. \tag{15}$$

By Eq. (15), we have

$$\int_0^{1/2} (1-2u) \log\left(\frac{f_Y(F_Y^{-1}(u))}{f_Y(F_Y^{-1}(1-u))}\right) du = 0. \tag{16}$$

By assumption f_Y belongs to C_2, consequently, from Eq. (16) we arrive at

$$f_Y(F_Y^{-1}(u)) = f_Y(F_Y^{-1}(1-u)),$$

for all $u \in (0, 1/2)$. The proof is completed by Lemma 2.3. □

We have the next result based on the Rényi entropy of order α, where $\alpha > 0$ and $\alpha \neq 1$.

Theorem 2.5. *The following two statements are equivalent for any f_Y belongs to C_2:*
(i) Y has a symmetric distribution;
(ii) $H_\alpha(Y_{[r:n]}) = H_\alpha(Y_{[n-r+1:n]})$, for some fixed positive integers r and n, such that $1 \leq r \leq n/2$.

Proof. From Eqs. (2) and (9), the Rényi entropy of $Y_{[r:n]}$ is given by

$$\begin{aligned} H_\alpha(Y_{[r:n]}) &= \frac{1}{1-\alpha}\log\int_{S_Y}[f_{Y_{[r:n]}}(y)]^\alpha dy \\ &= \frac{1}{1-\alpha}\log\int_0^1 (1+\lambda_{[r,n]}(1-2u))^\alpha\,[f_Y(F_Y^{-1}(u))]^{\alpha-1}du,\ \alpha>0,\ \alpha\neq 1. \end{aligned} \tag{17}$$

Similarly, we get the following statement for the Rényi entropy of $Y_{[n-r+1:n]}$

$$H_\alpha(Y_{[n-r+1:n]}) = \frac{1}{1-\alpha}\log\int_0^1 (1+\lambda_{[r,n]}(1-2u))^\alpha\,[f_Y(F_Y^{-1}(1-u))]^{\alpha-1}du. \tag{18}$$

If Y has a symmetric distribution, then from Eqs. (17) and (18), we immediately have $H_\alpha(Y_{[r:n]}) = H_\alpha(Y_{[n-r+1:n]})$ for all r and n, $1\leq r\leq n/2$. Now, we show that (*i*) follows by (*ii*). If $H_\alpha(Y_{[r:n]}) = H_\alpha(Y_{[n-r+1:n]})$, then from this and Eqs. (17) and (18), it can be shown that

$$\begin{aligned} &\int_0^1 (1+\lambda_{[r,n]}(1-2u))^\alpha\,\{[f_Y(F_Y^{-1}(u))]^{\alpha-1} - [f_Y(F_Y^{-1}(1-u))]^{\alpha-1}\}du = \\ &\int_0^{1/2}\Big\{(1+\lambda_{[r,n]}(1-2u))^\alpha - (1-\lambda_{[r,n]}(1-2u))^\alpha\Big\}\{[f_Y(F_Y^{-1}(u))]^{\alpha-1} - [f_Y(F_Y^{-1}(1-u))]^{\alpha-1}\}du = 0. \end{aligned} \tag{19}$$

First suppose $\lambda\in(0,1)$, since by assumption $r\leq n/2$, then $0<\lambda_{[r,n]}<1$ and $0<\lambda_{[r,n]}(1-2u)<1$ for $u\in(0,1/2)$. Consequently, we have the following inequality

$$(1+\lambda_{[r,n]}(1-2u))^\alpha > (1-\lambda_{[r,n]}(1-2u))^\alpha,\ \alpha>0, \tag{20}$$

for $u\in(0,1/2)$. By assumptions f_Y belongs to C_2, thus Eqs. (19) and (20) conclude that

$$[f_Y(F_Y^{-1}(u))]^{\alpha-1} - [f_Y(F_Y^{-1}(1-u))]^{\alpha-1} = 0,\ \alpha>0,\ \alpha\neq 1,$$

for all $u\in(0,1/2)$. Then in this case, the proof is completed by Lemma 2.3. Now, suppose $\lambda\in(-1,0)$, then we have $0<-\lambda_{[r,n]}<1$. Consequently, we arrive at the following inequality

$$(1+\lambda_{[r,n]}(1-2u))^\alpha < (1-\lambda_{[r,n]}(1-2u))^\alpha,\ \alpha>0,$$

for $u\in(0,1/2)$. Then the expression in the first curly braces in Eq. (19) becomes negative. So, the proof is completed by noting that f_Y belongs to C_2 and using Lemma 2.3. □

From Eqs. (3) and (9), the Tsallis entropy of $Y_{[r:n]}$ is given by

$$T_\alpha(Y_{[r:n]}) = \frac{1}{\alpha-1}\left(1-\int_0^1 (1+\lambda_{[r,n]}(1-2u))^\alpha\,[f_Y(F_Y^{-1}(u))]^{\alpha-1}du\right). \tag{21}$$

Using Eq. (21), we obtain a similar result as in Theorem 2.5 based on Tsallis entropy of order α which is stated in the next theorem. The proof is similar, so it is omitted.

Theorem 2.6. *The following two statements are equivalent for any f_Y belongs to C_2:*
(i) Y has a symmetric distribution;
(ii) $T_\alpha(Y_{[r:n]}) = T_\alpha(Y_{[n-r+1:n]})$, for some fixed positive integers r and n, such that $1\leq r\leq n/2$.

Analogous to the cross entropy (5) between two density functions f and g, a measure of inaccuracy associated with the distribution of $Y_{[r:n]}$ and the parent distribution is given by

$$K(Y_{[r:n]},Y) = -\int_{S_Y} f_{Y_{[r:n]}}(y)\log f_Y(y)dy. \tag{22}$$

Next theorem provides a result for symmetric distributions based on the cross entropy of concomitants of order statistics in FGM family.

Theorem 2.7. *The following two statements are equivalent for any f_Y belongs to C_2:*
(i) Y has a symmetric distribution;
(ii) $K(Y_{[r:n]}, Y) = K(Y_{[n-r+1:n]}, Y)$, for some fixed positive integers r and n such that $1 \le r \le n/2$.

Proof. We prove $(ii) \Rightarrow (i)$. For FGM family, from Eqs. (9) and (22) we get

$$\begin{aligned} K(Y_{[r:n]}, Y) &= -\int_{S_Y} f_Y(y)[1 + \lambda_{[r,n]}(1 - 2F_Y(y))] \log f_Y(y)dy \\ &= -\int_0^1 [1 + \lambda_{[r,n]}(1 - 2u)] \log f_Y(F_Y^{-1}(u))du. \end{aligned} \tag{23}$$

Similarly, $K(Y_{[n-r+1:n]}, Y)$ is given by

$$\begin{aligned} K(Y_{[n-r+1:n]}, Y) &= -\int_0^1 [1 + \lambda_{[n-r+1,n]}(1 - 2u)] \log f_Y(F_Y^{-1}(u))du \\ &= -\int_0^1 [1 - \lambda_{[r,n]}(1 - 2u)] \log f_Y(F_Y^{-1}(u))du \\ &= -\int_0^1 [1 + \lambda_{[r,n]}(1 - 2u)] \log f_Y(F_Y^{-1}(1 - u))du. \end{aligned} \tag{24}$$

By assumptions, Eqs. (23) and (24), we arrive at

$$\int_0^1 (1 - 2u) \log\left(\frac{f_Y(F_Y^{-1}(u))}{f_Y(F_Y^{-1}(1 - u))}\right) du = 0. \tag{25}$$

So, by Eq. (25), the rest of the proof is similar to the proof of Theorem 2.4. □

3. Results based on concomitants of k-records

Let $\{X_i,\ i \ge 1\}$ be a sequence of iid random variables with an absolutely continuous cdf $F_X(x)$ and pdf $f_X(x)$ with support S_X. Upper k-record process is defined in terms of the k-th largest X yet seen, $k \ge 1$. For the continuous case, let $T^U_{1,k} = k, R^U_{1,k} = X_{1:k}$ and for $n \ge 2$, let $T^U_{n,k} = \min\{j : j > T_{n-1,k},\ X_j > X_{T^U_{n-1,k}-k+1:T^U_{n-1,k}}\}$, where $X_{i:m}$ denotes the i-th order statistic in a sample of size m. The sequence of *upper k-records* is then defined by $R^U_{n,k} = X_{T_{n,k}-k+1:T_{n,k}}$ for $n \ge 1$, see Arnold *et al.* (1998, p. 43). The pdf of $R^U_{n,k}$ is given by

$$f_{R^U_{n,k}}(u) = \frac{(-k \log \bar{F}_X(u))^{n-1}}{(n-1)!} k(\bar{F}_X(u))^{k-1} f_X(u),\ u \in S_X, \tag{26}$$

where $\bar{F}_X = 1 - F_X$ is the survival function of X. An analogous definition can be given for *lower k-record values*, let us denote the n-th lower k-record values by $R^L_{n,k}$. Then, the pdf of $R^L_{n,k}$ is

$$f_{R^L_{n,k}}(l) = \frac{(-k \log F_X(l))^{n-1}}{(n-1)!} k(F_X(l))^{k-1} f_X(l),\ l \in S_X. \tag{27}$$

See, Arnold et al. (1998) and references therein for more details on the theory and applications of record values. The superscript 'U' and 'L' stand for the upper record and the lower record, respectively.

Now, let $(X_1, Y_1), (X_2, Y_2), \ldots$ be a sequence of iid pairs of random variables with common absolutely continuous joint cdf $F_{X,Y}$ and $\{R^U_{n,k}, n \ge 1\}$ be the sequence of upper k-record values in the sequence of X's. Then the Y-variate associated with the X-value which qualified as the n-th upper k-record will be called the concomitant of the n-th upper k-record and will be denoted by $R^U_{[n,k]}$. Then, the pdf of $R^U_{[n,k]}$ is given by

$$f_{R^U_{[n,k]}}(y) = \int_{S_X} f_{Y|X}(y|x) f_{R^U_{n,k}}(x)dx,\quad y \in S_Y, \tag{28}$$

see Arnold et al. (1998, p. 272) for the case of ordinary records (case $k = 1$). When the bivariate sequence $\{(X_i, Y_i), i \geq 1\}$ follows a bivariate FGM distribution, then from Eqs. (8), (26) and (28), we obtain the pdf of $R^U_{n,k}$ to be

$$f_{R^U_{[n,k]}}(y) = f_Y(y)\left\{1 + \lambda_{(n,k)}(1 - 2F_Y(y))\right\}, \quad y \in S_Y, \tag{29}$$

where $\lambda_{(n,k)} = (2\left(\frac{k}{k+1}\right)^n - 1)\lambda$, see also Theorem 1 in Chacko and Mary (2013). Similarly, one can define the concomitant corresponding to the lower k-records. Denote the concomitant of the n-th lower k-record by $R^L_{[n,k]}$, then the pdf of $R^L_{[n,k]}$ is

$$f_{R^L_{[n,k]}}(y) = \int_{S_X} f_{Y|X}(y|x) f_{R^L_{n,k}}(x)dx, \quad y \in S_Y. \tag{30}$$

For FGM family, from Eqs. (8), (27) and (30), we obtain the pdf of $R^L_{[n,k]}$ to be

$$f_{R^L_{[n,k]}}(y) = f_Y(y)\left\{1 - \lambda_{(n,k)}(1 - 2F_Y(y))\right\}, \quad y \in S_Y. \tag{31}$$

In the remainder of this section, we establish some characterization results for symmetric distributions based on the properties of concomitants of k-records in FGM type bivariate distributions. In all theorems, it is assumed that $\{(X_i, Y_i), i \geq 1\}$ is a sequence of iid pairs of random variables form bivariate FGM distribution function (1).

Theorem 3.1. *The following two statements are equivalent:*
(i) Y has a symmetric distribution about μ;
(ii) $\mu - R^U_{[n,k]} \stackrel{d}{=} \mu + R^L_{[n,k]}$ for some fixed positive integers $n, k \geq 1$.

Proof. Suppose the marginal distribution of Y is symmetric around μ, then from Eq. (29) we have

$$\begin{aligned} f_{R^U_{[n,k]}}(\mu - y) &= f_Y(\mu - y)\left\{1 + \lambda_{(n,k)}(1 - 2F_Y(\mu - y))\right\} \\ &= f_Y(\mu + y)\left\{1 + \lambda_{(n,k)}(2F_Y(\mu + y) - 1))\right\} \\ &= f_{R^L_{[n,k]}}(\mu + y), \quad \text{for } y \in S_Y, \text{ by (31).} \end{aligned}$$

This means that $\mu - R^U_{[n,k]} \stackrel{d}{=} \mu + R^L_{[n,k]}$ for all $n, k \geq 1$. For the converse from Eq. (29), we obtain the cdf of $\mu - R^U_{[r:n]}$ to be

$$F_{\mu - R^U_{[r:n]}}(y) = \frac{1}{2}\int_{-1}^{1-2F_Y(\mu-y)} (1 + \lambda_{(n,k)}u)du. \tag{32}$$

Similarly, from Eq. (31), we have the cdf of $\mu + R^L_{[n,k]}$ to be

$$F_{\mu + R^L_{[n,k]}}(y) = \frac{1}{2}\int_{-1}^{2F_Y(y-\mu)-1} (1 + \lambda_{(n,k)}u)du. \tag{33}$$

Then, Eqs. (32) and (33) resulted that $F_Y(\mu - t) = 1 - F_Y(t - \mu)$, for all $t \in \mathbb{R}$. Hence, the proof is complete. □

We have the next result based on the moments of the concomitants of $k-$records for FGM family. Here also let the class C_1 be as in the previous section, i.e. the class of all continuous cdf, F, with $F^{-1}(u) \geq -F^{-1}(1-u)$ for all $u \in (0, 1/2)$ or $F^{-1}(u) \leq -F^{-1}(1 - u)$ for all $u \in (0, 1/2)$.

Theorem 3.2. *Suppose $E(Y^m)$ exists for some positive integer number m. Then, the following two statements are equivalent for any F_Y belongs to C_1:*
(i) Y has a symmetric distribution;
(ii) $E\left(R^U_{[n,k]}\right)^m = (-1)^m E\left(R^L_{[n,k]}\right)^m$, for some fixed positive integers $n, k \geq 1$.

Proof. From Eq. (29), it is easy to show that

$$E[\left(R^U_{[n,k]}\right)^m] = \int_0^1 \left\{1 + \lambda_{(n,k)}(1-2u)\right\} [F_Y^{-1}(u)]^m du. \tag{34}$$

Similarly from Eq. (31), we have

$$E[\left(R^L_{[n,k]}\right)^m] = \int_0^1 \left\{1 + \lambda_{(n,k)}(1-2u)\right\} [F_Y^{-1}(1-u)]^m du. \tag{35}$$

By Eqs. (34) and (35), we obtain

$$\begin{aligned} E[\left(R^U_{[n,k]}\right)^m] - E[\left(-R^L_{[n,k]}\right)^m] &= \int_0^1 \left\{1 + \lambda_{(n,k)}(1-2u)\right\} \left([F_Y^{-1}(u)]^m - [-F_Y^{-1}(1-u)]^m\right) du \\ &= 2\lambda_{(n,k)} \int_0^{1/2} (1-2u) \left([F_Y^{-1}(u)]^m - [-F_Y^{-1}(1-u)]^m\right) du. \end{aligned}$$

Now, suppose (*ii*) holds, then

$$\int_0^{1/2} (1-2u) \left([F_Y^{-1}(u)]^m - [-F_Y^{-1}(1-u)]^m\right) du = 0.$$

By assumptions F_Y belongs to C_1, this implies that

$$[F_Y^{-1}(u)]^m - [-F_Y^{-1}(1-u)]^m = 0,$$

for all $u \in (0, 1/2)$. The rest of the proof is similar to the proof of Theorem 2.2. □

We have the next result regarding the entropies properties of concomitants of k-records form FGM family of bivariate distributions. Here also, let C_2 be the same as in the previous section, i.e., C_2 is the class of all continuous pdf, f, with connected support and $f(F^{-1}(u)) \geq f(F^{-1}(1-u))$ for all $u \in (0, 1/2)$ or $f(F^{-1}(u)) \leq f(F^{-1}(1-u))$ for all $u \in (0, 1/2)$.

Theorem 3.3. *The following two statements are equivalent for f_Y belongs to C_2:*
(i) Y has a symmetric distribution;
(ii) $H(R^U_{[n,k]}) = H(R^L_{[n,k]})$, for some fixed positive integers $n, k \geq 1$.

Proof. From Eq. (29), the entropy of $R^U_{[n,k]}$ is given by

$$H(R^U_{[n,k]}) = -\int_0^1 \left(1 + \lambda_{(n,k)}(1-2u)\right) \{\log f_Y(F_Y^{-1}(u)) - \log\left(1 + \lambda_{(n,k)}(1-2u)\right)\} du. \tag{36}$$

Similarly, from Eq. (31), the entropy of $R^L_{[n,k]}$ is

$$H(R^L_{[n,k]}) = -\int_0^1 \left(1 + \lambda_{(n,k)}(2u-1)\right) \{\log f_Y(F_Y^{-1}(u)) - \log\left(1 + \lambda_{(n,k)}(2u-1)\right)\} du. \tag{37}$$

If Y has a symmetric distribution, then from Eqs. (36) and (37), we readily find that $H(R^U_{[n,k]}) = H(R^L_{[n,k]})$ for all $n, k \geq 1$. Now, suppose (*ii*) holds, then from Eqs. (36) and (37) we obtain

$$\int_0^1 (1-2u) \log\left(\frac{f_Y(F_Y^{-1}(u))}{f_Y(F_Y^{-1}(1-u))}\right) du = 0.$$

The rest of the proof is the same as in Theorem 2.4. □

Theorem 3.4. *The following two statements are equivalent for f_Y belongs to C_2:*
(i) Y has a symmetric distribution;
(ii) f_Y belongs to C_2 and $H_\alpha(R^U_{[n,k]}) = H_\alpha(R^L_{[n,k]})$, for some fixed positive integers $n, k \geq 1$.

Proof. From Eqs. (2) and (29), the Rényi entropy of $R^U_{[n,k]}$ is given by

$$\begin{aligned} H_\alpha(R^U_{[n,k]}) &= \frac{1}{1-\alpha}\log\int_{S_Y}[f_{R^U_{[n,k]}}(u)]^\alpha du \\ &= \frac{1}{1-\alpha}\log\int_0^1\Big(1+\lambda_{(n,k)}(1-2u)\Big)^\alpha[f_Y(F_Y^{-1}(u))]^{\alpha-1}du. \end{aligned} \tag{38}$$

Similarly, from Eqs. (2) and (31), the Rényi entropy of $R^L_{[n,k]}$ is

$$H_\alpha(R^L_{[n,k]}) = \frac{1}{1-\alpha}\log\int_0^1\Big(1+\lambda_{(n,k)}(1-2u)\Big)^\alpha[f_Y(F_Y^{-1}(1-u))]^{\alpha-1}du. \tag{39}$$

Suppose (*ii*) holds, then from Eqs. (38) and (39) we get

$$\int_0^1\Big(1+\lambda_{(n,k)}(1-2u)\Big)^\alpha\Big\{[f_Y(F_Y^{-1}(u))]^{\alpha-1}-[f_Y(F_Y^{-1}(1-u))]^{\alpha-1}\Big\}du = 0.$$

This can be rewritten as

$$\int_0^{1/2}\Big\{\Big(1+\lambda_{(n,k)}(1-2u)\Big)^\alpha-\Big(1-\lambda_{(n,k)}(1-2u)\Big)^\alpha\Big\}\{[f_Y(F_Y^{-1}(u))]^{\alpha-1}-[f_Y(F_Y^{-1}(1-u))]^{\alpha-1}\}du = 0. \tag{40}$$

If $\lambda_{(n,k)} \in (-1, 0]$, then $0 < \Big(1+\lambda_{(n,k)}(1-2u)\Big) < 1$ and $1 < \Big(1-\lambda_{(n,k)}(1-2u)\Big) < 2$ for all $u \in (0, 1/2)$, then, the first term in the integrand (40) is negative. Also, if $\lambda_{(n,k)} \in (0, 1)$, then $1 < \Big(1+\lambda_{(n,k)}(1-2u)\Big) < 2$ and $0 < \Big(1-\lambda_{(n,k)}(1-2u)\Big) < 1$ for all $u \in (0, 1/2)$, these imply that the first term in the integrand (40) is positive. Consequently, by assumptions and Lemma 2.3, the proof is completed. □

We have a similar result as in Theorem 3.4 based on Tsallis entropy of concomitants of k-records in FGM family which is stated in the next theorem. The proof is similar, so it is omitted.

Theorem 3.5. *The following two statements are equivalent for f_Y belongs to C_2:*
(i) Y has a symmetric distribution;
(ii) $T_\alpha(R^U_{[n,k]}) = T_\alpha(R^L_{[n,k]})$, for some fixed positive integers $n, k \geq 1$.

Similar to the cross entropy (5) between two pdfs f and g, a measure of inaccuracy associated with distribution of $R^U_{[n,k]}$ and the parent distribution is given by

$$K(R^U_{[n,k]}, Y) = -\int_{S_Y} f_{R^U_{[n,k]}}(y)\log f_Y(y)dy. \tag{41}$$

Next theorem provides a result for symmetric distributions based on Kerridge inaccuracy of concomitants of k-records in FGM family.

Theorem 3.6. *The following two statements are equivalent for f_Y belongs to C_2:*
(i) Y has a symmetric distribution;
(ii) $K(R^U_{[n,k]}, Y) = K(R^L_{[n,k]}, Y)$, for some fixed positive integers $n, k \geq 1$.

Proof. We prove (*ii*) ⇒ (*i*). For FGM family, from Eqs. (9) and (41) we get

$$\begin{aligned} K(R^U_{[n,k]}, Y) &= -\int_{S_Y} f_Y(u)[1 + \lambda_{(n,k)}(1 - 2F_Y(u))] \log f_Y(y)dy \\ &= -\int_0^1 [1 + \lambda_{(n,k)}(1 - 2u)] \log f_Y(F_Y^{-1}(u))du. \end{aligned} \tag{42}$$

Similarly, $K(R^L_{[n,k]}, Y)$ is given by

$$\begin{aligned} K(R^L_{[n,k]}, Y) &= -\int_0^1 [1 + \lambda_{(n,k)}(2u - 1)] \log f_Y(F_Y^{-1}(u))du \\ &= -\int_0^1 [1 + \lambda_{(n,k)}(1 - 2u)] \log f_Y(F_Y^{-1}(1 - u))du. \end{aligned} \tag{43}$$

By assumptions, Eqs. (42) and (43), we arrive at

$$\int_0^1 [1 + \lambda_{(n,k)}(1 - 2u)] \log\left(\frac{f_Y(F_Y^{-1}(u))}{f_Y(F_Y^{-1}(1 - u))}\right)du = 0. \tag{44}$$

So, by Eq. (44), the rest of the proof is similar to the proof of Theorem 2.4. □

4. Conclusion

In this paper, we have provided various new characterization results for symmetric distributions based on some properties of concomitants of order statistics as well as on the basis of concomitants of *k*-records in FGM family. It is well known that characterizations of distributions often provide useful tools for constructing goodness-of-fit statistics. It may be noted that testing for symmetry is one of the oldest classical nonparametric problems and has an extensive literature. It has been investigated by several authors based on characterizations results, see for example, Baringhaus and Henze (1992), Nikitin and Ahsanullah (2015), Milošević and Obradović (2016), Allison and Pretorius (2017) and Božin et al. (2018). As mentioned by Józefczyk (2012), an answer to the question about symmetry is usually essential for many problems in econometrics, computer science, engineering and social sciences. So, the results obtained in this article may be useful in constructing goodness-of-fit test for symmetry.

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References

[1] J. Ahmadi, M. Fashandi, Some characterization and ordering results based on entropies of current records. Statistics and Probability Letters, 79 (2009) 2053–2059.
[2] J. Ahmadi, M. Fashandi, Characterization of symmetric distributions based on some information measures properties of order statistics. Physica A: Statistical Mechanics and its Applications, 517 (2019) 141–152.
[3] M. Ahsanullah, Characterizations of Univariate Continuous Distributions. Atlantis Press, 2017.
[4] J. S. Allison, C. Pretorius, A Monte Carlo evaluation of the performance of two new tests for symmetry. Computational Statistics, 32 (2017) 1323–1338.
[5] M. Amiri, B. E., Khaledi, A new test for symmetry against right skewness. Journal of Statistical Computation and Simulation, 86 (2016) 1479–1496.
[6] B. C. Arnold, N. Balakrishnan, H. N. Nagaraja, A First Course in Order Statistics. John Wiley & Sons, New York, 1992.
[7] B. C. Arnold, N. Balakrishnan, H. N. Nagaraja, Records. John Wiley & Sons, New York, 1998.

[8] I. Bairamov, S. Kotz, Dependence structure and symmetry of Huang-Kotz FGM distributions and their extensions. Metrika, 56 (2002) 55–72.
[9] N. Balakrishnan, C. D. Lai, Continuous Bivariate Distributions, second ed. Springer, New York, 2009.
[10] N. Balakrishnan, A. Selvitella, Symmetry of a distribution via symmetry of order statistics. Statistics and Probability Letters, 129 (2017) 367–372.
[11] L. Baringhaus, N. Henze, A characterization of and new consistent tests for symmetry. Communications in Statistics-Theory and Methods, 21 (1992) 1555–1566.
[12] S. Baratpour, J. Ahmadi, N. R. Arghami, Some characterizations based on entropy of order statistics and record values. Communications in Statistics-Theory and Methods, 36 (2007) 47–57.
[13] S. Baratpour, J. Ahmadi, N. R. Arghami, Characterizations based on Rényi entropy of order statistics and record values. Journal of Statistical Planning and Inference, f 138 (2008) 2544–2551.
[14] V. Božin, B. Milošević, Y. Y. Nikitin, M. Obradović, New characterization based symmetry tests. Bulletin of the Malaysian Mathematical Sciences Society, (2018) https://doi.org/10.1007/s40840-018-0680-3
[15] M. Chacko, M. S. Mary, Concomitants of k-record values arising from Morgenstern family of distributions and their applications in parameter estimation. Statistical Papers, 54 (2013) 21–46.
[16] Cover, T. M. and Thomas, J. A. (1991) Elements of Information Theory. A Wiley-Interscience Publication, Wiley, New York.
[17] H. D. David, H. N. Nagaraja, Order Statistics, third ed. John Wiley & Sons, Hoboken, New Jersey, 2003.
[18] D. Drouet Mari, S. Kotz, Correlation and Dependence. Imperial College Press, London, 2001.
[19] M. Fashandi, J. Ahmadi, Characterizations of symmetric distributions based on Rényi entropy. Statistics and Probability Letters, 82 (2012) 798–804.
[20] J. L. Johnson, S. Kotz, N. Balakrishnan, Continuous Univariate Distributions. Vol. 2 Houghton Mifflin co., Boston, 1995.
[21] J. Józefczyk, Data driven score tests for univariate symmetry based on non-smooth functions. Probability and Mathematical Statistics, 32 (2012) 301–322.
[22] A. Ghosh, C. Kundu, On generalized conditional cumulative past inaccuracy measure. Applications of Mathematics, 63 (2018) 167–193.
[23] S. Kotz, S. Nadarajah, Extreme Value Distributions: Theory and Applications. Imperial College Press, London, 2000.
[24] D. F. Kerridge, Inaccuracy and inference. Journal of the Royal Statistical Society, Series B, 23 (1961) 184–194.
[25] V. Kumar, Characterization results based on dynamic Tsallis cumulative residual entropy. Communications in Statistics-Theory and Methods, 46 (2017) 8343–8354.
[26] P. Nath, Inaccuracy and coding theory. Metrika, 13 (1968) 123–135.
[27] Y. Y. Nikitin, M. Ahsanullah, New U-empirical tests of symmetry based on extremal order statistics, and their efficiencies. In: Hallin, M., Mason, D.M., Pfeifer, D., Steinebach, J.G. (eds.) Mathematical Statistics and Limit Theorems, Festschrift in Honour of Paul Deheuvels, pp. 231–248. Springer, Berlin, 2015.
[28] B. Milošević, M. Obradović, Characterization based symmetry tests and their asymptotic efficiencies. Statistics and Probability Letters, 119 (2016) 155–162.
[29] R. Thapliyal, H. C. Taneja, V. Kumar, Characterization results based on non-additive entropy of order statistics. Physica A: Statistical Mechanics and its Applications, 417 (2015) 297–303.
[30] C. Tsallis, Possible generalization of Boltzmann-Gibbs statistics. Journal of Statistical Physics, 52 (1988) 479–487.
[31] T. G. Veena, P. Y. Thomas, Role of concomitants of order statistics in determining parent bivariate distributions. Communications in Statistics-Theory and Methods, 46 (2017) 7976–7997.