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On a Type of Spacetimes

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Abstract. In the present paper we characterize a type of spacetimes, called almost pseudo $\mathcal{Z}$-symmetric spacetimes $A(PZS)_4$. At first, we obtain a condition for an $A(PZS)_4$ spacetime to be a perfect fluid spacetime and Roberson-Walker spacetime. It is shown that an $A(PZS)_4$ spacetime is a perfect fluid spacetime if the $\mathcal{Z}$ tensor is of Codazzi type. Next we prove that such a spacetime is the Roberson-Walker spacetime and can be identified with Petrov types I, D or O[3], provided the associated scalar ϕ is constant. Then we investigate $A(PZS)_4$ spacetimes satisfying $divC = 0$ and state equation is derived. Also some physical consequences are outlined. Finally, we construct a metric example of an $A(PZS)_4$ spacetime.

1. Introduction

The basic difference between the Riemannian and semi-Riemannian geometry is the existence of a null vector, that is, a vector v satisfying $g(v, v) = 0$, where g is the metric tensor. The signature of the metric g of a Riemannian manifold is $(+, +, +, ...+, +, +)$ and of a semi-Riemannian manifold is $(-, -, -, ...+, +, +)$. Lorentzian manifold is a special case of semi-Riemannian manifold. The signature of the metric of a Lorentzian manifold is $(-, +, +, ...+, +, +)$. In a Lorentzian manifold three types of vectors exist such as timelike, spacelike and null vector. In general , a Lorentzian manifold (M, g) may not have a globally timelike vector field. If (M, g) admits a globally timelike vector field, it is called time orientable Lorentzian manifold, physically known as spacetime. The foundations of general relativity are based on a 4-dimensional spacetime manifold.

Perfect fluid play a crucial role in general relativity being the natural sources of Einstein's field equation compatible with the Bianchi's identities. A spacetime is called perfect fluid if the energymomentum tensor is of the form[27]

$$T(X, Y) = (\mu + p)A(X)A(Y) + pg(X, Y),$$

where μ is the energy density, p is the isotropic pressure, ρ is a unit timelike vector field ($g(\rho, \rho) = -1$) metrically equivalent to the 1-form A. The fluid is called perfect because of the absence of heat conduction terms and stress terms corresponding to viscosity[18].

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In addition, p and μ are related by an equation of state governing the particular sort of perfect fluid under consideration. In general, this is an equation of the form $p = p(\mu, T_0)$, where T_0 is the absolute temperature. However, we shall only be concerned with situations in which T_0 is effectively constant so that the equation of state reduces to $p = p(\mu)$. In this case, the perfect fluid is called isentropic[18]. Moreover, if $p = \mu$, then the perfect fluid is termed as stiff matter(see [32] , page 66).

The notion of an almost pseudo Ricci symmetric manifold was introduced by Chaki and Kawaguchi[9]. It was a generalization of the notion of pseudo Ricci symmetric manifolds[8] and was defined as follows:

A non-flat semi-Riemannian manifold (M^n, g), $(n > 2)$ is called an almost pseudo Ricci symmetric manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies

$$(\nabla_X S)(Y, U) = [A(X) + B(X)]S(Y, U) + A(Y)S(X, U) + A(U)S(Y, X), \tag{1}$$

where A and B are two 1-forms. In such a case A and B are called the associated 1-forms and an n-dimensional manifold of this kind is denoted by $A(PRS)_n$. If $B = A$, then the (1) takes the following form:

$$(\nabla_X S)(Y, U) = 2A(X)S(Y, U) + A(Y)S(X, U) + A(U)S(Y, X), \tag{2}$$

which is called a pseudo Ricci symmetric manifold introduced by Chaki. Let $g(X, P) = A(X)$ and $g(X, Q) = B(X)$, for all X. Then P, Q are called the basic vector fields of the manifold corresponding to the associated 1-forms A and B, respectively.

Tamassy and Binh[33] introduced the notion of weakly Ricci symmetric manifolds which are the generalizations of pseudo Ricci symmetric manifolds.

On the other hand, in 2012 Mantica and Molinari[22] defined a generalized (0,2) symmetric $\mathcal{Z}$ tensor given by

$$\mathcal{Z}(X, Y) = S(X, Y) + \phi g(X, Y), \tag{3}$$

where ϕ is an arbitrary scalar function. In Refs. ([24], [25]) various properties of the $\mathcal{Z}$ tensor were pointed out. The scalar Z is obtained by contracting (3) over X and Y as follows:

$$Z = r + n\phi. \tag{4}$$

A manifold is called almost pseudo $\mathcal{Z}$ symmetric and denoted by $A(PZS)_n$ if the generalized $\mathcal{Z}$ tensor is non-zero and satisfies the condition (1), that is,

$$(\nabla_X \mathcal{Z})(Y, U) = [A(X) + B(X)]\mathcal{Z}(Y, U) + A(Y)\mathcal{Z}(X, U) + A(U)\mathcal{Z}(Y, X). \tag{5}$$

If $B = A$ in (5) then the manifold reduces to pseudo $\mathcal{Z}$ symmetric manifold introduced by Mantica and Suh[24]. Moreover in [22] Mantica and Molinari studied weakly $\mathcal{Z}$ symmetric manifolds.

On the other hand, generalized Robertson-Walker (GRW) spacetimes were introduced in 1995 by Alias, Romero and Sánchez ([1],[2]). Generalized Robertson-Walker (GRW) spacetime extends the notion of RW spacetime by allowing for spatial non-homogeneity. A GRW spacetime M is the warped product $-I \times f^2 M^*$, where I is an open interval of real line and (M^*, g^*) (the fibre) is a Riemannian manifold of dimension $(n-1)$; $f > 0$ is a smooth warping function. If M^* is a manifold of constant curvature, then M is a RW spacetime. Einstein static universe is the simplest example of a RW spacetime. Also RW spacetime is conformally flat and globally hyperbolic. Robertson-Walker spacetimes have been studied by several authors such as ([11], [12], [19]) and many others. Recently Mantica and Molinari published a survey article on GRW spacetime[26].

Several authors studied spacetimes in different way such as ([15], [16], [20], [24], [34]) and many others. In [10] Chaki and Ray studied spacetimes with covariant constant energy momentum tensor.

Motivated by the above studies in the present paper we characterize almost pseudo $\mathcal{Z}$ symmetric spacetimes $A(PZS)_4$.

The paper is organized as follows:
After introduction and preliminaries, in Section 3, we study $A(PZS)_4$ spacetimes and prove that if the $\mathcal{Z}$ tensor in an $A(PZS)_4$ spacetime is of Codazzi type, then the spacetime is a perfect fluid spacetime and such a spacetime is Yang pure space under certain condition. Section 4 deals with the study of $A(PZS)_4$ satisfying $divC = 0$. In this section we obtain several interesting results. Finally, we construct an example of a $A(PZS)_4$ spacetime.

2. Preliminaries

Let S and r denote the Ricci tensor of type (0,2) and the scalar curvature respectively. L denotes the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S, that is,

$$g(LX, Y) = S(X, Y), \tag{6}$$

for any vector fields X, Y . Let $\bar{A}$ and $\bar{B}$ are two 1-forms defined by $A(LX) = \bar{A}(X)$, $B(LX) = \bar{B}(X)$. Then $\bar{A}$ and $\bar{B}$ are called auxiliary 1-forms corresponding to the 1-forms A and B respectively. We have from (3)

$$Z(X, Y) = Z(Y, X), \tag{7}$$

and

$$\mathcal{Z}(Y, Q) = \bar{B}(Y) + \phi B(Y). \tag{8}$$

Also we obtain from (5)

$$(\nabla_X \mathcal{Z})(Y, W) - (\nabla_W \mathcal{Z})(X, Y) = B(X)\mathcal{Z}(Y, W) - B(W)\mathcal{Z}(X, Y). \tag{9}$$

Using (3) in (9) we get

$$(\nabla_X S)(Y, W) + (X\phi)g(Y, W) - (\nabla_W S)(X, Y) - (W\phi)g(X, Y) = B(X)\mathcal{Z}(Y, W) - B(W)\mathcal{Z}(Y, X). \tag{10}$$

Now contracting (10) over Y, W and using (4) and (8) we get

$$dr(X) = \{2r + 2(n-1)\phi\}B(X) - 2\bar{B}(X) - 2(n-1)(X\phi). \tag{11}$$

3. An $A(PZS)_4$ spacetime with Codazzi type of $\mathcal{Z}$ tensor

A $(1, 1)$-tensor field T on a Riemannian or a semi-Riemannian manifold (M, g) is said to be of Codazzi type[5] if it satisfies the condition

$$(\nabla_X T)Y = (\nabla_Y T)X,$$

where ∇ is the Riemannian or semi-Riemannian connection on g and X, Y are arbitrary vector fields on M. A (0,2)-tensor is said to be of Codazzi type if the metrically associated $(1, 1)$-tensor is of Codazzi type[5].

In this section we suppose that the $\mathcal{Z}$ tensor in an $A(PZS)_4$ spacetime is of Codazzi type[5], i.e., $(\nabla_X \mathcal{Z})(Y, W) = (\nabla_Y \mathcal{Z})(X, W)$. Then from (5) we get

$$(\nabla_X \mathcal{Z})(Y, W) - (\nabla_Y \mathcal{Z})(X, W) = B(X)\mathcal{Z}(Y, W) - B(Y)\mathcal{Z}(X, W).$$

Since $\mathcal{Z}$ tensor is of Codazzi type the above equation implies

$$B(X)\mathcal{Z}(Y, W) = B(Y)\mathcal{Z}(X, W). \tag{12}$$

Replacing X by Q in (12) and using $B(Q) = g(Q,Q) = -1$, we get

$$\mathcal{Z}(Y,W) = -B(Y)\mathcal{Z}(Q,W). \tag{13}$$

Taking a frame field and putting $Y = W = e_i$ in (12), $1 \leq i \leq 4$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the spacetime, we obtain

$$B(X)Z = \mathcal{Z}(X,Q), \tag{14}$$

where $Z = \sum_{i=1}^{4} \epsilon_i \mathcal{Z}(e_i,e_i)$, $\epsilon_i = g(e_i,e_i) = \pm 1$.
Using (14) in (13) we infer that

$$\mathcal{Z}(Y,W) = -ZB(Y)B(W),$$

from which we get

$$S(Y,W) = \alpha g(Y,W) + \beta B(Y)B(W), \tag{15}$$

which implies that the spacetime under consideration is a perfect fluid spacetime, where $\alpha = -\phi$ and $\beta = -Z$.

Thus we can state the following:

Theorem 3.1. *If the $\mathcal{Z}$ tensor in an $A(PZS)_4$ spacetime is of Codazzi type, then the spacetime is a perfect fluid spacetime.*

By hypothesis the $\mathcal{Z}$ tensor is of Codazzi type. Hence from the definition of the $\mathcal{Z}$ tensor we obtain

$$\begin{aligned}(\nabla_X \mathcal{Z})(Y,W) - (\nabla_W \mathcal{Z})(X,Y) &= (\nabla_X S)(Y,W) - (\nabla_W S)(X,Y) \\ &+ (X\phi)g(Y,W) - (W\phi)g(X,Y).\end{aligned} \tag{16}$$

If $\phi = constant$, then the above equation reduces to

$$(\nabla_X \mathcal{Z})(Y,W) - (\nabla_W \mathcal{Z})(X,Y) = (\nabla_X S)(Y,W) - (\nabla_W S)(X,Y),$$

that is, if the $\mathcal{Z}$ tensor is of Codazzi type, then the Ricci tensor S is of Codazzi type. In [29] the author proves that in a perfect fluid spacetime with divergence-free projective curvature tensor the local cosmological structure of such a spacetime can be identified as Petrov types I, D, or O[3]. It is worth to note that the discussions about Petrov type is valid only in $n = 4$ dimensions. Since divergence-free projective curvature tensor and the Codazzi type of Ricci tensor are equivalent, hence we can conclude the following theorem:

Theorem 3.2. *If the $\mathcal{Z}$ tensor with the associated scalar $\phi = constant$ in an $A(PZS)_4$ spacetime is of Codazzi type, then the spacetime can be identified with Petrov types I, D or O.*

It is known[13] that in an n-dimensional Lorentzian manifold

$$(divC)(X,Y)W = \frac{n-3}{n-2}[(\nabla_X S)(Y,W) - (\nabla_Y S)(X,W) - \frac{1}{2(n-1)}\{g(Y,W)dr(X) - g(X,W)dr(Y)\}]. \tag{17}$$

Thus if the Ricci tensor is of Codazzi type, then from (17) we obtain $(divC)(X,Y)W = 0$, since $r = constant$.

Now we consider Einstein's field equation without cosmological constant, that is,

$$S(X,Y) - \frac{r}{2}g(X,Y) = \kappa T(X,Y), \tag{18}$$

being κ the Einstein's gravitational constant, T is the energymomentum tensor([32], [27]) describing the matter content of the spacetime.
From (15) and (18) we infer that

$$\kappa T(X,Y) = (-\phi - \frac{r}{2})g(X,Y) - ZB(X)B(Y),$$

being Q defined by $g(X,Q) = B(X)$ for all X is a unit timelike vector field.
The above equation is of the form of a perfect fluid spacetime

$$T(X,Y) = (p+\mu)B(X)B(Y) + pg(X,Y),$$

where $\kappa p = -\phi - \frac{r}{2}$ and $\kappa(p+\mu) = -Z$ from which it follows that $p = \frac{1}{\kappa}(-\phi - \frac{r}{2})$ and $\mu = \frac{1}{\kappa}(\phi + \frac{r}{2} - Z)$. Since the scalar $Z = r + 4\phi$, therefore $\mu = -\frac{1}{2\kappa}(r + 6\phi)$. Hence the state equation is

$$p = (\frac{r+2\phi}{r+6\phi})\mu. \tag{19}$$

Since $\phi = constant$ and $r = constant$, therefore the state equation is $p = c\mu$, c being a constant, that is, $p = p(\mu)$. Hence the spacetime is isentropic.

Remark: The general equation of state $p = c\mu$ provides a variety of physical solutions for a proper Ricci inheritance vector(**RIV**) which is a proper conformal Killing vector(**CKV**), including $p = \mu$.

It is well known that a RW spacetime is perfect fluid[27]. We recall Shepley and Taub's theorem for a 4-dimensional perfect fluid spacetime to be a RW spacetime.
Theorem:[31] A 4-dimensional perfect fluid spacetime with $divC = 0$ and subject to an equation of state $p = p(\mu)$ is conformally flat and the metric is RW, the flow is irrotational, shear free and geodesic.
From the above discussions we can state the following:

Theorem 3.3. *If the $\mathcal{Z}$ tensor with the associated scalar $\phi = constant$ in an $A(PZS)_4$ spacetime is of Codazzi type, then the spacetime is conformally flat, RW spacetime, the flow is irrotational, shear free and geodesic.*

In [14] Guilfoyle and Nolan named "Yang pure space" a 4-dimensional Lorentzian manifold (M,g) whose metric tensor solves Yang's equations:

$$(\nabla_X S)(Y,W) = (\nabla_Y S)(X,W).$$

This is equivalent to $r = constant$ and $divC = 0$.
The following theorem was stated[14] as:
A 4-dimensional perfect fluid spacetime (M,g) with $p + \mu \neq 0$ is a Yang pure space if and only if (M,g) is a RW spacetime.
Thus we have

Theorem 3.4. *If in an $A(PZS)_4$ spacetime the $\mathcal{Z}$ tensor is of Codazzi type, then the spacetime is a Yang pure space, provided $\phi = constant$.*

In [21] the authors proved the following:

Proposition 3.5. *[21] A perfect fluid spacetime in dimension $n \geq 4$, with differentiable equation of state $p = p(\mu)$, $p + \mu \neq 0$ and $divC = 0$ is a generalized Robertson-Walker spacetime.*

Recently in [23] it was proved that a GRW in $n = 4$ dimensions with $divC = 0$ is a RW spacetimes.
From the above results we can conclude the following:

Theorem 3.6. *An $A(PZS)_4$ spacetime with Codazzi type of $\mathcal{Z}$ tensor is a RW spacetime, provided $\phi = constant$.*

4. $A(PZS)_4$ spacetimes satisfying $divC = 0$

This section deals with an $A(PZS)_4$ spacetimes satisfying $divC = 0$, where C denotes the conformal curvature tensor and 'div' denotes divergence.

Hence we have[13]

$$(\nabla_X S)(Y, W) - (\nabla_Y S)(X, W) = \frac{1}{6}[g(Y, W)dr(X) - g(X, W)dr(Y)]. \tag{20}$$

Using (10) and (11) in (20) yields

$$\begin{aligned} & B(X)\mathcal{Z}(Y, W) - B(Y)\mathcal{Z}(X, W) - (X\phi)g(Y, W) + (Y\phi)g(X, W) \\ &= \frac{1}{6}[B(X)g(Y, W)\{2r + 6\phi\} - 6(X\phi)g(Y, W) - 2\bar{B}(X)g(Y, W) \\ & -B(Y)g(X, W)\{2r + 6\phi\} + 6(Y\phi)g(X, W) + 2\bar{B}(Y)g(X, W)]. \end{aligned} \tag{21}$$

Now putting $W = Q$ in (21), we get

$$B(X)\bar{B}(Y) = \bar{B}(X)B(Y). \tag{22}$$

Again putting $X = Q$ in (22) infers

$$\bar{B}(Y) = -tB(Y), \tag{23}$$

where $t = \bar{B}(Q)$ is a scalar.
Now using (23) in (11) we obtain

$$dr(X) = 2\{r + t + 3\phi\}B(X) - 6(X\phi).$$

Replacing X by Q in (21) and using (23) yields

$$\begin{aligned} -\mathcal{Z}(Y, W) - B(Y)\mathcal{Z}(Q, W) = \frac{1}{6}[&-\{2r + 6\phi\}g(Y, W) + 2tg(Y, W) \\ &-\{2r + 6\phi\}B(Y)B(W) + 2tB(Y)B(W)]. \end{aligned} \tag{24}$$

Using (3), (23) in (24) we get

$$S(Y, W) = \frac{r - t}{3}g(Y, W) + \frac{r - 4t}{3}B(Y)B(W), \tag{25}$$

which implies that the spacetime under consideration is a perfect fluid spacetime.
Therefore we have the following:

Theorem 4.1. *An $A(PZS)_4$ spacetime satisfying $divC = 0$ is a perfect fluid spacetime.*

Remark: The Weyl conformal tensor of a general perfect fluid spacetime M is divergence-free iff M is shear-free, irrotational and its energy density is constant over the spacelike hypersurface orthogonal to the 4-velocity vector[30].

Now from Einstein's equation (18) and (25) we get as in Section 3 that

$$\kappa p = \frac{r - t}{3} - \frac{r}{2}$$

and

$$\kappa(p + \mu) = \frac{r - 4t}{3}.$$

From the above relations we get the state equation $p = \frac{1}{3}\mu$, that is, $p = p(\mu)$. $p = \frac{1}{3}\mu$ represents a model for incoherent radiation[32]. Hence from a theorem of Mantica et al[21], we infer the following:

Theorem 4.2. *An $A(PZS)_4$ spacetime satisfying $divC = 0$ is a GRW spacetime.*

The matter content in general relativity theory is described by a second order symmetric tensor, the energy momentum tensor. Under limiting processes, one would like to know which energy momentum tensors might arise. A step in this study is the investigation of the limits of classes of energy-momentum tensors. A classification of this tensor is known according to its Segre type. It seems, therefore, important to investigate the relations among the Segre types under limiting processes. Spacetimes are sometimes classified according to the nature of the Segre' characteristic [28] of the Ricci tensor. We now investigate the nature of the Segre' characteristic of the Ricci tensor for perfect fluid $A(PZS)_4$ spacetime.
Equation (25) yields

$$S(Y,Q) = tg(Y,Q). \tag{26}$$

From (26) it follows that t is an eigen value of the Ricci tensor and Q is an eigen vector corresponding to this eigen value. For simplicity we assume that $t = -\frac{r}{2}$.
Let ξ be another eigen vector of S different from Q. Then ξ must be orthogonal to Q. Hence $g(Q,\xi) = 0$. That is,

$$B(\xi) = 0. \tag{27}$$

Putting $Y = \xi$ in (25) we obtain

$$S(X,\xi) = \frac{r}{2}g(X,\xi). \tag{28}$$

From (28) it follows that $\frac{r}{2}$ is another eigen value of S and ξ is an eigen vector corresponding to this eigen value. Since for a given eigen vector there is only one eigen value and $-\frac{r}{2}$ and $\frac{r}{2}$ are different, it follows that the Ricci tensor has only two distinct eigen values, namely $-\frac{r}{2}$ and $\frac{r}{2}$.
Let the multiplicity of $-\frac{r}{2}$ be m. Then the multiplicity of $\frac{r}{2}$ is $(4-m)$, since the dimension of the spacetime is 4.
Hence, $m(-\frac{r}{2}) + (4-m)\frac{r}{2} = 0$ which gives $m = 2$. Therefore, the multiplicity of $-\frac{r}{2}$ is 2 and the multiplicity of $\frac{r}{2}$ is 2. $m = 4$ implies that there is only one eigen value $\frac{r}{2}$ of multiplicity 4. But we have proved that there exists two eigen values $-\frac{r}{2}$ and $\frac{r}{2}$. So we can not take $m = 4$. Hence the Segre' characteristic of S is [(11),(11)]. This leads to the following result:

Theorem 4.3. *An $A(PZS)_4$ spacetime satisfying $divC = 0$ is of Segre' characteristic [(11),(11)].*

5. f(r,T)-gravity model

Very recently, Harko et al.[17] proposed the theory of $f(r,T)$-gravity which is the generalization or modification of general relativity. In this theory, gravitational Lagrangian is considered as an arbitrary function of r and T, where r is the trace of the Ricci tensor R_{ij} and T is the trace of stress-energy tensor T_{ij}. The field equations for this theory are derived from the Hilbert-Einstein type variational by considering the action

$$A = \frac{1}{16\pi}\int [f(r,T) + L_m]\sqrt{(-g)}d^4x, \tag{29}$$

where L_m is the matter Lagrangian density. The stress-energy tensor of the matter is given by

$$T_{ij} = \frac{-2}{\sqrt{(-g)}}\frac{\delta(\sqrt{(-g)})L_m}{\delta^{ij}}. \tag{30}$$

Let us suppose that the matter Lagrangian density depends only on g_{ij}, then the field equations of $f(r,T)$-gravity written as

$$\begin{aligned} f_r(r,T)R_{ij} - \frac{1}{2}f(r,T)g_{ij} + (g_{ij}\nabla_k\nabla^k - \nabla_i\nabla_j)f_r(r,T) \\ = 8\pi T_{ij} - f_T(r,T)T_{ij} - f_T(r,T)\Theta_{ij}, \end{aligned} \tag{31}$$

where f_r and f_T denote the partial derivative of f with respect to r and T respectively and

$$\Theta_{ij} = -2T_{ij} + g_{ij}L_m - 2g^{lk}\frac{\partial^2 L_m}{\partial g^{ij}\partial g^{lk}}. \tag{32}$$

Here ∇^i represents the covariant derivative. If we consider $f(r,T) = f(r)$, then the equations (29) and (30) give the field equations of $f(r)$-gravity[7]. It is to be noted that Buchdahl[4] introduced the notion of $f(r)$-gravity in 1970.

Let us consider $A(PZS)_4$ spacetime satisfies $divC = 0$. Then it is a perfect fluid. In local coordinate system equation (25) can be written as

$$R_{ij} = ag_{ij} + bB_iB_j, \tag{33}$$

where $a = \frac{r-t}{3}$ and $b = \frac{r-4t}{3}$. As we know that there is no unique definition of matter Lagrangian, thus we assume $L_m = -p$. Then the variation of stress-energy tensor of the fluid takes the form

$$\Theta_{ij} = -2T_{ij} - pg_{ij}. \tag{34}$$

Generally the field equations depend on the physical nature of the matter field and therefore for each choice of $f(r,T)$, we get a theoretical model. For instant, we choose

$$f(r,T) = r + 2f(T), \tag{35}$$

where $f(T)$ is the arbitrary function of the trace of stress-energy tensor of the matter such that $f'(T) \neq -4\pi$. After considering equation (35), (31) assumes the form

$$R_{ij} - \frac{1}{2}rg_{ij} = 8\pi T_{ij} - 2f'(T)T_{ij} - 2f'(T)\Theta_{ij} + f(T)g_{ij}. \tag{36}$$

Equations(33) and (36) together yield

$$T_{ij} = \frac{a - \frac{r}{2} - 2pf'(T) - f(T)}{8\pi + 2f'(T)}g_{ij} + \frac{b}{8\pi + 2f'(T)}B_iB_j.$$

Thus we can state the following:

Theorem 5.1. *An $A(PZS)_4$ spacetime satisfying $divC = 0$ shows a perfect fluid stress-energy tensor for any f(r,T)-gravity model.*

Remark: Recently, Capozziello et al[6] prove that an n-dimensional GRW spacetime with divergence free conformal curvature tensor exhibits a perfect fluid stress-energy tensor for any $f(r)$ gravity model. Also in [23] it was proved that a GRW spacetime in $n = 4$ dimensions with $divC = 0$ is a RW spacetime.
Therefore from theorem 4.2, we conclude the following:
An $A(PZS)_4$ spacetime with $divC = 0$ demonstrates a perfect fluid stress-energy tensor for any $f(r)$-gravity model.

6. Example of an $A(PZS)_4$ spacetime

In this section we prove the existence of a $A(PZS)_4$ spacetime by constructing a non-trivial concrete example.

We consider a Lorentzian manifold (M^4, g) endowed with the Lorentzian metric g given by

$$ds^2 = g_{ij}dx^idx^j = (dx^1)^2 + (x^1)^2(dx^2)^2 + (x^2)^2(dx^3)^2 - (dx^4)^2, \tag{37}$$

where $i, j = 1, 2, 3, 4$.
The only non-vanishing components of the Christoffel symbols, the curvature tensor and the Ricci tensor are

$$\Gamma^1_{22} = -x^1,\ \Gamma^2_{33} = -\frac{x^2}{(x^1)^2},\ \Gamma^2_{12} = \frac{1}{x^1},\ \Gamma^3_{23} = \frac{1}{x^2},$$

$$R_{1332} = -\frac{x^2}{x^1},\ S_{12} = -\frac{1}{x^1x^2}.$$

We shall now show that this M^4 is an $A(PZS)_4$ i.e., it satisfies the defining relation (5).

In this example we consider the scalar ϕ as follows:

$$\phi = \begin{cases} \frac{5}{3x^1}, & \text{for non-zero components of the Ricci tensor} \\ 0, & \text{for vanishing components of the Ricci tensor.} \end{cases}$$

Then only the non vanishing component for $\mathcal{Z}$ tensor and its covariant derivatives are given by

$$\mathcal{Z}_{12} = -\frac{1}{x^1x^2},\ \mathcal{Z}_{12,1} = \frac{2}{(x^1)^2x^2},\ \mathcal{Z}_{12,2} = \frac{x^1 + x^2}{(x^1x^2)^2}.$$

We choose the 1-forms as follows:

$$A_i(x) = \begin{cases} \frac{1}{2x^1}, & \text{for i=1} \\ \frac{1}{2x^1x^2}, & \text{for i=2} \\ 0, & \text{for i=3,4} \end{cases}$$

and

$$B_i(x) = \begin{cases} -\frac{3}{x^1}, & \text{for i=1} \\ -\frac{x^1+x^2+1}{x^1x^2}, & \text{for i=2} \\ 0, & \text{for i=3,4} \end{cases}$$

at any point $x \in M$. In our (M^4, g), (5) reduces with these 1-forms to the following equations:

$$\mathcal{Z}_{12,1} = (A_1 + B_1)\mathcal{Z}_{12} + A_1\mathcal{Z}_{12} + A_2\mathcal{Z}_{11} \tag{38}$$

and

$$\mathcal{Z}_{12,2} = (A_2 + B_2)\mathcal{Z}_{12} + A_2\mathcal{Z}_{12} + A_1\mathcal{Z}_{22}. \tag{39}$$

It can be easily proved that the equations (38) and (39) are true.
So, the manifold under consideration is an $A(PZS)_4$ spacetime.
Thus the following theorem holds.

Theorem 6.1. *Let* $(\mathbb{R}^4, g)$ *be a 4-dimensional Lorentzian manifold with the Lorentzian metric g given by*

$$ds^2 = g_{ij}dx^idx^j = (dx^1)^2 + (x^1)^2(dx^2)^2 + (x^2)^2(dx^3)^2 - (dx^4)^2,$$

where $i,\ j = 1, 2, 3, 4$. *Then* $(\mathbb{R}^4, g)$ *is an* $A(PZS)_4$ *spacetime.*

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