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Some New Fixed Point Results for Multi-Valued Weakly Picard Operators

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Abstract. In this paper, we present some extensions of Banach contraction principle for multi-valued maps. Corresponding convergence theorems for the Picard iteration associated to a class of multi-valued operators are obtained in the setting of modular metric spaces. The presented results improve many recent fixed points results in the setting of modular metric spaces and also generalize some classical known results. Moreover, some examples are given.

1. Introduction

In 1969, Nadler [21] introduced the notion of multi-valued Lipschitz mappings as a generalization of the Banach contraction principle in the setting of complete metric spaces. Since then, several authors investigated fixed point results in this direction, see [5–7, 9, 10, 13, 14, 20, 28]. In 2010, Chistyakov [11, 12] introduced the concept of modular metric spaces. There are different approaches for this concept. The class of modular metric spaces is viewed as the nonlinear version of the classical modular spaces introduced in [18, 19, 24] (see also [8, 16]). Recently, Abdou and Khamsi [1] investigated the fixed point property in the setting of modular spaces and introduced the analog of the Banach contraction principle theorem in the setting of modular metric spaces. In 2014, Abdou and Khamsi [2] established some fixed point theorems for multi-valued Lipschitzian mappings defined on some subsets of modular metric spaces. In 2012, Samet *et al.* [27] introduced the notion of $\alpha-\psi-$contractive mappings and $\alpha-$admissible mappings in metric spaces and obtained many fixed point results. Recently, Ali *et al.* [3] generalized and extended the notion of $\alpha-\psi-$contractive mappings by introducing the notion of $(\alpha,\psi,\xi)-$contractive multi-valued mappings and gave fixed point theorems for such type mappings in metric spaces.

Motivated by [2], the purpose of this paper is to extend the results of Abdou and Khamsi [1] by using the concept of $\alpha-$admissible contractive mappings. We will establish some fixed point theorems involving such contractions in the setting of modular metric spaces.

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2. Preliminaries

Throughout the paper, $\mathbb{N}$ and $\mathbb{R}$ denote the set of positive integers and the set of all real numbers, respectively. In what follows, we recall some definitions and results we will need in the sequel.

Let X be a nonempty set. For a function $\omega : (0,\infty) \times X \times X \to (0,\infty)$, we denote by

$$\omega_\lambda(x,y) := \omega(\lambda,x,y) \quad \text{for all } \lambda > 0 \text{ and } x,y \in X.$$

Definition 2.1. *[11] A function $\omega : (0,\infty) \times X \times X \to [0,\infty]$ is said to be a modular metric on X (or simply a modular if no ambiguity arises) if for all $x,y,z \in X$, the following three axioms hold:*

(i) $x = y$ *if and only if* $\omega_\lambda(x,y) = 0$ *for all* $\lambda > 0$;

(ii) $\omega_\lambda(x,y) = \omega_\lambda(y,x)$ *for all* $\lambda > 0$;

(iii) $\omega_{\lambda+\mu}(x,y) \leq \omega_\lambda(x,z) + \omega_\mu(z,y)$ *for all* $\lambda,\mu > 0$.

The pair (X,ω) is then called a modular metric space.

Definition 2.2. *[11] Let (X,ω) be a modular metric space.*

(1) ω is said regular if the axiom (i) in Definition 2.1 is replaced by the following axiom:

$$x = y \quad \text{if and only if } \omega_\lambda(x,y) = 0 \quad \text{for some } \lambda > 0.$$

(2) ω is said convex if for $\lambda,\mu > 0$ and $x,y,z \in X$, the following inequality is satisfied:

$$\omega_{\lambda+\mu}(x,y) \leq \frac{\lambda}{\lambda+\mu}\omega_\lambda(x,z) + \frac{\mu}{\lambda+\mu}\omega_\mu(z,y).$$

Definition 2.3. *[11] Let (X,ω) be a modular metric space and $x_0 \in X$ an arbitrarily element. Let*

$$X_\omega = X_\omega(x_0) = \{x \in X : \lim_{\lambda\to\infty} \omega_\lambda(x,x_0) = 0\}$$

and

$$X_\omega^\star = X_\omega^\star(x_0) = \{x \in X : \exists \lambda = \lambda(x) \text{ such that } \omega_\lambda(x,x_0) < \infty\}.$$

The sets X_ω and $X_\omega^\star$ are called modular spaces (around x_0).

Note that if ω is a convex modular metric on X, then $X_\omega^\star = X_\omega$.

Proposition 2.4. *[11] If (X,ω) is a modular metric space, then the modular set X_ω is a metric space with metric given by*

$$d_\omega(x,y) = \inf\{\lambda > 0 : \omega_\lambda(x,y) \leq \lambda\}, \quad x,y \in X_\omega.$$

Proposition 2.5. *[11] Given a convex modular space (X,ω). Define*

$$d_\omega^\star(x,y) = \inf\{\lambda > 0 : \omega_\lambda(x,y) \leq 1\}, \quad x,y \in X_\omega^\star.$$

Then $(X_\omega^\star, d_\omega^\star)$ is a metric space.

Definition 2.6. *[11, 12](Topological concepts)*

Let (X,ω) be a modular metric space.

(1) The sequence $\{x_n\}$ in X_ω is said $\omega-$convergent to an element $x \in X_\omega$ if and only if

$$\lim_{n\to\infty} \omega_1(x_n,x) = 0.$$

We say then that x is an $\omega-$limit of $\{x_n\}$.

(2) The sequence $\{x_n\}$ in X_ω is said $\omega-$Cauchy if

$$\lim_{n,m\to\infty} \omega_1(x_n, x_m) = 0.$$

(3) We say that a subset M of X_ω is $\omega-$closed if for any sequence $\{x_n\}$ in M such that $\lim\limits_{n\to\infty} \omega_1(x_n, x) = 0$, then, $x \in M$.

(4) We say that a subset M of X_ω is $\omega-$complete if any $\omega-$Cauchy sequence in M is $\omega-$convergent in M.

(5) We say that a subset M of X_ω is $\omega-$bounded if

$$\delta_\omega(M) = \sup\{\omega_1(x, y) : x, y \in M\} < \infty.$$

(6) We say that a subset M of X_ω is $\omega-$compact if any sequence in M has a subsequence $\omega-$convergent in M.

(7) We say that ω satisfies the Fatou property if for every sequence $\{x_n\}$ in X_ω and all $x, y \in X_\omega$, we have

$$\lim_{n\to\infty} \omega_1(x_n, x) = 0 \Rightarrow \omega_1(x, y) \leq \liminf_{n\to\infty} \omega_1(x_n, y).$$

Definition 2.7. *[2] We say that ω satisfies the Δ_2-condition, if $\lim\limits_{n\to\infty} \omega_\lambda(x_n, x) = 0$, for some $\lambda > 0$ implies $\lim_{n\to\infty} \omega_\lambda(x_n, x) = 0$, for all $\lambda > 0$.*

Following [2, 11, 12], the $\omega-$convergence and $d_\omega-$convergence are equivalent if and only if ω satisfies the Δ_2-condition. Furthermore, if ω is a convex modular, then d_ω and $d^\star_\omega$ are equivalent.

Definition 2.8. *[2] Let (X, ω) be a modular metric space. We say that ω satisfies the Δ_2-type condition, if for every $\mu > 0$, there exits $C_\mu > 0$ such that*

$$\omega_{\lambda/\mu}(x, y) \leq C_\mu \omega_\lambda(x, y), \quad \text{for all } \lambda > 0,\ x, y \in X,\ \text{with } x \neq y.$$

Obviously, if ω satisfies the Δ_2-type condition, then it satisfies the Δ_2-condition.

Definition 2.9. *[2] Let (X, ω) be a modular metric space. Define the function Ω as*

$$\Omega(t) = \sup\left\{\frac{\omega_{\lambda/t}(x, y)}{\omega_\lambda(x, y)} : \lambda > 0,\ x, y \in X_\omega,\ x \neq y\right\} \quad \text{for every } t > 0.$$

Lemma 2.10. *[2] Let (X, ω) be a modular metric space. Assume that ω is a convex regular modular satisfying the Δ_2-type condition. Then*

(1) $\Omega(t) < \infty$, *for each* $t > 0$;

(2) Ω *is an increasing function with* $\Omega(1) = 1$;

(3) $\Omega(st) \leq \Omega(s)\Omega(t)$ *for each* $s, t > 0$;

(4) $\Omega^{-1}(s)\Omega^{-1}(t) \leq \Omega^{-1}(st)$, *where* Ω^{-1} *is the function inverse of* Ω;

(5) *for each* $x, y \in X_\omega$, *with* $x \neq y$, *we have*

$$d^\star_\omega(x, y) \leq \frac{1}{\Omega^{-1}(1/\omega_1(x, y))}.$$

We have the useful lemmas.

Lemma 2.11. *[2] Let (X, ω) be a modular metric space. Assume that ω is a convex regular modular satisfying the Δ_2-type condition. Let $\{x_n\}$ be a sequence in X_ω such that*

$$\omega_1(x_{n+1}, x_n) \leq C\alpha^n \quad \forall n = 0, 1, 2, \ldots$$

where C is an arbitrary constant and $\alpha \in [0, 1)$. Then $\{x_n\}$ is Cauchy for both ω and $d^\star_\omega$.

Let (X, ω) be a modular metric space and M be a nonempty subset of X_ω.
We denote by
(a) $K(M)$ the family of all nonempty and $\omega-$compact subsets of M;
(b) $CL(M)$ the family of all nonempty and $\omega-$closed subsets of M;
(c) $CB(M)$ the family of all nonempty, $\omega-$closed and $\omega-$bounded subsets of M.
For $A, B \in CB(M)$ and $x \in X$, set

$$H_\omega(A,B) = \max\{\sup\{\omega_1(x,B) : x \in A\}, \sup\{\omega_1(y,A) : y \in B\}\}$$

where $\omega_1(x,A) = \inf\{\omega_1(x,y) : y \in A\}$. H_ω is the Hausdorff modular metric on $CB(M)$, induced by the modular metric ω.

Proposition 2.12. *Let (X, ω) be a modular metric space. Consider the metric D_ω as*

$$D_\omega((x,y),(u,v)) = d_\omega(x,u) + d_\omega(y,v) \quad \text{for all } (x,y),(u,v) \in X_\omega \times X_\omega.$$

Then $(X_\omega \times X_\omega, D_\omega)$ is a metric space.

Definition 2.13. *Let (X, ω) be a modular metric space, M be a nonempty subset of X_ω and $T : M \to CL(M)$ be a multi-valued mapping. The graph of T denoted by $G(T)$ is the subset $\{(x,y) : x \in M,\ y \in Tx\}$ of $M \times M$. Then T is said to be closed if the graph of $G(T)$ is a closed subset of $(M \times M, D_\omega)$.*

Definition 2.14. *A function $f : X_\omega \to [0, \infty)$ is called lower semi-continuous (l.s.c) if, for any $x \in X_\omega$ and $\{x_n\} \subset X_\omega$ with $\lim_{n\to\infty} x_n = x$ in X_ω, we have*

$$f(x) \le \lim_{n\to\infty} f(x_n).$$

For a multi-valued map $T : M \to CB(M)$, let $f_T : M \to [0, \infty)$ be a function defined by

$$f_T(x) = \omega_1(x, Tx).$$

Definition 2.15. *[26] Let (X, d) be a metric space and $T : X \to CL(X)$ be a multivalued operator. We say that T is a multivalued weakly Picard (briefly, MWP) operator if for all $x \in X$ and $y \in Tx$, there exists a sequence $\{x_n\}$ such that*

(i) $x_0 = x$ *and* $x_1 = y$;

(ii) $x_{n+1} \in Tx_n$ *for all* $n = 0, 1, 2, \dots$;

(iii) $\{x_n\}$ *is convergent and its limit is a fixed point of T.*

A sequence $\{x_n\}$ satisfying conditions (i) and (ii) in Definition 2.15 is said a sequence of successive approximations of T, starting from x_0.

As in [4, 17], we give the following definition.

Definition 2.16. *Let (X, ω) be a modular metric space and M be a nonempty subset of X_ω. let $T : M \to CB(M)$ be a multi-valued mapping. Such T is called $\alpha-$admissible if, for each $x \in M$ and $y \in Tx$ with $\alpha(x,y) \ge 1$, we have $\alpha(y,z) \ge 1$ for all $z \in Ty$.*

In this paper, we investigate several types of multivalued weakly Picard operators, and so we ensure the existence of fixed points. Some consequences and examples have been provided.

3. Fixed point of multi-valued weak contraction mappings

We start with the following useful technical lemmas (corresponding to the ones given in [21] on modular metric spaces).

Lemma 3.1. *Let (X, ω) be a modular metric space, M be a nonempty subset of X_ω and $B \in CB(M)$. If $a \in M$ and $\omega_1(a, B) < c$ with $c > 0$, then there exists $b \in B$ such that $\omega_1(a, b) < c$.*

Lemma 3.2. *[2] Let (X, ω) be a modular metric space and M be a nonempty subset of X_ω. Let $A, B \in CB(M)$, then for every $\varepsilon > 0$ and $x \in A$, there exists $y \in B$ such that*

$$\omega_1(x, y) \leq H_\omega(A, B) + \varepsilon.$$

Moreover, if B is $\omega-$compact and ω satisfies the Fatou property, then for every $x \in A$ there exists $y \in B$ such that

$$\omega_1(x, y) \leq H_\omega(A, B).$$

Lemma 3.3. *[2] Let (X, ω) be a modular metric space. Assume that ω satisfies the Δ_2- condition. Let M be a nonempty subset of X_ω and let A_n be a sequence of sets in $CB(M)$. Suppose that $\lim_{n\to\infty} H_\omega(A_n, A) = 0$, for $A \in CB(M)$. Then if $x_n \in A_n$ and $\lim_{n\to\infty} x_n = x$, it follows that $x \in A$.*

3.1. Result-I

In this subsection, we first introduce the notion of $\omega-$quasi-contractions in modular metric spaces.

Definition 3.4. *Let (X, ω) be a modular metric space and M be a nonempty subset of X_ω. A multi-valued $T : M \to CL(M)$ is said an $\omega-$quasi-contraction if there exists a constant $k \in [0, 1)$ such that for any $x, y \in M$ with $y \in Tx$, there exists $z \in Ty$ such that*

$$\omega_1(y, z) \leq k \max\{\omega_1(x, y), \omega_1(x, Tx), \omega_1(y, Ty)\}. \tag{1}$$

Now, we state and prove our first result.

Theorem 3.5. *Let (X, ω) be a modular metric space. Assume that ω is a convex regular modular satisfying the Δ_2-condition. Let M be a nonempty $\omega-$complete subset of X_ω. Let $T : M \to CL(M)$ be a closed $\omega-$quasi-contraction. Then T is a MWP operator.*

Proof. Let $x_0 \in M$ and $x_1 \in Tx_0$. Clearly, if $x_0 = x_1$, then x_1 is a fixed point of T and so this completes the proof. Now, we assume that $x_0 \neq x_1$. Since T is $\omega-$quasi-contraction, there exists $x_2 \in Tx_1$ such that

$$\omega_1(x_1, x_2) \leq k \max\{\omega_1(x_0, x_1), \omega_1(x_0, Tx_0), \omega_1(x_1, Tx_1)\}.$$

If $x_2 = x_1$, then x_2 is a fixed point of T and so the proof is finished.

From now on, we assume that $x_2 \neq x_1$. It follows that

$$\omega_1(x_1, x_2) \leq k \max\{\omega_1(x_0, x_1), \omega_1(x_0, x_1), \omega_1(x_1, x_2)\} = k \max\{\omega_1(x_0, x_1), \omega_1(x_1, x_2)\}.$$

If $\max\{\omega_1(x_0, x_1), \omega_1(x_1, x_2)\} = \omega_1(x_1, x_2)$, then we obtain

$$\omega_1(x_1, x_2) \leq k\omega_1(x_1, x_2) < \omega_1(x_1, x_2)$$

which is a contradiction. Then we get

$$\omega_1(x_1, x_2) \leq k\omega_1(x_0, x_1).$$

Continuing in this fashion, we construct a sequence $\{x_n\}$ in M such that $x_{n+1} \in Tx_n$, $x_{n+1} \neq x_n$ and

$$\omega_1(x_n, x_{n+1}) \leq k\omega_1(x_{n-1}, x_n) \quad \forall n = 1, 2, 3, \ldots \tag{2}$$

By induction, we get

$$\omega_1(x_n, x_{n+1}) \leq k^n \omega_1(x_0, x_1) \quad \forall n = 0, 1, 2, \ldots \tag{3}$$

By Lemma 2.11, we conclude that $\{x_n\}$ is $\omega-$Cauchy. By completeness of (M, ω), there exists $u \in M$ such that $\lim_{n\to\infty} x_n = u$.

We shall prove that i is a fixed point of T. Since T is a closed multi-valued mapping and $x_{n+1} \in Tx_n$, then $(x_n, x_{n+1}) \in G(T)$. Moreover, we have $D_\omega((x_n, x_{n+1}), (u, u)) = d_\omega(x_n, u) + d_\omega(x_{n+1}, u)$. Since ω satisfies the Δ_2-condition, we get

$$\lim_{n\to\infty} D_\omega((x_n, x_{n+1}), (u, u)) = 0.$$

Finally, since $G(T)$ is closed, it follows that $(u, u) \in G(T)$. Hence $u \in Tu$, that is, u is a fixed point of T. □

3.2. Result II

In this subsection, we give another characterization of *MWP* operators.

Definition 3.6. *Let (X, ω) be a modular metric space and M be a nonempty subset of X_ω. A multi-valued $T : M \to CB(M)$ is said an $\alpha - \omega-$weak contraction if there exist a function $\alpha : M \times M \to [0, \infty)$ and two constants $k \in [0, 1)$ and $L \geq 0$ such that for any $x, y \in M$, with $\alpha(x, y) \geq 1$, we have*

$$H_\omega(Tx, Ty) \leq k\omega_1(x, y) + L\omega_1(y, Tx). \tag{4}$$

Definition 3.7. *Let (X, ω) be a modular metric space and M be a nonempty subset of X_ω. The pair (M, α) is said $\omega-$regular if the following condition holds: for any sequence $\{x_n\}$ in M with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in M$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all $k \in \mathbb{N}$.*

Remark 3.8. *Definition 3.7 is valid for* **Result II** *and* **Result III**.

We provide the following result.

Theorem 3.9. *Let (X, ω) be a modular metric space. Assume that ω is a convex regular modular satisfying the Δ_2-condition. Let M be a nonempty $\omega-$complete subset of X_ω. Let $T : M \to CB(M)$ be an $\alpha - \omega-$weak contraction. Suppose also that*
(1) T is $\alpha-$admissible;
(2) there exit $x_0 \in M$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;
(3) (M, α) is $\omega-$regular or f_T is lower semi-continuous.
Then T is a MWP operator.

Proof. Let r be a real number such that $0 < k < r < 1$. By condition (2), there exist $x_0 \in M$ and $x_1 \in Tx_0$ such that

$$\alpha(x_0, x_1) \geq 1.$$

If $x_0 = x_1$, then x_1 is a fixed point of T and the proof is finished.
Now, we assume that $x_0 \neq x_1$. Since $x_1 \in Tx_0$, by (9), we have

$$\begin{aligned}
\omega_1(x_1, Tx_1) &\leq H_\omega(Tx_0, Tx_1) \\
&\leq k\omega_1(x_0, x_1) + L\omega_1(x_1, Tx_0) \leq k\omega_1(x_0, x_1) + L\omega_1(x_1, x_1) \\
&= k\omega_1(x_0, x_1) < r\omega_1(x_0, x_1).
\end{aligned}$$

By Lemma 3.1, there exists $x_2 \in Tx_1$ such that

$$\omega_1(x_1, x_2) < r\omega_1(x_0, x_1).$$

T is $\alpha-$admissible and $x_2 \in Tx_1$, so

$$\alpha(x_1, x_2) \geq 1.$$

If $x_2 = x_1$, then x_2 is a fixed point of T and again the proof is finished. Now, we assume that $x_2 \neq x_1$.

Since $x_2 \in Tx_1$ and $\alpha(x_1, x_2) \geq 1$, by (9), we have

$$\begin{aligned}\omega_1(x_2, Tx_2) &\leq H_\omega(Tx_1, Tx_2)\\ &\leq k\omega_1(x_1, x_2) + L\omega_1(x_2, Tx_1) \leq k\omega_1(x_1, x_2) + L\omega_1(x_2, x_2)\\ &= k\omega_1(x_1, x_2) < r\omega_1(x_1, x_2).\end{aligned}$$

Iterating this process, we can define a sequence $\{x_n\}$ such that

$$\alpha(x_n, x_{n+1}) \geq 1,\ x_{n+1} \neq x_n,\ \ x_{n+1} \in Tx_n$$

and

$$\omega_1(x_n, x_{n+1}) \leq r\omega_1(x_{n-1}, x_n) \quad \forall n = 1, 2, 3, \ldots \tag{5}$$

Hence we obtain

$$\omega_1(x_n, x_{n+1}) \leq r^n\omega_1(x_0, x_1) \quad \forall n = 0, 1, 2, \ldots$$

By Lemma 2.11, we deduce that $\{x_n\}$ is $\omega-$Cauchy. Since M is $\omega-$complete, there exists $v \in M$ such that $\lim\limits_{n\to\infty} x_n = v$.

We shall prove that v is a fixed point of T. Since $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to v$, in view of the fact that (M, α) is $\omega-$regular, there exists a subsequence $\{x_{n(m)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(m)}, v) \geq 1$ for all $m \in \mathbb{N}$. We have for any $m \geq 0$

$$\begin{aligned}H_\omega(Tx_{n(m)}, Tv) &\leq k\omega_1(x_{n(m)}, v) + L\omega_1(x_n, Tx_{n(m)})\\ &\leq k\omega_1(x_{n(m)}, v) + L\omega_1(v, x_{n(m)+1}).\end{aligned}$$

Passing to limit as $m \to \infty$, we get $\lim\limits_{m\to\infty} H_\omega(Tx_{n(m)}, Tv) = 0$. Since $x_{n(m)+1} \in Tx_{n(m)}$, by Lemma 3.3, we conclude that $v \in Tv$, that is, v is a fixed point of T.

Now, passing to the case where f_T is lower semi-continuous, we have

$$\omega_1(v, Tv) = f_T(v) \leq \lim_{n\to\infty} \omega_1(x_n, Tx_n) \leq \lim_{n\to\infty} \omega_1(x_n, x_{n+1}) = 0.$$

Thus, $\omega_1(v, Tv) = 0$, and so $v \in Tv$.

□

We give the following illustrated examples.

Example 3.10. *Let $X = \mathbb{R}^+$, $M = [0, 1]$ and $\omega_\lambda(x, y) = \dfrac{|x - y|}{\lambda}$, $\forall x, y \in X$, $\forall \lambda > 0$. Mention that ω is a convex regular modular and satisfies the Δ_2-condition. Also, M is an $\omega-$complete subset of X_ω. Define a mapping $T : M \to CB(M)$ by*

$$Tx = \begin{cases}\{\frac{x}{3}\}, & 0 \leq x \leq \frac{1}{2}\\ [\frac{5}{6}, \frac{x+2}{3}], & \frac{1}{2} < x \leq 1.\end{cases}$$

Let $\alpha : M \times M \to [0, \infty)$ be defined by $\alpha(x, y) = 1$ for all $x, y \in M$.

Condition (2) of Theorem 3.9 is satisfied and (M, α) is $\omega-$regular.

We show that (9) of Theorem 3.9 is satisfied for all $x, y \in M$ with $k \in [\frac{1}{3}, 1]$ and for $L \geq 3$.

We consider the following cases:

<u>*Case*1</u> : $(x, y) \in [0, \frac{1}{2}] \times [\frac{1}{2}, 1]$. *In this case, condition (9) reduces*

$$\begin{aligned}H_\omega(Tx, Ty) &= \max\{\max\{\omega_1(\frac{x}{3}, [\frac{y+2}{3}]), \sup\{\omega_1(a, \frac{x}{3}) : \frac{5}{6} \leq a \leq \frac{y+2}{3}\}\}\\ &= \max\{|\frac{x}{3} - \frac{5}{6}|, |\frac{x}{3} - \frac{y}{3} - \frac{2}{3}|\} = |\frac{x}{3} - \frac{y}{3} - \frac{2}{3}|\\ &\leq k|x - y| + L|y - \frac{x}{3}|.\end{aligned}$$

We have $|\frac{x}{3} - \frac{y}{3} - \frac{2}{3}| \leq 1$ *and* $|y - \frac{x}{3}| \geq \frac{1}{3}$ *for all* $x, y \in [0, \frac{1}{2}] \times [\frac{1}{2}, 1]$. *In order the previous inequality holds, it suffices to take* $L \geq 3$ *and* $k \in [0, 1)$ *to be arbitrary.* <u>*Case2*</u> : $(x, y) \in (\frac{1}{2}, 1] \times [0, \frac{1}{2}]$. *In this case, condition (9) is reduced to*

$$H_\omega(Tx, Ty) = |\frac{x}{3} + \frac{2}{3} - \frac{y}{3}| \leq k|x - y| + L|\frac{5}{6} - y|.$$

We have $|\frac{x}{3} + \frac{2}{3} - \frac{y}{3}| \leq 1$ *and* $|\frac{5}{6} - y| \geq \frac{1}{3}$ *for all* $x, y \in (\frac{1}{2}, 1] \times [0, \frac{1}{2}]$. *Again, in order the previous inequality holds, it suffices to take* $L \geq 3$ *and* $k \in [0, 1)$ *to be arbitrary.* <u>*Case3*</u> : $x, y \in [0, \frac{1}{2}]$. *In this case, the condition (9) is reduced*

$$H_\omega(Tx, Ty) = \omega_1(\frac{x}{3}, \frac{y}{3}) = \frac{1}{3}|x - y| \leq k|x - y| + L|y - \frac{x}{3}|$$

and so condition (9) is satisfied with $k \in [\frac{1}{3}, 1)$ *and* $L \geq 0$.
<u>*Case4*</u> : $x, y \in (\frac{1}{2}, 1]$. *In this case, condition (9) becomes*

$$H_\omega(Tx, Ty) = H_\omega([\frac{5}{6}, \frac{x+2}{3}], [\frac{5}{6}, \frac{y+2}{3}]) = \frac{1}{3}|x - y| \leq k|x - y| + L|y - \frac{x}{3}|.$$

and so condition (9) is satisfied with $k \in [\frac{1}{3}, 1)$ *and* $L \geq 0$. *Now, by summarizing all cases, we conclude that the condition (9) is satisfied with* $k \in [\frac{1}{3}, 1)$ *and* $L \geq 3$.

Hence, all hypotheses of Theorem 3.9 are satisfied and T has fixed points. Note that $Fix(T) = \{0\} \cup [\frac{5}{6}, 1]$ *where* $Fix(T)$ *denotes the set of fixed points of T.*

On the other hand, the main result of Abdou and Khamsi [2] is not applicable. In fact, taking $x = 0$ *and* $y = 1$, *we have* $H_\omega(Tx, Ty) = 1 > k = k\omega_1(0, 1)$ *for each* $k \in [0, 1)$.

Example 3.11. *Let* $X = \mathbb{R}$, $M = [0, 2]$ *and* $\omega_\lambda(x, y) = \dfrac{|x - y|}{\lambda}$, $\forall x, y \in X$, $\forall \lambda > 0$. *Mention that* ω *is convex regular modular satisfying the* Δ_2-*condition. Also,* $M = [0, 2]$ *is an* $\omega-$*complete subset of* X_ω. *Define a mapping* $T : M \to CB(M)$ *by*

$$Tx = \begin{cases} \{0, \frac{1+x}{2}\}, & 0 \leq x \leq \frac{1}{2} \\ \{0, \frac{2-x}{2}\}, & \frac{1}{2} < x \leq 1 \\ [0, 1], & 1 < x \leq 2. \end{cases}$$

Let $\alpha : M \times M \to [0, \infty)$ *be defined by*

$$\alpha(x, y) = \begin{cases} 1, & 0 \leq x, y \leq 1 \\ 0, & \textit{otherwise}. \end{cases}$$

Condition (2) of Theorem 3.9 is satisfied with $x_0 = 0$ *and* $x_1 = \frac{1}{2}$. *Obviously,* (M, α) *is* $\omega-$*regular.*

We show that (9) of Theorem 3.9 is satisfied for all $x, y \in M$ *such that* $\alpha(x, y) \geq 1$ *with* $k \in [\frac{1}{2}, 1]$ *and for all* $L \geq 0$.

Let $x, y \in M$ *be such that* $\alpha(x, y) \geq 1$. *Then* $0 \leq x, y \leq 1$. *We consider the following cases:*
<u>*case1*</u> : $x, y \in [0, \frac{1}{2}]$. *We have*

$$\begin{aligned} H_\omega(Tx, Ty) &= \max\{\max\{\omega_1(0, Ty), \omega_1(\frac{1+x}{2}, Ty)\}, \max\{\omega_1(0, Tx), \omega_1(\frac{1+y}{2}, Tx)\}\} \\ &= \max\{\omega_1(\frac{1+x}{2}, Ty), \omega_1(\frac{1+y}{2}, Tx)\} \\ &= \max\{\min\{\frac{1+x}{2}, \frac{1}{2}|x - y|\}, \min\{\frac{1+y}{2}, \frac{1}{2}|x - y|\}\} \\ &= \frac{1}{2}|x - y|. \end{aligned}$$

<u>case2</u> : $x \in [0, \frac{1}{2}]$, $y =\in [\frac{1}{2}, 1]$. *We have*

$$\begin{aligned} H_\omega(Tx,Ty) &= \max\{\max\{\omega_1(0,Ty), \omega_1(\frac{1+x}{2}, Ty)\}, \max\{\omega_1(0,Tx), \omega_1(\frac{2-y}{2}, Tx)\}\} \\ &= \max\{\min\{\frac{1+x}{2}, \frac{1}{2}|x+y-1|\}, \min\{\frac{2-y}{2}, \frac{1}{2}|x+y-1|\}\} \\ &= \frac{1}{2}|x+y-1| \leq \frac{1}{2}|x-y|. \end{aligned}$$

<u>case3</u> : $x, y \in [\frac{1}{2}, 1]$. *We have*

$$\begin{aligned} H_\omega(Tx,Ty) &= \max\{\max\{\omega_1(0,Ty), \omega_1(\frac{1+x}{2}, Ty)\}, \max\{\omega_1(0,Tx), \omega_1(\frac{2-y}{2}, Tx)\}\} \\ &= \max\{\min\{\frac{2-x}{2}, \frac{1}{2}|x-y|\}, \min\{\frac{2-y}{2}, \frac{1}{2}|x-y|\}\} \\ &= \frac{1}{2}|x-y|. \end{aligned}$$

Thus, all hypotheses of Theorem 3.9 are satisfied and T has two fixed points, that is, Fix$(T) = \{0, \frac{3}{2}\}$.

We can derive the following results.

Corollary 3.12. *Let (X, ω) be a modular metric space. Assume that ω is a convex regular modular satisfying the Δ_2-condition. Let M be a nonempty $\omega-$complete subset of X_ω. Suppose that $T : M \to CB(M)$ is a given mapping. Suppose that there exist a function $\alpha : M \times M \to [0, \infty)$ and two constants $k \in [0, 1)$ and $L \geq 0$ such that for any $x, y \in M$, we have*

$$\alpha(x,y)H_\omega(Tx,Ty) \leq k\omega_1(x,y) + L\omega_1(y,Tx). \tag{6}$$

Suppose also that
(1) *T is $\alpha-$admissible;*
(2) *there exit $x_0 \in M$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;*
(3) *(M, α) is $\omega-$regular or f_T is lower semi-continuous.*
Then T is a MWP operator.

Proof. Let $x, y \in M$ be such that $\alpha(x, y) \geq 1$. Then if (7) holds, we have

$$H_\omega(Tx,Ty) \leq \alpha(x,y)H_\omega(x,y) \leq k\omega_1(x,y) + L\omega_1(y,Tx).$$

The proof is concluded from Theorem 3.9.
□

Corollary 3.13. *([2], Theorem 4.1) Let (X, ω) be a modular metric space. Assume that ω is a convex regular modular satisfying the Δ_2-condition. Let M be a nonempty $\omega-$complete subset of X_ω. Given $T : M \to CB(M)$. Suppose that there exists a constant $k \in [0, 1)$ such that for any $x, y \in M$, we have*

$$H_\omega(Tx,Ty) \leq k\omega_1(x,y). \tag{7}$$

Then T is a MWP operator.

Corollary 3.14. *Let (X, ω) be a modular metric space. Assume that ω is a convex regular modular satisfying the Δ_2-condition. Let M be a nonempty $\omega-$complete subset of X_ω. Given $T : M \to M$. Suppose that there exist two constants $k \in [0, 1)$ and $L \geq 0$ with $0 \leq k + L < 1$ such that for any $x, y \in M$, we have*

$$\omega_1(Tx,Ty) \leq k\omega_1(x,y) + L\omega_1(y,Tx). \tag{8}$$

Then T has a unique fixed point in X_ω.

Proof. We assume that there exist $x, y \in X$ such that $x = Tx$ and $y = Ty$ with $x \neq y$. We have

$$0 < \omega_1(x,y) = \omega_1(Tx,Ty) \leq k\omega_1(x,y) + L\omega_1(y,Tx) = (k+L)\omega_1(x,y) < \omega_1(x,y),$$

which is a contradiction. Hence $x = y$. □

3.3. Result III

In this part, we give a general class of *MWP* operators.

Definition 3.15. *Let (X, ω) be a modular metric space and M be a nonempty subset of X_ω. A multi-valued $T : M \to CB(M)$ is said an $\alpha - \omega-$generalized weak contraction if there exist a function $\alpha : M \times M \to [0, \infty)$, a function $\theta : [0, \infty) \to [0, 1)$ satisfying $\limsup_{r \to t^+} \theta(r) < 1$, for every $t \in [0, \infty)$ and a constant $L \geq 0$ such that for any $x, y \in M$ with $\alpha(x, y) \geq 1$, we have*

$$H_\omega(Tx, Ty) \leq \theta(\omega_1(x, y))\omega_1(x, y) + L\omega_1(y, Tx). \tag{9}$$

Theorem 3.16. *Let (X, ω) be a modular metric space. Assume that ω is a convex regular modular satisfying the Δ_2-condition. Let M be a nonempty $\omega-$complete subset of X_ω. Let $T : M \to CB(M)$ be a $\alpha - \omega-$generalized weak contraction.*

Suppose that
(1) *T is $\alpha-$admissible;*
(2) *there exit $x_0 \in M$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;*
(3) *(M, α) is $\omega-$regular or f_T is lower semi-continuous.*
Then T is a MWP operator.

Proof. The proof is inspired from Theorem 2.1 in [14].
By condition (2), there exist $x_0 \in M$ and $x_1 \in Tx_0$ such that

$$\alpha(x_0, x_1) \geq 1.$$

If $x_0 = x_1$, then x_1 is a fixed point of T and so the proof is finished. Now, we assume that $x_0 \neq x_1$. Select a positive integer n_1 such that

$$\theta^{n_1}(\omega_1(x_0, x_1)) < [1 - \theta(\omega_1(x_0, x_1))]\omega_1(x_0, x_1).$$

By Lemma 3.2, we can select $x_2 \in Tx_1$ such that

$$\omega_1(x_1, x_2) \leq H_\omega(Tx_0, Tx_1) + \theta^{n_1}(\omega_1(x_0, x_1)).$$

Since $\alpha(x_0, x_1) \geq 1$ then by (9), we have

$$\begin{aligned} \omega_1(x_1, x_2) &\leq H_\omega(Tx_0, Tx_1) + \theta^{n_1}(\omega_1(x_0, x_1)) \\ &\leq \theta(\omega_1(x_0, x_1))\omega_1(x_0, x_1) + L\omega_1(x_1, Tx_0) + \theta^{n_1}(\omega_1(x_0, x_1)) \\ &= \theta(\omega_1(x_0, x_1))\omega_1(x_0, x_1) + \theta^{n_1}(\omega_1(x_0, x_1)) \\ &< \omega_1(x_0, x_1). \end{aligned}$$

The mapping T is $\alpha-$admissible and $x_2 \in Tx_1$, so

$$\alpha(x_1, x_2) \geq 1.$$

If $x_2 = x_1$, then x_2 is a fixed point of T and again the proof is finished.
Now, we assume that $x_2 \neq x_1$. We choose a positive integer $n_2 > n_1$ such that

$$\theta^{n_2}(\omega_1(x_1, x_2)) < [1 - \theta(\omega_1(x_1, x_2))]\omega_1(x_1, x_2).$$

By Lemma 3.2, we may select $x_3 \in Tx_2$ such that

$$\omega_1(x_2, x_3) \leq H_\omega(Tx_1, Tx_2) + \theta^{n_2}(\omega_1(x_1, x_2)).$$

Since $\alpha(x_1, x_2) \geq 1$, by (9), we have

$$\begin{aligned} \omega_1(x_2, x_3) &\leq H_\omega(Tx_1, Tx_2) + \theta^{n_2}(\omega_1(x_1, x_2)) \\ &\leq \theta(\omega_1(x_1, x_2))\omega_1(x_1, x_2) + L\omega_1(x_2, Tx_1) + \theta^{n_2}(\omega_1(x_1, x_2)) \\ &= \theta(\omega_1(x_1, x_2))\omega_1(x_1, x_2) + \theta^{n_2}(\omega_1(x_1, x_2)) \\ &< \omega_1(x_1, x_2). \end{aligned}$$

Continuing in this process, we may select a positive integer n_k such that

$$\theta^{n_k}(\omega_1(x_{k-1}, x_k)) < [1 - \theta(\omega_1(x_{k-1}, x_k))]\omega_1(x_{k-1}, x_k)$$

where $x_{k-1} \neq x_k$ and $\alpha(x_{k-1}, x_k) \geq 1$. By Lemma 3.2, we may select $x_{k+1} \in Tx_k$ such that

$$\omega_1(x_k, x_{k+1}) \leq H_\omega(Tx_{k-1}, Tx_k) + \theta^{n_k}(\omega_1(x_{k-1}, x_k)).$$

In view of T is $\alpha-$admissible and $x_{k+1} \in Tx_k$, we have

$$\omega_1(x_k, x_{k+1}) \geq 1.$$

Since $\alpha(x_{k-1}, x_k) \geq 1$, by (9), we get

$$\omega_1(x_k, x_{k+1}) < \omega_1(x_{k-1}, x_k) \quad \text{for all } k = 1, 2, 3, \ldots$$

It follows that $\{a_k \equiv \omega_1(x_k, x_{k+1})\}$ is a nonincreasing sequence of nonnegative numbers. Then there exists a constant $c \geq 0$ such that $\lim_{k\to\infty} a_k = c$. By assumption, $\limsup_{t\to c^+} \theta(t) < 1$. This implies that there exists a positive integer N such that for $k \geq N$, we have $\theta(\omega_1(x_k, x_{k+1})) < h$ where $\limsup_{t\to c^+} \theta(t) < h < 1$.

Now, by (9), we have for $k \geq N$

$$\begin{aligned} a_k = \omega_1(x_k, x_{k+1}) &\leq \theta(a_k)a_{k-1} + \theta^{n_k}(a_{k-1}) \\ &\leq \theta(a_k)\theta(a_{k-1})a_{k-2} + \theta(a_k)\theta^{n_{k-1}}(a_{k-2}) + \theta^{n_k}(a_{k-1}) \\ &\vdots \\ &\leq \prod_{i=1}^{k} \theta(a_i)a_0 + \sum_{m=1}^{k-1} \prod_{i=m+1}^{k} \theta(a_i)\theta^{n_m}(a_m) + \theta^{n_k}(a_{k-1}) \\ &\leq \prod_{i=1}^{k} \theta(a_i)a_0 + \sum_{m=1}^{k-1} \prod_{i=\max\{k_0, m+1\}}^{k} \theta(a_i)\theta^{n_m}(a_m) + \theta^{n_k}(a_{k-1}) \\ &\leq h^{k-k_0+1} \prod_{i=1}^{k_0-1} \theta(a_i)a_0 + (k_0 - 1)h^{k-k_0+1} \sum_{m=1}^{k_0-1} \theta^{n_m}(a_m) \\ &+ \sum_{m=k_0}^{k-1} h^{k-m}\theta^{n_m}(a_m) + \theta^{n_k}(a_{k-1}) \\ &\leq C_1h^k + C_2h^k + C_3h^k + h^{n_k} \leq Ch^k, \end{aligned}$$

where C_1, C_2, C_3, C are appropriate constants.

For $k \geq N$, we have

$$\omega_1(x_k, x_{k+1}) \leq Ch^k.$$

Proceeding as in proof of Theorem 3.9, we may prove that x_n is $\omega-$Cauchy in M. By completeness of (M, ω), there exists a $u \in M$ such that $\lim_{n\to\infty} x_n = u$. We shall prove that u is a fixed point of T. Since $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, $x_n \to u$ and (M, α) is $\omega-$regular, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, u) \geq 1$ for all $k \in \mathbb{N}$.

We have for any $k \geq 0$

$$\begin{aligned} H_\omega(Tx_{n(k)}, Tu) &\leq \theta(\omega_1(x_{n(k)}, u))\omega_1(x_{n(k)}, u) + L\omega_1(x_n, Tx_{n(k)}) \\ &\leq \omega_1(x_{n(k)}, u) + L\omega_1(u, x_{n(k)+1}). \end{aligned}$$

Passing to limit as $k \to \infty$, we get $\lim_{k\to\infty} H_\omega(Tx_{n(k)}, Tu) = 0$.

Since $x_{n(k)+1} \in Tx_{n(k)}$, by Lemma 3.3, we conclude that $u \in Tu$, that is, u is a fixed point of T.

Now, passing to the case where f_T is lower semi-continuous, we get again $u \in Tu$. □

3.4. Result IV

In [15], Edelstein proved that if X is a complete $\varepsilon-$chainable metric space and $f : X \to X$ is an $(\varepsilon, \lambda)-$uniformly locally contractive mapping, then f has a fixed point. Nadler [21] extended this result to multi-valued mappings and he proved that if (X, d) is a complete $\varepsilon-$chainable metric space and $F : X \to 2^X$ is an $(\varepsilon, \lambda)-$uniformly locally contractive multi-valued mapping, then F has a fixed point. We generalize this result to $\alpha - \omega-$weak contractions in the setting of modular metric spaces.

Definition 3.17. *Let (X, ω) be a modular metric space. A nonempty subset M of X_ω is called to be finitely $\varepsilon-$chainable (where $\varepsilon > 0$ is fixed) if and only if given $x, y \in M$, there is an $\varepsilon-$chain from x to y (that is, a finite set of points $x_0, x_1, \ldots, x_p \in M$ such that $x_0 = x, x_p = y$, and $\omega_1(x_{i-1}, x_i) < \varepsilon$ for all $i = 1, 2, \ldots, p$).*

Definition 3.18. *Let (X, ω) be a modular metric space and M be a nonempty subset of X_ω. A multi-valued $T : M \to CB(M)$ is called to be an $(\varepsilon, k, L) - \omega-$uniformly locally weak contraction if there exist two constants $k \in [0, 1)$, $L \geq 0$ such that for any $x, y \in M$, we have*

$$H_\omega(Tx, Ty) \leq k\omega_1(x, y) + L\omega_1(y, Tx), \quad \textit{whenever} \quad \omega_1(x, y) < \varepsilon. \tag{10}$$

Theorem 3.19. *Let (X, ω) be a modular metric space. Assume that ω is a convex regular modular satisfying the Δ_2-type condition and the Fatou property. Let M be a nonempty $\omega-$complete and $\omega-$compact subset of X_ω, which is finitely $\varepsilon-$chainable, for some $\varepsilon > 0$. Let $T : M \to CB(M)$ be an $(\varepsilon, k, L) - \omega-$uniformly locally weak contraction.*

Then T is a MWP operator.

Proof. Let $(x, y) \in M \times M$. Take

$$\omega^\varepsilon(x, y) = \inf\{\Sigma_{i=1}^p \omega_1(x_{i-1}, x_i) : x_0 = x, x_1, \ldots, x_p = y \text{ is an } \varepsilon - \text{chain from } x \text{ to } y\}.$$

It is clear that $\omega^\varepsilon(x, y) < \infty$ for every $x, y \in M$ and $\omega^\varepsilon(x, y) = \omega_1(x, y)$ for all $x, y \in M$ such that $\omega_1(x, y) < \varepsilon$. Moreover, by definition of ω, we have for all $x, y \in M$

$$\omega_n(x, y) \leq \inf\{\omega_1(x, x_1) + \omega_1(x_1, x_2) + \ldots + \omega_1(x_{n-1}, y)\} = \omega^\varepsilon(x, y).$$

Let $z_0 \in M$ and $z_1 \in Tz_0$. Letting $x_0, x_1, \ldots, x_p$ be a $\varepsilon-$chain from z_0 to z_1. Since Tx_1 is $\omega-$compact, by Lemma 3.2, there exists $y_1 \in Tx_1$ such that

$$\omega_1(z_1, y_1) \leq H_\omega(Tz_0, Tx_1).$$

Similarly, there exists $y_2 \in Tx_2$ such that

$$\omega_1(y_1, y_2) \leq H_\omega(Tx_1, Tx_2).$$

Continuing in this fashion, we can find $y_3, \ldots, y_n$ such that $y_i \in Tx_i$ and

$$\begin{aligned} \omega_1(y_i, y_{i+1}) &\leq H_\omega(Tx_i, Tx_{i+1}) \leq k\omega_1(x_i, x_{i+1}) + L\omega_1(x_{i+1}, Tx_i) \\ &\leq k\omega_1(x_i, x_{i+1}) + L\omega_1(x_{i+1}, x_{i+1}) = k\omega_1(x_i, x_{i+1}). \end{aligned}$$

Obviously, $z_0, y_1, \ldots, y_p$ is a $\varepsilon-$chain from z_0 to y_p with $y_n \in Tz_1$. Take $z_2 = y_p$.

Since T is an $(\varepsilon, k, L) - \omega-$uniformly locally weak contraction, we obtain

$$\omega^\varepsilon(z_1, z_2) \leq k\omega^\varepsilon(z_0, z_1).$$

Proceeding as above, we may construct a sequence $\{z_n\}$ in M such that for all $n \geq 1$, we have $z_{n+1} \in Tz_n$ and

$$\omega^{\varepsilon}(z_n, z_{n+1}) \leq k\omega^{\varepsilon}(z_{n-1}, z_n).$$

By induction, we get

$$\omega^{\varepsilon}(z_n, z_{n+1}) \leq k^n \omega^{\varepsilon}(z_0, z_1).$$

Considering that ω satisfies the Δ_2-type condition, so there exists a constant $C > 0$ such that for all $n \geq 1$

$$\omega_1(z_n, z_{n+1}) \leq C\omega_n(z_n, z_{n+1}) \leq C\omega^{\varepsilon}(z_n, z_{n+1}) \leq Ck^n \omega^{\varepsilon}(z_0, z_1).$$

Using Lemma 2.11, we conclude that $\{z_n\}$ is $\omega-$Cauchy. By completeness of (M, ω), there exists a point $z \in M$ such that $\lim_{n\to\infty} z_n = z$.

Proceeding as the above, there exists $y_n \in Tz$ such that

$$\omega_1(x_{n+1}, y_n) \leq H_{\omega}(x_n, z).$$

The mapping T is an $(\varepsilon, k, L) - \omega-$uniformly locally weak contraction, so

$$\omega^{\varepsilon}(x_{n+1}, y_n) \leq k\omega^{\varepsilon}(x_n, z).$$

Since $\lim_{n\to\infty} z_n = z$, there exists $N \geq 1$ such that for $n \geq N$, we have $\omega_1(x_n, z) < \varepsilon$. It follows that $\omega^{\varepsilon}(x_n, z) = \omega_1(x_n, z)$.

We have for $n \geq N$,

$$\begin{aligned}
\omega_{p+1}(y_n, z) &\leq \omega_p(y_n, x_{n+1}) + \omega_1(x_{n+1}, z) \leq \omega^{\varepsilon}(x_{n+1}, y_n) + \omega_1(x_{n+1}, z) \\
&\leq \omega_1(x_{n+1}, z) + k\omega_1(x_n, z).
\end{aligned}$$

Passing to limit as $n \to \infty$, we get $\lim_{n\to\infty} \omega_{p+1}(y_n, z)$. Since ω satisfies the Δ_2-type condition and Tz is closed, we conclude that $z \in Tz$. □

4. Fixed point theory in ordered modular metric spaces

The study of fixed points in partially ordered sets was developed in [22, 23, 25]. In this section, we give some results of fixed point for multi-valued mappings in the concept of partially ordered modular metric spaces. We say that $x, y \in X_{\omega}$ are comparable if $x \preceq y$ or $y \preceq x$ holds. Moreover, for $A, B \subseteq X_{\omega}$, we have $A \preceq B$ whenever for each $x \in A$, there exists $y \in B$ such that $x \preceq y$. Now, we introduce the following concepts.

Definition 4.1. *Let (X, ω) be a modular metric space and M be a nonempty subset of X_{ω}. A multi-valued $T : M \to CB(M)$ is called to be weak continuous if the following condition holds: if $\{x_n\}$ is an $\omega-$convergent sequence in M to $x \in M$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} H_{\omega}(Tx_{n(k)}, Tx) = 0$.*

Definition 4.2. *Let (X, ω) be a modular metric space and M be a nonempty subset of X_{ω}. The pair $(M, \preceq)$ is said to be $\omega-$regular if the following condition holds: for any sequence $\{x_n\}$ in M with $Tx_n \preceq Tx_{n+1}$, for all $n \in \mathbb{N}$ and $x_n \to x \in M$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $Tx_{n(k)} \preceq Tx$, for all $k \in \mathbb{N}$.*

We have the following theorem.

Theorem 4.3. *Let $(X, \omega, \preceq)$ be a partially ordered modular metric space. Assume that ω is a convex regular modular satisfying the Δ_2-condition. Let M be a nonempty $\omega-$complete subset of X_{ω}. Suppose that $T : M \to CB(M)$ is a multi-valued mapping. Assume that there exist a function $\theta : [0, \infty) \to [0, 1)$ satisfying $\limsup_{r\to t^+} \theta(r) < 1$ for every $t \in [0, \infty)$ and a constant $L \geq 0$ such that*

$$H_{\omega}(Tx, Ty)) \leq \theta(\omega_1(x, y))\omega_1(x, y) + L\omega_1(y, Tx) \tag{11}$$

for all $x, y \in M$, with $Tx \preceq Ty$.

Suppose also that

1. *there exist* $x_0 \in M$ *and* $x_1 \in Tx_0$ *such that* $Tx_0 \preceq Tx_1$;

2. *for each* $x \in M$ *and* $y \in Tx$ *with* $Tx \preceq Ty$, *we have* $Ty \preceq Tz$ *for all* $z \in Ty$;

3. T *is weak continuous.*

Then T *is a MWP operator.*

Proof. Given $\alpha : M \times M \to [0,\infty)$ as

$$\alpha(x,y) = \begin{cases} 1 \text{ } if \text{ } Tx \preceq Ty \\ 0 \quad otherwise. \end{cases}$$

The multi-valued mapping T is $\alpha-$admissible. In fact, if $x \in M$ and $y \in Ty$ with $\alpha(x,y) \geq 1$, then $Tx \preceq Ty$. By condition (2), we have $Ty \preceq Tz$ for all $z \in Ty$, then $\alpha(y,z) = 1$. Also, by (11), T verifies (9). Proceeding as in proof of Theorem 3.16, we may construct an $\omega-$convergent sequence $\{x_n\}$ to $x \in M$ such that $x_{n+1} \in Tx_n$ for all $n \in \mathbb{N}$. Finally, by condition (3) and Lemma 3.3, we conclude that x is a fixed point of T. □

Theorem 4.3 remains true if the continuity hypothesis is replaced by the $\omega-$regularity of $(M, \preceq)$.
This statement is given as follows.

Theorem 4.4. *Let* $(X, \omega, \preceq)$ *be a partially ordered modular metric space. Assume that* ω *is a convex regular modular satisfying the* Δ_2-*condition. Let* M *be a nonempty* $\omega-$*complete subset of* X_ω. *Suppose that* $T : M \to CB(M)$ *is a multi-valued mapping. Assume that there exist a function* $\theta : [0,\infty) \to [0,1)$ *satisfying* $\limsup_{r\to t^+} \theta(r) < 1$ *for every* $t \in [0,\infty)$ *and a constant* $L \geq 0$ *such that*

$$H_\omega(Tx, Ty)) \leq \theta(\omega_1(x,y))\omega_1(x,y) + L\omega_1(y, Tx) \tag{12}$$

for all $x, y \in M$, *with* $Tx \preceq Ty$.
Suppose also that

1. *there exist* $x_0 \in M$ *and* $x_1 \in Tx_0$ *such that* $Tx_0 \preceq Tx_1$;

2. *for each* $x \in M$ *and* $y \in Tx$ *with* $Tx \preceq Ty$, *we have* $Ty \preceq Tz$ *for all* $z \in Ty$;

3. $(M, \preceq)$ *is* $\omega-$*regular.*

Then T *is a MWP operator.*

Proof. As in proof of the above theorem, we define the function $\alpha : M \times M \to [0,\infty)$ as follows

$$\alpha(x,y) = \begin{cases} 1 \text{ } if \text{ } Tx \preceq Ty \\ 0 \quad otherwise. \end{cases}$$

It is clear that the multi-valued mapping T is $\alpha-$admissible. Also, by (12), T verifies the contraction (9). Finally, by condition (3), the sequence (M, α) is $\omega-$regular. Thus, all hypotheses of Theorem 3.16 are satisfied and hence T has a fixed point. □

Authors contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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