



Lacunary Distributional Convergence in Topological Spaces

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Abstract. Most of the summability methods cannot be defined in an arbitrary Hausdorff topological space unless one introduces a linear or a group structure. In the present paper, using distribution functions over the Borel σ -field of the topology and lacunary sequences we define a new type of convergence method in an arbitrary Hausdorff topological space and we study some inclusion theorems with respect to the resulting summability method. We also investigate the inclusion relation between lacunary sequence and lacunary refinement of it.

1. Preliminaries

Studying summability in a topological space has always been a difficult issue due to the lack of the linearity because many of the summability methods need linear structure on the space. Therefore, many authors have restricted the scope by assuming either the topological space to have a group structure or a linear structure. Recently, some authors have studied some summability methods that directly can be defined in arbitrary Hausdorff spaces such as A -statistical convergence and A -distributional convergence (see e.g., [2, 15, 17–19]).

The idea of convergence of a real sequence was extended to statistical convergence by Fast [8]. This concept has been further investigated from various points of view later on. For example, properties of statistically convergent scalar sequences investigated by Fridy [10] and Salát [16]. Furthermore, the concept of statistical convergence has been studied in topological spaces by many authors (see e.g., [2, 15, 18]). Moreover, a generalization of the statistical convergence which is called ideal convergence can be studied in topological spaces (see e.g., [6, 7]).

As the structure of statistical convergence is compatible with topological structure, i.e., it can be characterized considering the elements of the base of the topology, it can be studied in arbitrary Hausdorff topological spaces whereas similar idea vanishes when the issue is to study many other convergence methods in topology such as strong convergence or matrix summability [3–5]. If the sequence $Ax := \{(Ax)_n\}$ is convergent to L then we say that x is A -summable to L where the series

$$(Ax)_n = \sum_k a_{nk} x_k$$

2010 *Mathematics Subject Classification.* 54A20, 40J05, 40A05

Keywords. Distributional convergence, statistical convergence, lacunary sequence

Received: 16 February 2019; Revised: 27 April 2019; Accepted: 05 March 2019

Communicated by Eberhard Malkowsky

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is convergent for any positive integer n . A summability matrix A is said to be regular if $\lim_n (Ax)_n = L$ whenever $\lim_k x_k = L$ [1].

Let (X, τ) be a Hausdorff topological space and let $A = (a_{nk})$ be a non-negative regular summability matrix. Then a sequence $x = (x_k)$ in X is said to be A -statistically convergent to $\alpha \in X$ [2, 15] if for any open set U that contains α

$$\lim_n \sum_{k: x_k \notin U} a_{nk} = 0.$$

Note that the definition of the concept of A -statistical convergence can be given with the elements of the base of the topology instead of open sets. Considering the base of the standard topology of the space of real numbers we can easily get the following well-known definition of the A -statistical convergence of a real valued sequence [3, 8, 10, 11, 16]: Let $x = (x_k)$ be a real sequence and let $A = (a_{nk})$ be a non-negative regular summability matrix. Then x is said to be A -statistically convergent to the real number L if for any $\varepsilon > 0$

$$\lim_n \sum_{k: |x_k - L| > \varepsilon} a_{nk} = 0.$$

If we consider the Cesàro matrix, $C = (c_{nk})$, then A -statistical convergence reduces to the statistical convergence where

$$c_{nk} := \begin{cases} \frac{1}{n} & , \quad \text{if } k \leq n \\ 0 & , \quad \text{otherwise.} \end{cases}$$

In 1993, Fridy and Orhan introduced the concept of lacunary statistical convergence that has a strong relationship with statistical convergence [12]. An increasing integer sequence $\theta = \{k_r\}$ is called a lacunary sequence if it satisfies that $k_0 = 0$ and $h_r := k_r - k_{r-1} \rightarrow \infty$ as $r \rightarrow \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r^\theta := (k_{r-1}, k_r]$ and the ratio k_r/k_{r-1} will be abbreviated by q_r for $r \geq 2$ and $q_1 = 0$. The lacunary sequence $\theta' = \{k'_r\}$ is called a lacunary refinement of the lacunary sequence $\theta = \{k_r\}$ if $\{k_r\} \subseteq \{k'_r\}$ [9]. We denote the cardinality of a subset $E \subseteq \mathbb{N}$ by $|E|$.

Fridy and Orhan [12] introduced the concept of lacunary statistical convergence. Let θ be a lacunary sequence. Then a real sequence $x = (x_k)$ is said to be lacunary statistically convergent to L provided that for each $\varepsilon > 0$,

$$\lim_r \frac{1}{h_r} \left| \left\{ k \in I_r^\theta : |x_k - L| \geq \varepsilon \right\} \right| = 0. \tag{1.1}$$

Actually, lacunary statistical convergence coincides with A -statistical convergence for $A = C_\theta$ where C_θ is the matrix given by

$$C_\theta [n, k] := \begin{cases} \frac{1}{h_r} & , \quad \text{if } k \in I_r^\theta \\ 0 & , \quad \text{if } k \notin I_r^\theta \end{cases}$$

The definition of lacunary statistical convergence can be extended to Hausdorff topological spaces as follows: Let (X, τ) be a Hausdorff topological space and let $\theta = \{k_r\}$ be a lacunary sequence. Then a sequence $x = (x_k)$ in X is said to be lacunary statistically convergent to $\alpha \in X$ [13] if for any open set U that contains α

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r^\theta} \chi_{\{x_k \notin U\}} = 0.$$

Distributional convergence is another summability method that can be defined in topological spaces. There is a close relationship between the concepts of distributional convergence and statistical convergence.

In 2014, Ünver et. al. [18] investigated the relationship between these concepts and they observed that A -distributional convergence is equivalent to A -statistical convergence for a particular degenerate distribution. In [12], the authors established some inclusion relations between the concept of statistical convergence and lacunary statistical convergence.

Consider a set function $F : \sigma(\tau) \rightarrow [0, 1]$ such that $F(X) = 1$ and if G_1, G_2, \dots are disjoint sets in $\sigma(\tau)$ then

$$F\left(\bigcup_{i=1}^{\infty} G_i\right) = \sum_{i=1}^{\infty} F(G_i)$$

where $\sigma(\tau)$ is the Borel σ -algebra of τ . Such a function is called a probability measure or a distribution. Now, let $A = (a_{nk})$ be a non-negative regular matrix such that each row adds up to one and F be a probability measure on $\sigma(\tau)$. Then the sequence $x = (x_k)$ in X is said to be A -distributionally convergent to F if for all $G \in \sigma(\tau)$ with $F(\partial G) = 0$ we have

$$\lim_n \sum_{k:x_k \in G} a_{nk} = F(G)$$

where ∂G is the boundary of G [18].

The aim of the present paper is to study a variant of distributional convergence in which the non-negative regular summability matrix $A = (a_{nk})$ is replaced by C_θ matrix and to obtain similar results given in [12] for topological spaces. Thus, we define a new concept of summability which can be directly studied in Hausdorff topological spaces. Analogously in [14], we also study the same concept with respect to the lacunary refinement of a lacunary sequence. Moreover, we obtain generalizations of some well-known results of summability theory for topological spaces.

Definition 1.1. Let X be a Hausdorff topological space, let F be a distribution on $\sigma(\tau)$ and let $\theta = \{k_r\}$ be a lacunary sequence. Then the sequence $x = (x_k)$ in X is said to be lacunary distributionally convergent to F if for all $G \in \sigma(\tau)$ with $F(\partial G) = 0$ we have

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r^\theta} \chi_{\{x_k \in G\}} = F(G).$$

Giving a characterization for A -statistical convergence, Ünver et. al. [18] has proved that A -statistical convergence is the special case of A -distributional convergence. This result with Definition 1.1 entails the following remark immediately:

Remark 1.2. Let X be a Hausdorff topological space, let $x = (x_k)$ be a sequence in X and let $\theta = \{k_r\}$ be a lacunary sequence. Then, x is lacunary statistically convergent to $\alpha \in X$ if and only if it is lacunary distributionally convergent to $F : \sigma(\tau) \rightarrow [0, 1]$ defined with

$$F(G) := \begin{cases} 0 & , \alpha \notin G \\ 1 & , \alpha \in G. \end{cases} \tag{1.2}$$

For the sake of completeness we keep the following definition of Cesàro distributional convergence which is a special case of A -distributional convergence for Cesàro matrix:

Definition 1.3. Let X be a Hausdorff topological space, let F be a distribution on $\sigma(\tau)$. Then a sequence $x = (x_k)$ in X is said to be Cesàro distributionally convergent to F if for all $G \in \sigma(\tau)$ with $F(\partial G) = 0$ we have

$$\lim_n \frac{1}{n} \sum_{k=1}^n \chi_{\{x_k \in G\}} = F(G).$$

2. Inclusion Theorems

Here, we prove some inclusion relations between the concepts of lacunary distributional convergence and Cesàro distributional convergence. Throughout this section, we assume that X is a Hausdorff topological space with topology τ .

Theorem 2.1. *Let $x = (x_k)$ be a sequence in X , let $\theta = \{k_r\}$ be a lacunary sequence and let F be a distribution on $\sigma(\tau)$ and assume that $\liminf_r q_r > 1$. If for all $G \in \sigma(\tau)$ with $F(\partial G) = 0$ we have*

$$\limsup_n \frac{1}{n} \sum_{k=1}^n \chi_{\{x_k \in G\}} \leq F(G) \tag{2.1}$$

then there exist $M > 1$ such that

$$\limsup_r \frac{1}{h_r} \sum_{k \in I_r^\theta} \chi_{\{x_k \in G\}} \leq F(G) M.$$

Proof. Assume that $\liminf_r q_r = \alpha > 1$ and write $\beta = (\alpha - 1) / 2 > 0$. Then, there exists a positive integer r_0 such that $q_r \geq 1 + \beta$ whenever $r > r_0$. Therefore, for each $r > r_0$ we have

$$\frac{h_r}{k_r} = 1 - \frac{1}{q_r} \geq 1 - \frac{1}{\beta + 1} = \frac{\beta}{\beta + 1}.$$

Now if $G \in \sigma(\tau)$ with $F(\partial G) = 0$, then we have

$$\frac{1}{k_r} \sum_{k \leq k_r} \chi_{\{x_k \in G\}} \geq \frac{\beta}{\beta + 1} \frac{1}{h_r} \sum_{k \in I_r^\theta} \chi_{\{x_k \in G\}}$$

for each $r > r_0$. Hence, we get

$$\limsup_r \frac{1}{k_r} \sum_{k \leq k_r} \chi_{\{x_k \in G\}} \geq \limsup_r \frac{\beta}{\beta + 1} \frac{1}{h_r} \sum_{k \in I_r^\theta} \chi_{\{x_k \in G\}}.$$

Finally, from (2.1) we obtain

$$\limsup_r \frac{1}{h_r} \sum_{k \in I_r^\theta} \chi_{\{x_k \in G\}} \leq F(G) M$$

where $M = \frac{\beta + 1}{\beta} > 1$. \square

Next proposition is a version of Lemma 2 of [12] for topological spaces which is obtained from Theorem 2.1.

Proposition 2.2. *Let $x = (x_k)$ be a sequence in X and let $\theta = \{k_r\}$ be a lacunary sequence with $\liminf_r q_r > 1$. If x is statistically convergent to α then it is lacunary statistically convergent to α .*

Proof. Suppose that x is statistically convergent to α . Then, we get from Proposition 1 of [18] that the sequence x is Cesàro distributionally convergent to F defined by (1.2). Let U be an open set that contains α . Then $V := U^c$ is a closed set that does not contain α where U^c is the complement of U . Therefore, we can

write $F(V) = 0$. As V is closed we have $\partial V \subseteq V$ which entails $\alpha \notin \partial V$. Thus, we get $F(\partial V) = 0$. Now, since x is Cesàro distributionally convergent to F we have

$$\lim_n \frac{1}{n} \sum_{k=1}^n \chi_{\{x_k \in V\}} = F(V).$$

Since, $\liminf_r q_r > 1$, Theorem 2.1 yields that there exists $M > 1$ such that

$$\limsup_r \frac{1}{h_r} \sum_{k \in I_r^\theta} \chi_{\{x_k \in V\}} \leq F(V)M = 0.$$

Thus, we get

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r^\theta} \chi_{\{x_k \in V\}} = 0$$

i.e.,

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r^\theta} \chi_{\{x_k \notin U\}} = 0.$$

Therefore, x is lacunary statistically convergent to α . \square

Theorem 2.3. Let $x = (x_k)$ be a sequence in X , let $\theta = \{k_r\}$ be a lacunary sequence and let F be a distribution on $\sigma(\tau)$. Assume that $\limsup_r q_r < \infty$. If for all $G \in \sigma(\tau)$ with $F(\partial G) = 0$

$$\limsup_r \frac{1}{h_r} \sum_{k \in I_r^\theta} \chi_{\{x_k \in G\}} \leq F(G)$$

then there exists $H > 0$ such that

$$\limsup_n \frac{1}{n} \sum_{k=1}^n \chi_{\{x_k \in G\}} \leq F(G)H.$$

Proof. Since, $\limsup_r q_r < \infty$ there exist $H > 0$ such that $q_r < H$ for any r . Take an arbitrary $G \in \sigma(\tau)$ with $F(\partial G) = 0$. Suppose that

$$\limsup_r \frac{1}{h_r} \sum_{k \in I_r^\theta} \chi_{\{x_k \in G\}} = T \leq F(G) \tag{2.2}$$

and write $N_r := \sum_{k \in I_r^\theta} \chi_{\{x_k \in G\}}$. From (2.2), for all $\varepsilon > 0$, there exists a positive integer r_0 such that

$$\left| \sup_{r > r_0} \frac{N_r}{h_r} - T \right| < \varepsilon.$$

Now, let $M := \max \{N_r : 1 < r \leq r_0\}$ and let n be a positive integer satisfying $k_{r-1} < n \leq k_r$; then we can write

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^n \chi_{\{x_k \in G\}} &\leq \frac{1}{k_{r-1}} \sum_{k=1}^{k_r} \chi_{\{x_k \in G\}} \\ &= \frac{1}{k_{r-1}} \{N_1 + N_2 + \dots + N_{r_0} + \dots + N_r\} \\ &\leq \frac{M}{k_{r-1}} r_0 + \frac{1}{k_{r-1}} \left\{ h_{r_0+1} \frac{N_{r_0+1}}{h_{r_0+1}} + \dots + h_r \frac{N_r}{h_r} \right\} \\ &\leq \frac{M}{k_{r-1}} r_0 + \frac{1}{k_{r-1}} \left(\sup_{r>r_0} \frac{N_r}{h_r} \right) \{h_{r_0+1} + \dots + h_r\} \\ &\leq \frac{M}{k_{r-1}} r_0 + (\varepsilon + F(G)) \frac{k_r - k_{r_0}}{k_{r-1}} \\ &\leq \frac{M}{k_{r-1}} r_0 + (\varepsilon + F(G)) q_r \\ &< \frac{M}{k_{r-1}} r_0 + (\varepsilon + F(G)) H. \end{aligned}$$

Since, $\lim_r k_r = \infty$ and ε is an arbitrary, the proof is completed. \square

The following proposition is a version of Lemma 3 of [12] for topological spaces which is obtained from Theorem 2.3.

Proposition 2.4. *Let $x = (x_k)$ be a sequence in X and let $\theta = \{k_r\}$ be a lacunary sequence with $\limsup_r q_r < \infty$. If x is lacunary statistically convergent to α then it is statistically convergent to α .*

Proof. Suppose that $x = (x_k)$ is lacunary statistically convergent to α . Then, we get from Proposition 1 of [18] that the sequence x is lacunary distributionally convergent to F defined by (1.2). Let U be an open set that contains α . As in the proof of Proposition 2.2, $V := U^c$ is a closed set that does not contain α with $F(\partial V) = 0$. Since, x is lacunary distributionally convergent to F we have

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r^\theta} \chi_{\{x_k \in V\}} = F(V)$$

for $V \in \sigma(\tau)$ with $F(\partial V) = 0$. Now, since $\limsup_r q_r < \infty$, we have from Theorem 2.3 that there exists $H > 0$ such that

$$\limsup_n \frac{1}{n} \sum_{k \leq n} \chi_{\{x_k \in V\}} \leq F(V) H = 0.$$

Thus, we get

$$\lim_n \frac{1}{n} \sum_{k \leq n} \chi_{\{x_k \in V\}} = 0$$

i.e.

$$\lim_n \frac{1}{n} \sum_{k \leq n} \chi_{\{x_k \notin U\}} = 0.$$

Therefore, x is statistically convergent to α . \square

Now, combining Proposition 2.2 and Proposition 2.4 we get the version of Theorem 4 of [12] for topological spaces:

Theorem 2.5. *Let $x = (x_k)$ be a sequence in X and let $\theta = \{k_r\}$ be a lacunary sequence with*

$$1 < \liminf q_r \leq \limsup q_r < \infty.$$

Then x is statistically convergent to α if and only if it is lacunary statistically convergent to α .

In the following theorem we study the equality of distributional limits.

Theorem 2.6. *Let $x = (x_k)$ be a sequence in X , let $\theta = \{k_r\}$ be a lacunary sequence and let F_1 and F_2 be two distribution on $\sigma(\tau)$. If x is Cesàro distributionally convergent to F_1 and it is lacunary distributionally convergent to F_2 then $F_1(G) = F_2(G)$ for any $G \in \sigma(\tau)$ with $F_1(\partial G) = F_2(\partial G) = 0$.*

Proof. If x is Cesàro distributionally convergent to F_1 and lacunary distributionally convergent to F_2 then we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \chi_{\{x_k \in G\}} = F_1(G)$$

for all $G \in \sigma(\tau)$ with $F_1(\partial G) = 0$ and

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r^\theta} \chi_{\{x_k \in G\}} = F_2(G)$$

for all $G \in \sigma(\tau)$ with $F_2(\partial G) = 0$. Suppose that $F_1(G_1) \neq F_2(G_1)$ for some $G_1 \in \sigma(\tau)$ such that $F_1(\partial G_1) = F_2(\partial G_1) = 0$. Then, we get

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \chi_{\{x_k \in G_1\}} \neq F_2(G_1) \tag{2.3}$$

for $G_1 \in \sigma(\tau)$. On the other hand one can have

$$\begin{aligned} \frac{1}{k_m} \sum_{k \leq k_m} \chi_{\{x_k \in G_1\}} &= \frac{1}{k_m} \sum_{r=1}^m \sum_{k \in I_r^\theta} \chi_{\{x_k \in G_1\}} \\ &= \frac{1}{k_m} \sum_{r=1}^m h_r \frac{1}{h_r} \sum_{k \in I_r^\theta} \chi_{\{x_k \in G_1\}} \\ &= \frac{1}{m} \sum_{r=1}^m h_r t_r \end{aligned} \tag{2.4}$$

where $t_r = \frac{1}{h_r} \sum_{k \in I_r^\theta} \chi_{\{x_k \in G_1\}}$. Since x is lacunary distributionally convergent to F_2 , we get $\lim_r t_r = F_2(G_1)$. As the transformation in (2.4) is regular [12], the right hand side of (2.4) tends to $F_2(G_1)$ as $r \rightarrow \infty$. This contradicts with (2.3). \square

The following is a version of Theorem 6 of [12] for topological spaces which is obtained from Theorem 2.6.

Theorem 2.7. Let $x = (x_k)$ be a sequence in X and let $\theta = \{k_r\}$ be an arbitrary lacunary sequence. If x is both statistically convergent and lacunary statistically convergent then the statistical limit and the lacunary statistical limit of x coincide.

Proof. Assume that x is statistically convergent to α and lacunary statistically convergent to β with $\alpha \neq \beta$. Then, we get from Proposition 1 of [18] that x is Cesàro distributionally convergent to F_1 and lacunary distributionally convergent to F_2 where $F_{1,2} : \sigma(\tau) \rightarrow [0, 1]$ are distributions defined with

$$F_1(G) := \begin{cases} 0 & , \alpha \notin G \\ 1 & , \alpha \in G \end{cases}$$

and

$$F_2(G) := \begin{cases} 0 & , \beta \notin G \\ 1 & , \beta \in G. \end{cases}$$

Since $\alpha \neq \beta$ and (X, τ) is Hausdorff, there exist an open set U that contains α and an open set V that contains β such that $U \cap V = \emptyset$. Then the set $W := U^c$ is closed and $V \subset W$. Since $\beta \in V$, we have $\beta \in W$ and $\beta \in W^o$ which implies $\beta \notin (W^o)^c$ where W^o is the interior of W . In that case we get $\beta \notin W \cap (W^o)^c = \partial W$. Thus $F_2(W) = 1$ and $F_2(\partial W) = 0$. On the other hand, it is obvious that $\alpha \notin W$. Therefore we can write $F_1(W) = 0$. As W is closed we have $\partial W \subseteq W$ which implies $\alpha \notin \partial W$. Hence we get $F_1(\partial W) = 0$. From Theorem 2.6, since $F_1(\partial W) = F_2(\partial W) = 0$, we have $F_1(W) = F_2(W)$. This is a contradiction. \square

Now, we study lacunary refinement distributional convergence.

Theorem 2.8. Let $x = (x_k)$ be a sequence in X , let F be a distribution on $\sigma(\tau)$ and assume that $\theta' = \{k'_r\}$ is a lacunary refinement of a given lacunary sequence $\theta = \{k_r\}$. If there exists $\delta > 0$ such that

$$\frac{|J_j^{\theta'}|}{|I_i^{\theta}|} \geq \delta$$

for each $J_j^{\theta'} \subseteq I_i^{\theta}$, then for all $G \in \sigma(\tau)$ with $F(\partial G) = 0$

$$\limsup_r \frac{1}{|I_r^{\theta}|} \sum_{k \in I_r^{\theta}} \chi_{\{x_k \in G\}} \leq F(G)$$

implies

$$\limsup_r \frac{1}{|J_r^{\theta'}|} \sum_{k \in J_r^{\theta'}} \chi_{\{x_k \in G\}} \leq \frac{F(G)}{\delta}.$$

Proof. Suppose that

$$\limsup_r \frac{1}{|I_r^{\theta}|} \sum_{k \in I_r^{\theta}} \chi_{\{x_k \in G\}} \leq F(G)$$

for an arbitrary $G \in \sigma(\tau)$ with $F(\partial G) = 0$. For any positive integer j there exists positive integer i such that

$J_j^{\theta'} \subseteq I_i^\theta$. Then, we have

$$\begin{aligned} \frac{1}{|J_j^{\theta'}|} \sum_{k \in J_j^{\theta'}} \chi_{\{x_k \in G\}} &= \frac{|I_i^\theta|}{|J_j^{\theta'}|} \frac{1}{|I_i^\theta|} \sum_{k \in J_j^{\theta'}} \chi_{\{x_k \in G\}} \\ &\leq \frac{|I_i^\theta|}{|J_j^{\theta'}|} \frac{1}{|I_i^\theta|} \sum_{k \in I_i^\theta} \chi_{\{x_k \in G\}} \\ &\leq \frac{1}{\delta} \frac{1}{|I_i^\theta|} \sum_{k \in I_i^\theta} \chi_{\{x_k \in G\}}. \end{aligned}$$

which implies

$$\limsup_j \frac{1}{|J_j^{\theta'}|} \sum_{k \in J_j^{\theta'}} \chi_{\{x_k \in G\}} \leq \frac{F(G)}{\delta}.$$

□

The following theorem is a version of Theorem 1 of [14] for topological spaces.

Theorem 2.9. *Let $x = (x_k)$ be a sequence in X and assume that $\theta' = \{k_r\}$ is a lacunary refinement of the lacunary sequence $\theta = \{k_r\}$. If there exists $\delta > 0$ such that*

$$\frac{|J_j^{\theta'}|}{|I_i^\theta|} \geq \delta \tag{2.5}$$

for every $J_j^{\theta'} \subseteq I_i^\theta$, then lacunary statistical convergence of x with respect to θ implies lacunary statistical convergence of x with respect to θ' .

Proof. Suppose that x is lacunary statistically convergent to α with respect to θ . Then we get from Proposition 1 of [18] that the sequence x is lacunary distributionally convergent to F defined by (1.2). Let U be an open set that contains α . As in the proof of Proposition 2.2, $V := U^c$ is a closed set that does not contain α with $F(\partial V) = 0$. Since x is lacunary distributionally convergent to F we have

$$\lim_r \frac{1}{|I_r^\theta|} \sum_{k \in I_r^\theta} \chi_{\{x_k \in V\}} = F(V) = 0$$

for $V \in \sigma(\tau)$ with $F(\partial V) = 0$ and since there exists $\delta > 0$ such that (2.5) holds, we get from Theorem 2.8 that

$$\limsup_j \frac{1}{|J_j^{\theta'}|} \sum_{k \in J_j^{\theta'}} \chi_{\{x_k \in V\}} \leq \frac{F(V)}{\delta} = 0.$$

□

3. Conclusion

In this paper, we define the concept of lacunary distributional convergence which is a generalization of lacunary statistical convergence in Hausdorff topological spaces that need not to have linear or group structure. This new type of convergence is one of the convergence methods that can be directly studied in arbitrary Hausdorff topological spaces. We also obtain some inclusion theorems between this concept and statistical convergence which are more general than the classical ones [12]. Later, we get the inclusion relation between lacunary statistical convergence and lacunary refinement of it.

4. Acknowledgement

We gratefully thank to the referee for the valuable comments and suggestions which substantially helped improving the quality of the paper.

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