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Generalized Form of Fixed Point Theorems in Banach algebras Under Weak Topology with an Application

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Abstract. In this manuscript, by means of the technique of measures of weak noncompactness, we establish a generalized form of fixed point theorems for a 2×2 block operator matrix involving multivalued maps acting on suitable Banach algebras. The results obtained are then applied to a coupled system of nonlinear integral equations.

1. Introduction

Many problems arising in diverse areas of natural science consider the existence of solutions of nonlinear equations of the form

$$x = Ax \cdot Bx + Cx; \quad x \in S, \tag{1}$$

where $A, C : X \longrightarrow X$ and $B : S \longrightarrow X$ are nonlinear mappings, here S is a nonempty, closed, and convex subset of a Banach algebra X. A useful prototype for solving equations of the type (1) is the celebrated fixed point theorem due to Dhage [15], see also for example [3, 6, 7, 16, 25] and the references therein, it stated that $A \cdot B + C$ has at least one fixed point in S, when A, B and C fulfill: B is completely continuous, A and C are Lipschitzians with Lipschitz constants α and β respectively with $\alpha M + \beta < 1$, where $M = \|B(S)\|$, and $A(S) \cdot B(S) + C(S) \subseteq S$. The proof of the above relies on the use of the advanced notions of the nonlinear functional analysis such as measures of noncompactness and condensing mappings. We should notice that the strategy of the authors in the above-mentioned works consists in giving sufficient conditions which ensure the invertibility of $\left(\frac{I-C}{A}\right)$ in order to deal with the mapping $\left(\frac{I-C}{A}\right)^{-1} B$. Another research direction is done in the setting of Hausdorff locally convex topological vector spaces. In [11], A. Ben Amar et al. asked that A, B and C are weakly sequentially continuous and weakly compact on S, A and C are $\mathcal{D}$-Lipschitzian with the $\mathcal{D}$-functions ϕ_A and ϕ_C respectively with $M\phi_A(r) + \phi_C(r) < r$, for all $r \in \mathbb{R}_+^*$, A must be a regular operator and $A(S) \cdot B(S) + C(S) \subset S$. They then obtained a version of hybrid fixed point theorem using the weak topology of a Banach algebra. The proof is based upon Schauder fixed point principle and uses the weak compactness of $\left(\frac{I-C}{A}\right)^{-1} B$. The main goal of this work is to prove some existence principles on Banach algebras of operators defined by a 2×2 block operator matrix

$$\begin{pmatrix} A & B \cdot B' \\ C & D \end{pmatrix} \tag{2}$$

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in critical case, i.e. when the operator $\left(\frac{I-A}{B(I-D)^{-1}C}\right)$ may not be invertible and we investigate this kind of generalization by looking for the multi-valued mapping $\left(\dfrac{I-A}{B(I-D)^{-1}C}\right)^{-1} B'(I-D)^{-1}C$. Note that our analysis is based on the De Blasi measure of weak noncompactness which allows us to cover earlier results in the litterature [8, 23]. Precisely, we prove that if S is a nonempty, bounded, closed, and convex subset of a Banach algebra X satisfying the sequential condition $(\mathcal{P})$ and $A, C : S \longrightarrow X$ and $B, B', D : X \longrightarrow X$ are five weakly sequentially continuous operators satisfying the following conditions:
(i) A is a 1-set-weakly contractive,
(ii) $(I-D)^{-1}C$ is weakly compact and $C(S) \subseteq (I-D)(S)$,
(iii) $B(I-D)^{-1}C$ is regular on $B'(S)$,
(iv) the set $\left\{x \in X \;:\; x = \lambda Ax + B(I-D)^{-1}Cx \cdot B'(I-D)^{-1}Cy\right\}$ is convex for all $y \in S$, and
(v) $B'(I-D)^{-1}C(S) \subset \left(\dfrac{I-\lambda A}{B(I-D)^{-1}C}\right)(S)$,
then, the block operator matrix (2) has, at least, one fixed point in $S \times X$.

In [20, 22], the authors have established some fixed point results for the block operator matrix (2), where the inputs are nonlinear mappings based on the convexity of the bounded domain, on the well-known Schauder's fixed point theorem, and also on the properties of the inputs (cf. completely continuous [22, 25], weakly sequentially continuous [20], etc, ...). In this direction, A. Jeribi, B. Krichen and B. Mefteh, in [21], have also established some new variants of fixed point theorems for the operator (2), where $B' = 1$. Due to the lack of compactness in L_1 of the operator $C(\lambda - A)^{-1}$, their analysis was carried out via arguments of weak topology and particulary the notion of the measure of weak noncompactness. We should notice that all their arguments are based on the invertibility of $(I - A)$. Our main results are applied to investigate the existence of solutions for the following two-dimensional nonlinear functional integral equation in a suitable Banach algebra.

$$\begin{cases} x(t) = \psi\left(t, \displaystyle\int_0^t \frac{t}{t+s} h(s, x(s))ds\right) \cdot u + Ty(t) \cdot \left[\displaystyle\int_0^{\sigma_1(t)} g(s, y(s))ds \cdot v\right] \\ y(t) = \displaystyle\int_0^{\sigma_2(t)} p\,(s, x(s))\,ds \cdot w + a(t)y(t), \\ (x(0), y(0)) = (x_0, y_0) \in \mathbb{R}^2 \end{cases} \tag{3}$$

where $\psi, h, g, p : [0,1] \times X \longrightarrow \mathbb{R}$ as well as $T : C([0,1], X) \longrightarrow C([0,1], X)$ are supposed to be weakly sequentially continuous, b, σ_1, σ_2 are continuous real functions on $[0,1]$ and u, v and w are three non vanishing vector of X. The main used tools are the fixed point theorems and the measure of weak noncompactness of De Blasi [14]. Note that the system (3) can be written as a fixed point problem:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \varphi(t,\cdot) & T \cdot G(t,\cdot) \\ K(t,\cdot) & a(t) \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

where

$$\begin{cases} \varphi(t, x(t)) = \psi\left(t, \displaystyle\int_0^t \frac{t}{t+s} h(s, x(s))ds\right) \cdot u, \\ G(t, y(t)) = \displaystyle\int_0^{\sigma_1(t)} g(s, y(s))ds \cdot v, \;\text{ and} \\ K(t, x(t)) = \displaystyle\int_0^{\sigma_2(t)} p\,(s, x(s))\,ds \cdot w. \end{cases}$$

The problem (3) has not been studied in the literature before, so the results of this paper are new to the theory of differential inclusions for a weak topology. For example, if $a(t) = 1$, $\varphi(t, x(t)) = c(t)$, $\sigma_2(t) = 0$ and $G(t, x(t)) = \left[q(t) + \int_0^{\sigma(t)} p(t, s, x(\zeta(s)), x(\eta(s)))ds\right] \cdot u$. Then the problem (3) reduces to the following functional integral equation with a delay

$$\begin{cases} x(t) = c(t) + (Tx)(t) \cdot G(t, x(t)) \\ x(0) = x_0 \in \mathbb{R} \end{cases} \tag{4}$$

where c, q, ζ, η and σ are continuous on $[0, 1]$, T and p are nonlinear functions and u is a non vanishing vector of X. Here, X is a Banach algebra satisfying satisfying certain topological conditions of sequential nature. The equation (4) has been studied in [12] via a variant of the Dhage's fixed point theorem.
In the special case when $a(t) = 1$, $\varphi(t, x(t)) = c(t)$, $\sigma_2(t) = 0$ and $(Tx)(t) = f(t, x(t))$ in (3), it reduces to the nonlinear quadratic integral equation, namely

$$\begin{cases} x(t) = c(t) + g(t, x(t)) \cdot \int_0^t f(s, x(s))ds \\ x(0) = x_0 \in \mathbb{R} \end{cases} \tag{5}$$

The equation (5) has been studied in [26] via the Tychonov's fixed point theorem where f and g satisfy Carathodory condition.

Our basic strategy is as follows. In section 2, we recall some definitions and give basic results for future use. In section 3, we will deal with some critical fixed point results for 2×2 block operator matrices, which consist of operators acting on closed and convex subsets in Banach algebras. In section 4, we use the results of Section 3 to discuss the existence of solutions of the system (3).

2. Notations, Basic definitions and auxiliary results

Throughout this section, X denotes a Banach algebra. For any $r > 0$, B_r denotes the closed ball in X centered at 0_X with radius r and $\mathcal{B}(X)$ denotes the collection of all nonempty bounded subsets of X. Also $\mathcal{W}(X)$ is the subfamily of $\mathcal{B}(X)$ consisting of all nonempty weakly compact subsets of X. We write $x_n \to x$ and $x_n \rightharpoonup x$ to denote the strong convergence and the weak convergence of a sequence $\{x_n\}$ to a point x.

The following concepts, due to Dhage [16], are crucial for our goal.

Definition 2.1. *An operator $T : X \longrightarrow X$ is called $\mathcal{D}$-Lipschitzian if there exists a continuous and nondecreasing function $\phi_T : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that*

$$\|Tx - Ty\| \leq \phi_T(\|x - y\|), \text{ for each } x, y \in X,$$

where $\phi_T(0) = 0$. The function ϕ_T is then called a $\mathcal{D}$-function of T. In addition, if ϕ_T satisfies $\phi_T(r) < r$ for all $r > 0$, then T is called a nonlinear contraction with a contraction function ϕ_T. Obviously every Lipschitzian mapping is a $\mathcal{D}$-Lipschitzian mapping, but the converse is not true. Indeed, we can take the mapping $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ defined as $f(x) = \sqrt{x}$ which is not Lipschitzian but it is $\mathcal{D}$-Lipschitzian with a $\mathcal{D}$-function $\phi_f(r) = \sqrt{r}$, see Remark 2.1 *in [5].* ◊

Definition 2.2. *An operator $T : X \longrightarrow X$ is said to be weakly compact, if $T(B)$ is relatively weakly compact for every nonempty bounded subset $B \subseteq X$.* ◊

Definition 2.3. *An operator $T : X \longrightarrow X$ is said to be weakly sequentially continuous on X if, for every sequence $\{x_n,\ n \in \mathbb{N}\}$ with $x_n \rightharpoonup x$, we have $Tx_n \rightharpoonup Tx$.* ◊

Because of the lack of stability of convergence for the product sequences under the weak topology, the authors in [11] have introduced a new class of Banach algebras satisfying the condition denoted $(\mathcal{P})$:

Definition 2.4. *A Banach algebra X is called satisfies condition $(\mathcal{P})$, if for any sequences $\{x_n,\ \ n \in \mathbb{N}\}$ and $\{y_n,\ \ n \in \mathbb{N}\}$ in X such that $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$, we have $x_n \cdot y_n \rightharpoonup x \cdot y$.* ◊

Remark 2.5. *If X is a Banach algebra satisfying the sequential condition $(\mathcal{P})$ and K is a compact Hausdorff space then, $C(K, X)$ is also a Banach algebra satisfying the condition $(\mathcal{P})$ in view of the Dobrakov's theorem [17]. Moreover, every finite dimensional Banach algebra satisfies condition $(\mathcal{P})$.* ◊

We recall that a function $\beta : \mathcal{B}(X) \longrightarrow \mathbb{R}_+$ is called to be a measure of weak noncompactness of X, if for every $S \in \mathcal{B}(X)$, the following properties are satisfied:

1. $\beta(M) = 0$ if, and only if, $\overline{M^w} \in \mathcal{W}(X)$. Here, $\overline{M^w} \in \mathcal{W}(X)$ denotes the weak closure of M.
2. $\beta(\overline{co}(M)) = \beta(M)$.

In what follows, we consider a measure of weak noncompactness β having some of the following properties

(P_1) If $M_1 \subseteq M_2$, then $\beta(M_1) \leq \beta(M_2)$, $M_1, M_2 \in \mathcal{B}(X)$.

(P_2) $\beta(M_1 \cup M_2) = \max\{\beta(M_1), \beta(M_2)\}$, $M_1, M_2 \in \mathcal{B}(X)$.

(P_3) $\beta(M_1 + M_2) \leq \beta(M_1) + \beta(M_2)$, $M_1, M_2 \in \mathcal{B}(X)$.

(P_4) $\beta(\alpha M) = \beta(M)$ for all $\alpha \in \mathbb{R}_+^*$, $M \in \mathcal{B}(X)$.

(P_5) $\beta(M_1 \cup \{x_0\}) = \beta(M_1)$ for all $x_0 \in X$.

As examples of measures of weak noncompactness which satisfy all the previous properties we recall the De Blasi MWNC defined on $\mathcal{B}(X)$ in [14] by the following way:

$$\beta(S) = \inf\Big\{r > 0 \text{ such that there exists } K \in \mathcal{W}(X) \text{ such that } S \subseteq K + \mathcal{B}_r\Big\}.$$

The function $\beta(S)$ possesses several useful properties which may be found in [1, 4]. Using the abstract measure of weak noncompactness introduced above, we say that $F : X \longrightarrow X$ is k-set-weakly contractive if it maps bounded sets into bounded sets, and there exists $k \in (0, 1)$ such that $\beta(F(M)) \leq k\beta(M)$, for any $M \subseteq \mathcal{B}(X)$. In the particular case, if F is bounded and $k = 0$, then F is said to be weakly compact.

A correspondence $G : X \longrightarrow \mathcal{P}(X)$ is called a multi-valued operator or a multivalued mapping on X into the class $\mathcal{P}(X)$ of all nonempty subsets of X. A point $x \in X$ is called a fixed point of G if $x \in G(x)$. We say that G has a weakly sequentially closed graph if for every sequence $\{x_n\}_{n=0}^{\infty} \subset S$ with $x_n \rightharpoonup x$ in S and for every sequence $y_n \in G(x_n)$ with $y_n \rightharpoonup y$ in X implies $y \in G(x)$.

3. Critical type of Fixed point theorems

In this section, we are examining a critical type of fixed point theorem for the block operator matrix (2). Precisely, the suggested findings provide the necessary hypotheses for the inputs so that the block operator matrix (2) has a fixed point in the case where $\left(\frac{I-A}{B(I-D)^{-1}C}\right)$ is non-invertible.

Theorem 3.1. *Let S be a nonempty, closed, and convex subset of a Banach algebra X. Assume that $A, B, C, D, B' : S \longrightarrow X$ are five operators satisfying the following conditions:*
(i) $(I - D)^{-1}$ exists on $C(S)$.
(ii) $B(I - D)^{-1}C$ is regular on S, (i.e. $B(I - D)^{-1}C$ maps S into the set of all invertible elements of S),
(iii) $B'(I - D)^{-1}C(S)$ is a relatively weakly compact subset of X,

(iv) If $\left(\frac{I-A}{B(I-D)^{-1}C}\right)x_n \rightharpoonup y$, *then there is a weakly convergent subsequence of* $(x_n)_n$,
(v) $\left(\frac{I-A}{B(I-D)^{-1}C}\right)^{-1} B'(I-D)^{-1}Cx$ *is convex for all* $x \in S$,
(vi) $B'(I-D)^{-1}C$ *and* $\left(\frac{I-A}{B(I-D)^{-1}C}\right)$ *are weakly sequentially continuous on* S,
(vii) $B'(I-D)^{-1}C(S) \subset \left(\frac{I-A}{B(I-D)^{-1}C}\right)(S)$.
Then the block operator matrix (2) *has, at least, one fixed point in* $S \times X$. ◇

Proof. First, we assume that the operator $\left(\frac{I-A}{B(I-D)^{-1}C}\right)$ is invertible. In order to achieve the proof, we will apply Theorem 2.5 in [9]. Hence, we only have to prove that the map which assigns to each $x \in S$ the value

$$T(x) = \left(\frac{I-A}{B(I-D)^{-1}C}\right)^{-1} B'(I-D)^{-1}Cx. \tag{6}$$

is weakly sequentially continuous on S such that $T(S)$ is a relatively weakly compact subset in X. In fact, consider $\{y_n,\ n \in \mathbb{N}\}$ be any sequence of $T(S)$, then

$$\left(\frac{I-A}{B(I-D)^{-1}C}\right)(y_n) = B'(I-D)^{-1}Cx_n, \text{ for some } \ x_n \in S.$$

From assumption *(iii)*, it follows that there exists a subsequence (x_{n_k}) of $\{x_n,\ n \in \mathbb{N}\}$ such that

$$B'(I-D)^{-1}C(x_{n_k}) \rightharpoonup z, \ \text{ for some } \ z \in S.$$

Thus, there is a weakly convergent subsequence of $\{y_n,\ n \in \mathbb{N}\}$ and consequently $T(S)$ is relatively weakly compact in view of the Eberlein- Šmulian's theorem [18]. Now, let $\{x_n,\ n \in \mathbb{N}\}$ be a weakly convergent sequence of S to a point x. From the above discussion, it is easy to see that there is a subsequence $(x_{\varphi_1(n)})$ of $\{x_n,\ n \in \mathbb{N}\}$ such that $T(x_{\varphi_1(n)}) \rightharpoonup y$, for some $y \in S$. Keeping in mind the weak sequential continuity of the operator $\left(\frac{I-A}{B(I-D)^{-1}C}\right)$, and using the equality:

$$\left(\frac{A-I}{B(I-D)^{-1}C} + I\right)T = -B'(I-D)^{-1}C + T, \tag{7}$$

we obtain

$$\left(\frac{A-I}{B(I-D)^{-1}C} + I\right)T(x_{\varphi_1(n)}) \rightharpoonup -B'(I-D)^{-1}Cx + y.$$

This implies that

$$\left(\frac{A-I}{B(I-D)^{-1}C} + I\right)y = -B'(I-D)^{-1}Cx + y.$$

Now a standard argument shows that $\{T(x_n),\ n \in \mathbb{N}\}$ converging weakly to $T(x)$. Suppose the contrary, then there would exist a weak neighborhood V^w of $T(x)$ and a subsequence (x_{n_k}) of $\{x_n,\ n \in \mathbb{N}\}$ such that $T(x_{n_k}) \notin V^w$, for all $k \geq 1$. An argument similar as before, we may extract a subsequence $(x_{n_{k_j}})$ of (x_{n_k}) such that $T(x_{n_{k_j}}) \rightharpoonup y$, which is a contradiction and consequently T is sequentially weakly continuous. Hence, T has, at least, one fixed point x in S in view of [9, Theorem 2.5].
In the second case, it is assumed that $\left(\frac{I-A}{B(I-D)^{-1}C}\right)$ is not invertible. Then $\left(\frac{I-A}{B(I-D)^{-1}C}\right)^{-1}$ may be viewed as a multi-valued mapping. Let us define the mapping $G : S \longrightarrow \mathcal{P}_{cv}(S)$ by the formula

$$G(x) = \left(\frac{I-A}{B(I-D)^{-1}C}\right)^{-1} B'(I-D)^{-1}Cx. \tag{8}$$

It should be noted that the multi-valued mapping G is well defined and $G(S)$ is relatively weakly compact. In order to apply Theorem 2.2 in [10], it is sufficient to demonstrate that the multi-valued G has a weakly sequentially closed graph. To do so, consider $\{x_n,\ n \in \mathbb{N}\}$ as a sequence in S that is weakly convergent to a point x and let $\{y_n,\ n \in \mathbb{N}\}$ be a weakly converging sequence in $G(x_n)$ to a point y. By the definition of G, we have

$$\left(\frac{I-A}{B(I-D)^{-1}C}\right) y_n = B'(I-D)^{-1}Cx_n.$$

Using the weak sequential continuity of $\left(\frac{I-A}{B(I-D)^{-1}C}\right)$ and $B'(I-D)^{-1}C$, we obtain

$$\left(\frac{I-A}{B(I-D)^{-1}C}\right) y = B'(I-D)^{-1}Cx.$$

This equality means that is G has a weakly sequentially closed graph. Hence, G has, at least, one fixed point x in S in view of [10, Theorem 2.2]. So, the vector $y = (I-D)^{-1}Cx$ solves the problem. Q.E.D.

As a consequence we have the following fixed point result.

Corollary 3.2. *Let S be a nonempty, closed, and convex subset of a Banach algebra X. Assume that $A, C : S \longrightarrow X$ and $B, B', D : X \longrightarrow X$ be five operators satisfying the following conditions:*
(i) A, B and C are $\mathcal{D}$-Lipschtzian with $\mathcal{D}$-functions ϕ_A, ϕ_B and ϕ_C respectively
(ii) D is a contraction with a constant k,
(iii) $(I-D)^{-1}C(S)$ is relatively weakly compact and $C(S) \subseteq (I-D)(S)$,
(iv) $B(I-D)^{-1}C$ is regular on $B'(S)$,
(v) $\left(\frac{I-A}{B(I-D)^{-1}C}\right)^{-1} B'(I-D)^{-1}C$ is weakly compact,
(vi) $\left(\frac{I-A}{B(I-D)^{-1}C}\right)$ and $B'(I-D)^{-1}C$ are weakly sequentially continuous,
(vii) $B'(I-D)^{-1}C(S) \subset \left(\frac{I-A}{B(I-D)^{-1}C}\right)(S)$.
Then the block operator matrix (2) *has, at least, one fixed point whenever $M\phi_{\mathcal{B}} \circ \left(\frac{1}{1-k}\phi_C\right)(r) + \phi_{\mathcal{A}}(r) < r$, $r > 0$.* ◇

Now, we may combine Theorem 3.1 and Remark 4.1 in [22], in order to obtain the following fixed point theorem.

Theorem 3.3. *Let S be a nonempty, closed, and convex subset of a Banach algebra X. Assume that $A, C : S \longrightarrow X$ and $B, B', D : X \longrightarrow X$ be five operators satisfying the following conditions:*
(i) A, B and C are $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-functions $\phi_{\mathcal{A}}$, $\phi_{\mathcal{B}}$ and ϕ_C respectively,
(ii) D is a contraction with a constant k and $C(S) \subseteq (I-D)(S)$,
(iii) $B(I-D)^{-1}C$ is regular on $B'(S)$,
(iv) $B'(I-D)^{-1}C(S)$ is a relatively weakly compact subset of X,
(v) $B'(I-D)^{-1}C$ and $\left(\frac{I-A}{B(I-D)^{-1}C}\right)$ are sequentially weakly continuous,
(vi) $Ax + B(I-D)^{-1}Cx \cdot y \in S$, for each $x \in S$ and $y \in B'(I-D)^{-1}C(S)$.
Then the block operator matrix (2) *has, at least, one fixed point whenever $M\phi_{\mathcal{B}} \circ \left(\frac{1}{1-k}\phi_C\right)(r) + \phi_{\mathcal{A}}(r) < r$, $r > 0$.* ◇

Proof. The use of assumption (ii) as well as Remark 4.1 in [22] allows us to say that the inverse operator $(I-D)^{-1}C$ exists and is $\mathcal{D}$-Lipschitzian with a $\mathcal{D}$-function $\left(\frac{1}{1-k}\phi_C\right)$.
Let y be fixed in $B'(I-D)^{-1}C(S)$ and let us define the mapping $\varphi_y : S \longrightarrow S$ by the formula

$$\varphi_y(x) = Ax + B(I-D)^{-1}Cx \cdot y.$$

Notice that φ_y is nonlinear contraction with a $\mathcal{D}$-function $\psi(r) = M\phi_{\mathcal{B}} \circ \left(\frac{1}{1-k}\phi_C(r)\right) + \phi_{\mathcal{A}}(r)$. By using the Browder's theorem [13], we deduce that φ_y has a unique fixed point in S, say x_y. From assumption (iii), it

follows that the inverse operator $\left(\frac{I-A}{B(I-D)^{-1}C}\right)^{-1}$ is well defined on $B'(I-D)^{-1}C(S)$.
Let us define a mapping

$$\begin{cases} N : S \longrightarrow S \\ y \mapsto x_y \end{cases}$$

Similarly to the proof of Theorem 3.1, the operator N is weakly sequentially continuous. Hence, N has, at least, one fixed point x in S in view of [9, Theorem 2.5]. Q.E.D.

A similar approach will be adopted to study the case where D is linear.

Theorem 3.4. *Let S be a nonempty, closed, and convex subset of a Banach algebra X satisfying the condition $(\mathcal{P})$. Assume that $A, B, C, D, B' : S \longrightarrow X$ be five operators satisfying the following conditions:*
(i) A, B, C and B' are weakly sequentially continuous,
(ii) D is linear, bounded and such that D^p is a separate contraction, for some $p \in \mathbb{N}^*$,
(iii) $B(I-D)^{-1}C$ is regular on S
(iv) The set $\Gamma_y = \left\{x \in S\ ;\ \ x = Ax + B(I-D)^{-1}Cx \cdot y\right\}$ is convex in X, for each $y \in B'(I-D)^{-1}C(S)$,
(v) $\left(\frac{I-A}{B(I-D)^{-1}C}\right)^{-1} B'(I-D)^{-1}C(S)$ is relatively weakly compact,
(vi) $x = Ax + B(I-D)^{-1}Cx \cdot B'(I-D)^{-1}Cy\ ;\ y \in S \Rightarrow x \in S$.
Then the block operator matrix (2) *has, at least, one fixed point in $S \times X$.* ◇

Proof. The use of assumption (iii) as well as Lemma 1.2 in [24] allows us to say that the inverse operator $(I-D)^{-1}C$ exists and is weakly continuous on S. In order to apply Theorem 2.2 in [10], it is sufficient to demonstrate that G defined in (10) has a weakly sequentially closed graph. To do so, let $\{\theta_n,\ \ n \in \mathbb{N}\}$ be a weakly convergent sequence in S to a point θ and let $\xi_n \in G(\theta_n)$ such that $\xi_n \rightharpoonup \xi$. Then $\{\xi_n,\ \ n \in \mathbb{N}\}$ satisfies the following equation

$$\xi_n = A\xi_n + B(I-D)^{-1}C\xi_n \cdot B'(I-D)^{-1}C\theta_n.$$

Using the condition $(\mathcal{P})$ combined with the fact that A, B, B' and $(I-D)^{-1}C$ are weakly sequentially continuous, we obtain $\xi \in G(\theta)$. Q.E.D.

A consequence of Theorem 3.4 is

Corollary 3.5. *Let S be a nonempty, closed, and convex subset of a Banach algebra X satisfying the condition $(\mathcal{P})$. Assume that $A, B, C, D, B' : S \longrightarrow X$ be five operators satisfying the following conditions:*
(i) A, B, C and B' are weakly sequentially continuous,
(ii) D is linear, bounded and such that D^p is a separate contraction, for some $p \in \mathbb{N}^*$,
(iii) $B(I-D)^{-1}C$ is regular on S
(iv) the set $\Gamma_y = \left\{x \in S\ ;\ \ x = Ax + B(I-D)^{-1}Cx \cdot y\right\}$ is convex in X, for each $y \in B'(I-D)^{-1}C(S)$,
(v) $A(S)$ and $C(S)$ are relatively weakly compact,
(vi) $x = Ax + B(I-D)^{-1}Cx \cdot B'(I-D)^{-1}Cy\ ;\ y \in S \Rightarrow x \in S$.
Then the block operator matrix (2) *has, at least, one fixed point in $S \times X$.* ◇

Proof. In view of Theorem 3.4, it is enough to prove that $T(S)$ is relatively weakly compact, where T is defined in (6). To do this, let $\{\xi_n,\ \ n \in \mathbb{N}\}$ be any sequence of $T(S)$. Then,

$$(I-A)(\xi_n) = B(I-D)^{-1}C(\xi_n) \cdot B'(I-D)^{-1}C(\theta_n), \quad \text{for some } \theta_n \in S.$$

Based on assumption (v), it follows that there is two subsequence $(\theta_{\varphi_1(n)})$ and $(\theta_{\varphi_2(n)})$ of $\{\theta_n,\ \ n \in \mathbb{N}\}$ such that

$$A\theta_{\varphi_1(n)} \rightharpoonup y \quad \text{and} \quad C\theta_{\varphi_2(n)} \rightharpoonup z, \quad \text{for some } y, z \in X.$$

Moreover, the use of assumption (i) and Lemma 1.2 in [24] allows us to have

$$B(I-D)^{-1}C\theta_{\varphi_2(n)} \rightharpoonup B(I-D)^{-1}z \quad \text{and} \quad B'(I-D)^{-1}C\theta_{\varphi_2(n)} \rightharpoonup B'(I-D)^{-1}z.$$

Using the condition ($\mathcal{P}$), we get

$$A\theta_{\varphi_1(n)} + B(I-D)^{-1}C\theta_{\varphi_2(n)} \cdot B'(I-D)^{-1}C\theta_{\varphi_2(n)} \rightharpoonup y + B(I-D)^{-1}z \cdot B'(I-D)^{-1}z.$$

Applying the Eberlein-Šmulian's theorem [18], we obtain that $T(S)$ is relatively weakly compact. Q.E.D.

Now our attention is directed towards the case where A is a 1-set-weakly contractive, to deduce the following critical type of fixed point theorem, meaning by this that $\left(\frac{I-\lambda A}{B(I-D)^{-1}C}\right)$ can be allowed to be not invertible.

Theorem 3.6. *Let S be a nonempty, bounded, closed, and convex subset of a Banach algebra X satisfying the sequential condition ($\mathcal{P}$). Assume that $A, C : S \longrightarrow X$ and $B, B', D : X \longrightarrow X$ are five weakly sequentially continuous operators satisfying the following conditions:*
(i) *A is a 1-set-weakly contractive and $\lambda A(S) \subset A(S)$, for all $\lambda \in (0,1)$,*
(ii) *$(I-D)^{-1}C$ is weakly compact and $C(S) \subseteq (I-D)(S)$,*
(iii) *$B(I-D)^{-1}C$ is regular on $B'(S)$,*
(iv) *the set $\left\{x \in X \ : \ x = \lambda Ax + B(I-D)^{-1}Cx \cdot B'(I-D)^{-1}Cy\right\}$ is convex for all $y \in S$,*
(v) *$x = \lambda Ax + B(I-D)^{-1}Cx \cdot B'(I-D)^{-1}Cy, \ y \in S \Rightarrow x \in S$.*
Then the block operator matrix (2) *has, at least, one fixed point in $S \times X$.* $\diamond$

Proof. Evidence for the above mentioned Theorem is provided through the following two cases. In the first case, $F_\lambda = \left(\frac{I-\lambda A}{B(I-D)^{-1}C}\right)$ is assumed to be invertible. Consider $\lambda \in (0,1)$. Then, for each $y \in S$, there exists a unique point $x_y^\lambda \in S$ such that $F_\lambda(x_y^\lambda) = B'(I-D)^{-1}C(y)$ or, equivalently

$$Ax_y^\lambda + B(I-D)^{-1}Cx_y^\lambda \cdot B(I-D)^{-1}Cy = x_y^\lambda.$$

Since assumption (v) holds, then $x_y^\lambda \in S$ and so the operator $T_\lambda := F_\lambda^{-1} \circ B'(I-D)^{-1}C : S \longrightarrow S$ is well defined. Keeping in mind the weak compactness of the operator $(I-D)^{-1}C$, and using the equality

$$T_\lambda = B(I-D)^{-1}CT_\lambda \cdot B'(I-D)^{-1}C + \lambda AT_\lambda \tag{9}$$

we obtain

$$\begin{aligned} \beta(T_\lambda(S)) &\leq \beta\left(\lambda A \circ T_\lambda(S)\right) + \beta\left(B(I-D)^{-1}C \circ T_\lambda(S) \cdot B'(I-D)^{-1}C(S)\right) \\ &\leq \lambda\beta\left(T_\lambda(S)\right) \\ &< \beta\left(T_\lambda(S)\right). \end{aligned}$$

which is a contradiction and consequently T_λ is weakly compact. Next we claim that T_λ is weakly sequentially continuous. To see this, consider $\{x_n, \ n \in \mathbb{N}\}$ as a sequence in S that is weakly convergent to a point $x \in S$. From the above discussion, it is easy to see that the set $\{T_\lambda(x_n), \ n \in \mathbb{N}\}$ is relatively weakly compact. Then, there is a subsequence $(x_{\varphi_1(n)})$ of $\{x_n, \ n \in \mathbb{N}\}$ such that $T_\lambda(x_{\varphi_1(n)}) \rightharpoonup y$, for some $y \in S$. Taking into account the weak compactness of $(I-D)^{-1}C$ and using the following equality:

$$(I-D)^{-1}C = C + D(I-D)^{-1}C,$$

to obtain there is a subsequence $(x_{\varphi_2(n)})$ of $\{x_n, \ n \in \mathbb{N}\}$ such that

$$(I-D)^{-1}C(x_{\varphi_2(n)}) \rightharpoonup Cx + Dz, \quad \text{for some } z \in S.$$

Thus, $z = Cx + Dz$. Now a standard argument shows that $\{(I-D)^{-1}C(x_n),\ n \in \mathbb{N}\}$ converging weakly to $\{(I-D)^{-1}Cx\}$. Suppose the contrary, then there would exist a weak neighborhood V^w of $\{(I-D)^{-1}Cx\}$ and a subsequence (x_{n_k}) such that $(I-D)^{-1}C(x_{n_k}) \notin V^w$, for all $k \geq 1$. Arguing as before we may extract a subsequence $(x_{n_{k_j}})$ of (x_{n_k}) such that $(I-D)^{-1}C(x_{n_{k_j}}) \rightharpoonup z_1$, which is a contradiction and consequently $(I-D)^{-1}C$ is weakly sequentially continuous. If we consider the weak sequential continuity of the maps A, B and B' and deploy the equality (9), we get

$$T_\lambda(x_{\varphi_1(n)}) \rightharpoonup \lambda A(y) + B(I-D)^{-1}C(y) \cdot B'(I-D)^{-1}C(x).$$

Thus, $y = \lambda A(y) + B(I-D)^{-1}C(y) \cdot B'(I-D)^{-1}C(x)$. By using a similar reasoning, we may prove that T_λ is weakly sequentially continuous. Accordingly, the operator T_λ has a fixed point x^λ in S by using Arino, Gautier and Penot's fixed point theorem [2]. Now, choose a sequence $\{\lambda_n,\ n \in \mathbb{N}\} \subset [0,1]$ such that $\lambda_n \to 1$ and consider the corresponding sequence $\{x_n,\ n \in \mathbb{N}\}$ of elements of S satisfying

$$x_n = \lambda_n A x_n + B(I-D)^{-1}Cx_n \cdot B'(I-D)^{-1}Cx_n.$$

From the above discussion, it is easy to see that the sequence $\{x_n,\ n \in \mathbb{N}\}$ is a relatively weakly compact subset of S and consequently $\{x_n,\ n \in \mathbb{N}\}$ possesses a weakly convergent subsequence to a point $x \in S$. Besides, since A, B, B' and $(I-D)^{-1}C$ are weakly sequentially continuous, it follows that

$$x = Ax + B(I-D)^{-1}Cx \cdot B'(I-D)^{-1}Cx.$$

In the second case, it is assumed that $F_\lambda = \left(\frac{I-\lambda A}{B(I-D)^{-1}C}\right)$ is not invertible. Let us define the multi-valued mapping $G_\lambda : S \longrightarrow \mathcal{P}(S)$ by the formula

$$G_\lambda(x) = \left(\frac{I-\lambda A}{B(I-D)^{-1}C}\right)^{-1} B'(I-D)^{-1}C(x). \tag{10}$$

It should be noted that G_λ defines a multi-valued map with convex values in light of assumption (*iv*). By using the second part of assumption (*i*), we infer that

$$\begin{aligned} G_\lambda(S) &\subseteq \lambda A \circ G_\lambda(S) + B(I-D)^{-1}C \circ G_\lambda(S) \cdot B'(I-D)^{-1}C(S) \\ &\subseteq A(S) + B(I-D)^{-1}C(S) \cdot B'(I-D)^{-1}C(S). \end{aligned}$$

Based on the subadditivity of the De Blasi's measure of weak non-compactness it is shown that

$$\begin{aligned} \beta(G_\lambda(S)) &\leq \beta\left(\lambda A \circ G_\lambda(S)\right) + \beta\left(B(I-D)^{-1}C \circ G_\lambda(S) \cdot B'(I-D)^{-1}C(S)\right) \\ &\leq \lambda \beta\left(G_\lambda(S)\right). \end{aligned}$$

So, if $\beta(G_\lambda(S)) > 0$, then we get a contradiction and consequently G_λ is weakly compact.
In order to complete our proof, it is necessary to show that G_λ has a weakly sequentially closed graph. For this, consider $\{x_n,\ n \in \mathbb{N}\}$ as a sequence in S that is weakly convergent to a point $x \in S$ and let $\{y_n,\ n \in \mathbb{N}\}$ be a weakly converging sequence in $G_\lambda(x_n)$ to a point y. By the definition of G_λ we have

$$\left(\frac{I-\lambda A}{B(I-D)^{-1}C}\right)(y_n) = B'(I-D)^{-1}C(x_n).$$

If we consider the weak sequential continuity of the maps λA and $(I-D)^{-1}C$ and deploy the sequential condition $(\mathcal{P})$, we get $y \in G_\lambda(x)$ and so G_λ has a weakly sequentially closed graph. The remained proof follows along the lines of Theorem 2.2 in [10]. Q.E.D.

A consequence of Theorem 3.6 is

Corollary 3.7. *Let S be a nonempty, bounded, closed, and convex subset of a Banach algebra X satisfying the sequential condition $(\mathcal{P})$. Assume that $A, C : S \longrightarrow X$ and $B, B', D : X \longrightarrow X$ are five weakly sequentially continuous operators satisfying the following conditions:*
(i) A is weakly compact,
(ii) B, C, B' and D are Lipschitzian with Lipschitz constants k_1, k_2, k_3 and k_4 respectively,
(iii) $B(I-D)^{-1}C$ is regular on $B'(S)$,
(vi) the operator inverse F_λ^{-1} exists on $B'(I-D)^{-1}C(S)$,
(v) $B'(I-D)^{-1}C(S) \subset F_\lambda(S)$.
Then the block operator matrix (2) has, at least, one fixed point in $S \times X$ provided that $\max\left\{k_4 + M_1k_1k_2, \frac{M_2k_2k_3}{1-(k_4+M_1k_1k_2)}\right\} < 1$, where $M_1 = \|B'(I-D)^{-1}C(S)\|$ and $M_2 = \|B(I-D)^{-1}C(S)\|$. $\diamond$

Proof. Consider $\lambda \in (0,1)$. It is easy to check that, for each $y \in S$, there is an unique point $x_y^\lambda \in X$ such that $F_\lambda(x_y^\lambda) = B'(I-D)^{-1}Cy$ in light of assumption (vi). Then $x_y^\lambda \in S$ and we have

$$x_y^\lambda = \lambda A x_y^\lambda + B(I-D)^{-1}Cx_y^\lambda \cdot B'(I-D)^{-1}Cy, \quad \text{for each } y \in S.$$

According to assumption (vi), we reach the result that the map which assigns to each $x \in S$ the value

$$T_\lambda(x) = F_\lambda^{-1} \circ B'(I-D)^{-1}C(x)$$

is well defined. Now, consider $(S_n)_{n\geq 1}$ as a decreasing sequence of subsets in X defined by

$$\begin{cases} S_1 = S \\ S_{n+1} = \overline{co}(T_\lambda(S_n)). \end{cases}$$

It should be noted that $(S_n)_{n\geq1}$ consists of nonempty, bounded, closed, and convex subsets of S. The use of the equality (9) allows us to have

$$\begin{aligned} T_\lambda(S_n) &\subseteq \lambda A T_\lambda(S_n) + B(I-D)^{-1}CT_\lambda(S_n) \cdot B'(I-D)^{-1}C(S_n) \\ &\subseteq \lambda A(S) + B(I-D)^{-1}CT_\lambda(S_n) \cdot B'(I-D)^{-1}C(S_n). \end{aligned}$$

Making use of Lemma 2.4 in [20] together with the assumptions on B, C, D and B' enables us to have

$$\begin{aligned} \beta(S_{n+1}) &\leq \|B'(I-D)^{-1}C(S_n)\|\left(\frac{k_1k_2}{1-k_4}\right)\beta(T_\lambda(S_n)) + \|B(I-D)^{-1}C(T_\lambda(S_n))\|\left(\frac{k_2k_3}{1-k_4}\right)\beta(S_n) \\ &\leq \left(\frac{M_1k_1k_2}{1-k_4}\right)\beta(S_{n+1}) + \left(\frac{M_2k_2k_3}{1-k_4}\right)\beta(S_n). \end{aligned}$$

This implies that

$$\beta(S_{n+1}) \leq \left(\frac{M_2k_2k_3}{1-(k_4+M_1k_1k_2)}\right)\beta(S_n).$$

It follows by induction that

$$\beta(S_n) \leq \left(\frac{M_2k_2k_3}{1-(k_4+M_1k_1k_2)}\right)^n \beta(S).$$

As $n \longrightarrow \infty$, we have $\lim\limits_{n\to\infty} \beta(S_{n+1}) = 0$, and consequently the set $S_\infty = \bigcap\limits_{n\geq1} S_n$ is nonempty, closed, convex, and weakly compact in S. Also it is easy to see that $T_\lambda(S_\infty) \subseteq S_\infty$. Using arguments similar to those used in the proof of Theorem 3.6, we can deduce that $T_\lambda : S_\infty \longrightarrow S_\infty$ is weakly sequentially continuous and the remained proof follows along the lines of Arino, Gautier and Penot's fixed point theorem [2]. Q.E.D.

4. Existence Theory

Let X be a Banach algebra satisfying the condition $(\mathcal{P})$ and let $C([0,1],X)$ be the Banach algebra of all continuous functions from $[0,1]$ to X endowed with the sup-norm $\|\cdot\|_\infty$.
The system (3) will be considered under the following assumptions:

(H_0) The functions $\sigma_1, \sigma_2 : [0,1] \longrightarrow [0,1]$ are continuous and nondecreasing with $\|\sigma_2\|_\infty > 1$.
(H_1) The function $a : [0,1] \longrightarrow \mathbb{R}$ is a continuous function with bound $\|a\|_\infty < 1$
(H_2) The function $g : [0,1] \times X \longrightarrow X$ is such that:
 (a) The partial function $t \mapsto g(t,x)$ is continuous uniformly.
 (b) The partial function $x \mapsto g(t,x)$ is weakly sequentially continuous.
 (c) The partial function $x \mapsto g(t,x)$ is a contraction with a constant $\|a\|_\infty$.
(H_3) The function $p : [0,1] \times X \longrightarrow \mathbb{R}$ is such that:
 (a) The partial $x \mapsto p(t,x)$ is affine on X, for $t \in [0,1]$.
 (b) p is weakly sequentially continuous with respect to the second variable, and
 (c) The partial function $x \longrightarrow p(t,x)$ is a contraction with a constant $\dfrac{1-\|a\|_\infty}{\|\sigma_2\|_\infty}$.
(H_4) The function $h : [0,1] \times X \longrightarrow \mathbb{R}$ is such that:
 (a) The partial function $x \mapsto h(t,x)$ is affine on X, for $t \in [0,1]$.
 (b) The partial function $t \mapsto h(t,x)$ is measurable, for all $x \in X$,
 (c) The partial function $x \mapsto h(t,x)$ is weakly sequentially continuous, for all $t \in [0,1]$,
 (d) There exists a continuous function $\gamma : J \longrightarrow \mathbb{R}_+$ such that

$$\left|h(t,x) - h(t,y)\right| \leq \gamma(t)\|x-y\|, \text{ for all } t \in [0,1] \text{ and } x,y \in X.$$

 (d) There exists a constant $b > 0$ and a function $m \in L^1([0,1])$ such that

$$|h(t,x)| \leq m(t) \text{ and } \int_0^1 \frac{1}{t+s} m(s)ds \leq b.$$

(H_5) The function $\psi : [0,1] \times X \longrightarrow \mathbb{R}$ is such that:
 (a) The partial $x \mapsto \psi(t,x)$ is affine on X, for $t \in [0,1]$.
 (b) ψ is weakly sequentially continuous with respect to the second variable,
 (c) There exists a constant L satisfying

$$\left|\psi(t,x) - \psi(t,y)\right| \leq L\|x-y\|, \text{ for all } t \in [0,1] \text{ and } x,y \in X.$$

(H_6) The operator $T : C([0,1],X) \longrightarrow C([0,1],X)$ satisfies:
 (a) T is affine,
 (b) T is regular on $C([0,1],X)$,
 (c) T is weakly sequentially continuous on $C([0,1],X)$.
 (d) there exists a continuous function $\rho : [0,1] \longrightarrow \mathbb{R}_+$ such that

$$\left|Tx(t) - Ty(t)\right| \leq \rho(t)\|x(t) - y(t)\|, \text{ for all } t \in [0,1] \text{ and } x,y \in C([0,1],X).$$

(H_7) There exists $M_1, M_2 \in \mathbb{R}_+^*$ such that:
 (a) for all $r > 0$, we have $|g(t,x)| \leq M_1$ and $|p(t,x)| \leq M_2$, where $\|x\| \leq r$, for each $t \in [0,1]$,
 (b) $\lambda L\|\gamma\|_\infty \ln(2) + \|\rho\|_\infty\|\sigma_1\|_\infty\|v\|_\infty M < \frac{\lambda}{2}$
 (c) for all $r > 0$, we have $\sup\limits_{t\in[0,1]} \|k(t,x)\| \leq (1 - \|b\|_\infty - \|u\|_\infty)r$, where $\|x\| \leq r$.

Theorem 4.1. *Let X a Banach algebra satisfying the condition $(\mathcal{P})$. Under the assumptions (H_0)-(H_7), the system* (3) *has, at least, one solution $(x,y) = (x(t), y(t))$ which belongs to the Banach algebra* $C([0,1],X) \times C([0,1],X)$. ◇

Proof. Let S be the closed ball $\mathcal{B}_{r_0}$ on $C([0,1],X)$ centered at origin of radius $r_0 > 0$. where

$$r_0 = \frac{b\|u\|_\infty}{1 - (\lambda L\|\gamma\|_\infty \ln(2)\|u\|_\infty + \|\rho\|_\infty\|\sigma_1\|_\infty\|v\|_\infty M)}.$$

We recall that the problem (3) can be written in the following form

$$\begin{cases} x(t) = Ax(t) + By(t) \cdot B'y(t) \\ y(t) = Cx(t) + Dy(t), \end{cases}$$

where

$$\begin{cases} (Ax)(t) = \psi\left(t, \int_0^t \frac{t}{t+s} h(s,x(s))ds\right) \cdot u \; ; \; t \in [0,1] \\ (Bx)(t) = (Tx)(t) \; ; \; t \in [0,1] \\ (B'x)(t) = \int_0^{\sigma_1(t)} g(t,s,y(s))ds \cdot v \; ; \; t \in [0,1]. \\ (Cx)(t) = \int_0^{\sigma_2(t)} p\,(s,x(s))\,ds \cdot w \; ; \; t \in [0,1] \\ (Dx)(t) = a(t)x(t) \; ; \; t \in J \end{cases} \tag{11}$$

In order to apply Theorem 3.6, we have to verify the following steps.

Claim 1: $(I-D)^{-1}C(S)$ is a relatively weakly compact subset of $C([0,1],X)$. Let $\{x_n,\ n \in \mathbb{N}\}$ be any sequence in S. From assumption (H_5), it follows that

$$\begin{aligned} \|y_n(t)\| &= \|(I-D)^{-1}C(x_n(t))\| \\ &\le \|Cx_n(t)\| + \|D(I-D)^{-1}Cx_n(t)\| \\ &\le \|Cx_n(t)\| + \|a\|_\infty\|(I-D)^{-1}Cx_n(t)\| + \|D(I-D)^{-1}C(0)(t)\|. \end{aligned}$$

Since $y_n \in C([0,1],X)$, there is $t^* \in [0,1]$ such that

$$\begin{aligned} \|y_n\|_\infty = \|y_n(t^*)\| &\le \frac{\|w\|_\infty}{1-\|a\|_\infty}\left[\int_0^{\sigma_2(t^*)} p\,(s,x_n(s))\,ds + \sup_{t\in[0,1]}|p(t,0)|\right] \\ &\le \frac{r_0\|w\|_\infty}{1-\|a\|_\infty}(\|\sigma_2\|_\infty + 1) = \varrho. \end{aligned}$$

This prove that $\{(I-D)^{-1}Cx_n(t),\ n \in \mathbb{N}\}$ is a uniformly bounded sequence in $(I-D)^{-1}C(S)$. As a result, $(I-D)^{-1}C(S)(t)$ is sequentially relatively weakly compact. Now, we proceed to show that $(I-D)^{-1}C(S)$ is weakly equi-continuous. If we take $\varepsilon > 0, x \in S, x^* \in X^*$ and $t, t' \in [0,1]$ (without loss of generality assume that $t < t'$), then we have

$$\begin{aligned} \left\|x^*\left(((I-D)^{-1}Cx_n)(t') - ((I-D)^{-1}Cx_n)(t)\right)\right\| &\le \frac{1}{1-\|a\|_\infty}\left|\int_{\sigma_2(t)}^{\sigma_2(t')} p\,(s,x(s))\,ds\right|\|x^*(w)\| \\ &\le \frac{r_0\|x^*(w)\|}{1-\|a\|_\infty}\sup\left\{|\sigma_2(t') - \sigma_2(t)|,\ |t-t'| \le \varepsilon\right\}. \end{aligned}$$

The uniform continuity of the function σ_2 on the set $[0,1]$ leads to

$$\left|x^*\left(((I-D)^{-1}Cx_n)(t) - ((I-D)^{-1}Cx_n)(t')\right)\right| \to 0, \text{ as } t \to t'.$$

An application of the Arzelà-Ascoli's theorem [19], we conclude that $(I-D)^{-1}C(S)$ is sequentially relatively weakly compact in X Again, an application of Eberlein-Šmulian's theorem [18] shows that $(I-D)^{-1}C(S)$ is relatively weakly compact.

Claim 2: A, B, C, D and B' are weakly sequentially continuous. Firstly, we verify that the mapping A is well defined. Let $x \in C([0,1], X)$ and $\{t_n\ ,\ n \in \mathbb{N}\}$ be any sequence in $[0,1]$ converging to a point t. Then,

$$\begin{aligned}
\|(Ax)(t_n) - (Ax)(t)\| &\leq |R_n(t) - R(t)|\,\|u\|_\infty \\
&\leq L\left|\int_0^{t_n} \frac{t_n}{t_n+s} h(s,x(s))ds - \int_0^t \frac{t}{t+s} h(s,x(s))ds\right|\|u\|_\infty \\
&\leq \left|\int_0^{t_n} \frac{s(t_n-t)}{(t_n+s)(t+s)} h(s,x(s))ds\right| + \left|\int_{t_n}^t \frac{t}{t+s} h(s,x(s))ds\right| \\
&\leq b|t_n - t| + \int_{t_n}^t m(s)ds.
\end{aligned}$$

This implies that the map Ax is continuous on $[0,1]$ in view of assumption $(H_5)(c)$. Using an argument similar to that above, we deduce that the maps Cx and $B'x$ are continuous on $[0,1]$.
In the proof of Theorem 3.6, we need to prove that A, C and D weakly sequentially continuous on S and B and B' are weakly sequentially continuous on $(I-D)^{-1}C(S)$. We begin to show the property for the operator A. To see this, let $\{\xi_n,\ n \in \mathbb{N}\}$ be a weakly converging sequence of S to a point ξ. Since S is bounded, we can apply the Dobrakov's theorem [17] in order to get

$$\xi_n(t) \rightharpoonup \xi(t) \quad \text{in } X.$$

The use of assumption $(\mathcal{H}_5)(b)$ allows us to have

$$\frac{t}{t+s} h(s, \xi_n(s)) \rightharpoonup \frac{t}{t+s} h\left(s, \xi(s)\right) \ \text{ in } \mathbb{R}.$$

Using both the Lebesgue dominated convergence theorem and the condition $(\mathcal{P})$, we can deduce

$$(A\xi_n)(t) \rightharpoonup (A\xi)(t) \ \text{ in } \ X.$$

Since $\{A\xi_n,\ n \in \mathbb{N}\}$ is bounded with a bound $[L\|m\|_{L^1} + \|h(t,0)\|_\infty]\,\|v\|_\infty$, then we can again apply the Dobrakov's theorem to obtain $A\xi_n \rightharpoonup A\xi$, that is, the operator A is weakly sequentially continuous on S. Now, By using assumptions (H_3) and (H_7) combined with the dominated convergence theorem, we obtain

$$\int_0^{\sigma_2(t)} p\left(s, \xi_n(s)\right) ds \to \int_0^{\sigma_2(t)} p\left(s, \xi(s)\right) ds, \quad \text{as } n \to \infty.$$

Using the condition $(\mathcal{P})$ combined with the fact that $\{C\xi_n,\ n \in \mathbb{N}\}$ is bounded with a bound $\|\sigma_2\|_\infty M_2 \|w\|_\infty$ we obtain $C\xi_n \to C\xi$ and so C is weakly sequentially continuous on S. Moreover, taking into account that $\{D\xi_n,\ n \in \mathbb{N}\}$ is bounded with a bound $\|a\|_\infty r_0$, and using the Dobrakov's theorem [17] we show that D is a weakly sequentially continuous operator on S.
Next, we claim that B' is weakly sequentially continuous on $(I-D)^{-1}C(S)$. Indeed, let $\{y_n,\ n \in \mathbb{N}\}$ be any sequence in $(I-D)^{-1}C(S)$ weakly converging to a point y. From the above discussion, it is easy to verify that $\{y_n,\ n \in \mathbb{N}\}$ is bounded with a bound ϱ, and so we can apply the Dobrakov's theorem in order to get $y_n(t) \rightharpoonup y(t)$ in X. For all $t \in [0,1]$, we have $B'y_n(t) = r_n(t) \cdot v$, where

$$r_n(t) = \int_0^{\sigma_1(t)} g\left(s, y_n(s)\right) ds.$$

Since $\{r_n(t),\ n \in \mathbb{N}\}$ is a bounded real sequence with a bound $M_1\|\sigma_1\|_\infty$, it follows that there is a renamed subsequence such that $r_n(t) \to r(t)$. Using the condition $(\mathcal{P})$ combined with the fact $\{B'y_n,\ n \in \mathbb{N}\}$ is bounded, we obtain $B'y_n \rightharpoonup B'y$ and the claim is proved.

Claim 3: The operator A is 1-set-weakly contractive. To see this, let us fix arbitrary $x, y \in C([0,1],\mathbb{R})$. If we take an arbitrary $t \in [0,1]$, then we get

$$\begin{aligned}
\|(Ax)(t) - (Ay)(t)\| &\leq \left|\psi\left(t, \int_0^t \frac{t}{t+s} h(s,x(s))ds\right) - \psi\left(t, \int_0^t \frac{t}{t+s} h(s,y(s))ds\right)\right| \|u\|_\infty \\
&\leq L\left|\int_0^t \frac{t}{t+s} h(s,x(s))ds - \int_0^t \frac{t}{t+s} h(s,y(s))ds\right| \|u\|_\infty \\
&\leq L\int_0^t \frac{t}{t+s}\left|h(s,x(s)) - h(s,y(s))\right| ds\|u\|_\infty \\
&\leq L\int_0^t \frac{1}{t+s}\gamma(s)\|x(s)-y(s)\|ds\|u\|_\infty \\
&\leq L\|\gamma\|_\infty \ln(2)\|x-y\|_\infty.
\end{aligned}$$

By using Proposition 3.1 in [12], we deduce that A is 1-set-weakly contractive.

Claim 4: The set $\left\{x \in X\ :\ x = \lambda Ax + B(I-D)^{-1}Cx \cdot B'(I-D)^{-1}Cy\right\}$ is convex for all $y \in S$. Indeed, let $x \in S$ be arbitrary and let $y, z \in \left(\dfrac{I-\lambda A}{B(I-D)^{-1}C}\right)^{-1} B'(I-D)^{-1}C(x)$, then

$$\begin{cases}
y = \lambda Ay + B(I-D)^{-1}Cy \cdot B'(I-D)^{-1}Cx \ \text{ and} \\
z = \lambda Az + B(I-D)^{-1}Cz \cdot B'(I-D)^{-1}Cx
\end{cases}$$

Since ψ, h and T are affine and $B'(I-D)^{-1}C$ is constant, it follows that for any $\mu \in [0,1]$, we have

$$\begin{aligned}
\left[\mu y + (1-\mu)z\right](t) &= \lambda\psi\left(t, \left[\int_0^t \frac{t}{t+s}\mu h(s,y(s))ds + \int_0^t \frac{t}{t+s}(1-\mu)h(s,z(s))ds\right]\right)\cdot u \\
&+ \left[T(\mu y_1 + (1-\mu)z_1)\right](t)\cdot\int_0^{\sigma_1(t)} g(t,s,x_1(s))ds\cdot v \\
&= \lambda\psi\left(t, \int_0^t \frac{t}{t+s} h_{\mu y+(1-\mu)z}(s)ds\right)\cdot u + B_{\mu y_1+(1-\mu)z_1}(t)\cdot B'_{x_1}(t) \\
&= \lambda A_{\mu y+(1-\mu)z}(t) + B_{\mu y_1+(1-\mu)z_1}(t)\cdot B'_{x_1}(t).
\end{aligned}$$

where

$$\begin{cases}
x_1(t) = (I-D)^{-1}Cx(t) = (1-a(t))^{-1}\left[\int_0^{\sigma_2(t)} p(s,x(s))ds\cdot w\right] \\
y_1(t) = (I-D)^{-1}Cy(t) = (1-a(t))^{-1}\left[\int_0^{\sigma_2(t)} p(s,y(s))ds\cdot w\right] \\
z_1(t) = (I-D)^{-1}Cz(t) = (1-a(t))^{-1}\left[\int_0^{\sigma_2(t)} p(s,z(s))ds\cdot w\right]
\end{cases}$$

Based on assumption $(H_3)(a)$, it follows that

$$[\mu y + (1-\mu)z](t) = \lambda A_{\mu y+(1-\mu)z}(t) + B(I-D)^{-1}C_{\mu y+(1-\mu)z}(t) \cdot B'(I-D)^{-1}C_x(t)$$

As a result $\mu y + (1-\mu)z \in \left(\frac{I-\lambda A}{B(I-D)^{-1}C}\right)^{-1} B'(I-D)^{-1}C(x)$, for each $x \in S$.

Claim 5: $B'(I-D)^{-1}C(S) \subset \left(\frac{I-\lambda A}{B(I-D)^{-1}C}\right)(S)$. To do it, let $x \in S$ be a fixed point.

Let us define the mapping $\varphi_x : C([0,1],X) \longrightarrow C([0,1],X)$ by the formula

$$\varphi_x(y)(t) = \lambda\psi\left(t, \int_0^t \frac{t}{t+s} h(s,y(s))ds\right) \cdot u + Ty_1(t) \cdot \left[\int_0^{\sigma_1(t)} g(s,x_1(s))ds \cdot v\right],$$

where $x_1, y_1 \in C([0,1],X)$ are unique and satisfying

$$\|x_1\|_\infty \leq \frac{r_0\|w\|_\infty}{1-\|a\|_\infty} \quad \text{and} \quad \|y_1\|_\infty \leq \frac{\|w\|_\infty}{1-\|a\|_\infty}\|y\|_\infty.$$

Let us fix arbitrary $y, z \in C([0,1],X)$ and $t \in [0,1]$. Then, we have

$$\begin{aligned}
\left\|\varphi_x(y) - \varphi_x(z)\right\| &\leq \|\lambda Ay(t) - \lambda Az(t)\| + \|B(I-D)^{-1}Cy(t) - B(I-D)^{-1}Cz(t)\|\|B'(I-D)^{-1}Cx(t)\| \\
&\leq \left\|\lambda\psi\left(t, \int_0^t \frac{t}{t+s} h(s,y(s))ds\right) \cdot u - \lambda\psi\left(t, \int_0^t \frac{t}{t+s} h(s,z(s))ds\right) \cdot u\right\| \\
&+ \|\rho\|_\infty \left\|(I-D)^{-1}Cy - (I-D)^{-1}Cz\right\| \left\|\int_0^{\sigma_1(t)} g(s,(I-D)^{-1}Cx(s))ds \cdot v\right\| \\
&\leq [\lambda L\|\gamma\|_\infty \ln(2)\|u\|_\infty + \|\rho\|_\infty\|\sigma_1\|_\infty\|v\|_\infty M] \|y-z\|.
\end{aligned}$$

According to assumption $(H_7)(b)$, we reach the result that φ_x is a contraction mapping, and then it implies that there is a unique point $y \in C([0,1],X)$ such that $\varphi_x(y) = y$. Hence,

$$B'(I-D)^{-1}C(S) \subset \left(\frac{I-\lambda A}{B(I-D)^{-1}C}\right)(C([0,1],X)).$$

Since $y \in C([0,1],X)$, there is $t^* \in [0,1]$ such that $\|y\|_\infty = \|y(t^*)\|$ and consequently,

$$\begin{aligned}
\|y\|_\infty &\leq \|\lambda Ay(t^*)\| + \|B(I-D)^{-1}Cy(t^*)\|\|B'(I-D)^{-1}Cx(t^*)\| \\
&\leq \lambda L\left[\int_0^{t^*} \frac{t^*}{t^*+s} \left|h(s,y(s)) - h(s,0)\right| ds + \int_0^{t^*} \frac{t^*}{t^*+s} |h(s,0)|\, ds\right] \|u\|_\infty \\
&+ \|\rho\|_\infty \left\|(I-D)^{-1}Cy(t^*) - (I-D)^{-1}C(0)(t^*)\right\| \left\|\int_0^{\sigma_1(t^*)} g(s,(I-D)^{-1}Cx(s))ds \cdot v\right\| \\
&\leq \lambda L\|\gamma\|_\infty \ln(2)\|u\|_\infty\|y\|_\infty + b\|u\|_\infty + \|\rho\|_\infty\|\sigma_1\|_\infty\|v\|_\infty M\|y\|_\infty.
\end{aligned}$$

This implies that

$$\|y\|_\infty \leq \frac{b\|u\|_\infty}{1 - (\lambda L\|\gamma\|_\infty \ln(2)\|u\|_\infty + \|\rho\|_\infty\|\sigma_1\|_\infty\|v\|_\infty M)}.$$

To end the proof, we apply Theorem 3.6, we deduce that the the problem (3) has, at least, one solution in $\mathbf{B}_{r_0} \times C([0,1],X)$. Q.E.D.

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