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On $[p,q]$-Order of Growth of Solutions of Complex Linear Differential Equations near a Singular Point

Jianren Long[a], **Sangui Zeng**[a]

[a] *School of Mathematical Science, Guizhou Normal University, Guiyang, 550025, P.R. China.*

Abstract. We investigate the $[p,q]$-order of growth of solutions of the following complex linear differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0,$$

where $A_j(z)$ are analytic in $\overline{\mathbb{C}} - \{z_0\}$, $z_0 \in \mathbb{C}$. Some estimations of $[p,q]$-order of growth of solutions of the equation are obtained, which is generalization of previous results from Fettouch-Hamouda.

1. Introduction and Main Results

For the following complex linear differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0, \tag{1}$$

where $A_j(z)$ are analytic in a complex domain, $j = 0, 1, \ldots, k-1$, $k \geq 2$. The growth of solutions of (1) is very interesting topic after Wittich's work [16], the main tool is Nevanlinna theory of meromorphic functions which can be found in [6, 10, 18]. Many results have been obtained by many different researchers, for the case of complex plane $\mathbb{C}$, see, for example, [10–13, 17] and therein references, for the case of unit disc $\mathbb{D}$, see, for example [1, 2, 4, 7, 14] and therein references. Recently, Fettouch and Hamouda investigated the growth of solutions of equation (1) by using a new idea, in which the coefficients are analytic function except a finite singular point, more details can be found in [3, 5]. The concepts of $[p,q]$-order and $[p,q]$-type of entire functions was introduced by Juneja et al. in [8, 9], more recently, some related development was founded by Srivastava et al., see [15] for more details. It inspired us to investigate the $[p,q]$-order of solutions of equation (1). We firstly recall some related notations for our results. Let $f(z)$ be meromorphic in $\overline{\mathbb{C}} - \{z_0\}$, where $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $z_0 \in \mathbb{C}$. Define the counting function of $f(z)$ near z_0 by

$$N_{z_0}(r, f) = -\int_{\infty}^{r} \frac{n(t,f) - n(\infty, f)}{t} dt - n(\infty, f)\log r,$$

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Email addresses: longjianren2004@163.com (Jianren Long), 1033325326@qq.com (Sangui Zeng)

where $n(t,f)$ denotes the number of poles of $f(z)$ in the region $\{z \in \mathbb{C} : t \le |z-z_0|\} \cup \{\infty\}$ counting its multiplicities; the proximity function near z_0 is defined by

$$m_{z_0}(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(z_0 - re^{i\varphi})| d\varphi.$$

The characteristic function of $f(z)$ near z_0 is defined by

$$T_{z_0}(r,f) = m_{z_0}(r,f) + N_{z_0}(r,f).$$

Similarly to the case of complex plane, for all $R \in (0,\infty)$, we define $\exp_1 R = e^R$ and $\exp_{p+1} R = \exp(\exp_p R)$, $\log_1 R = \log R$ and $\log_{p+1} R = \log(\log_p R)$. Let $f(z)$ be meromorphic in $\overline{\mathbb{C}} - \{z_0\}$, p and q be two integers with $p \ge q \ge 1$. The $[p,q]$-order of $f(z)$ near z_0 is defined by

$$\sigma_{[p,q],T}(f,z_0) = \limsup_{r\to 0} \frac{\log_p^+ T_{z_0}(r,f)}{\log_q \frac{1}{r}}. \tag{2}$$

For an analytic function $f(z)$ in $\overline{\mathbb{C}} - \{z_0\}$, the $[p,q]$-order of $f(z)$ is defined by

$$\sigma_{[p,q],M}(f,z_0) = \limsup_{r\to 0} \frac{\log_{p+1}^+ M_{z_0}(r,f)}{\log_q \frac{1}{r}}, \tag{3}$$

where $M_{z_0}(r,f) = \max\{|f(z)| : |z-z_0| = r\}$.

Remark 1.1. *Suppose that $f(z)$ is an analytic function in $\overline{\mathbb{C}} - \{z_0\}$. Then we get $\sigma_{[p,q],M}(f,z_0) = \sigma_{[p,q],T}(f,z_0)$ by using [3, Lemma 2.2]. Therefore, in the sequel, we denote $\sigma_{[p,q]}(f,z_0) = \sigma_{[p,q],M}(f,z_0) = \sigma_{[p,q],T}(f,z_0)$.*

Let $f(z) = e^{\frac{1}{(z_0-z)^n}}$, where n is a positive integer. Obviously, $f(z)$ is analytic in $\overline{\mathbb{C}} - \{z_0\}$. We get $M_{z_0}(r,f) = e^{\frac{1}{r^n}}$ and $T_{z_0}(r,f) = \frac{n}{\pi r^n}$. This shows that $\sigma_{[1,1],T}(f,z_0) = \sigma_{[1,1],M}(f,z_0) = n$.

We define $[p,q]$-type near z_0 by using similar reason as in the case of complex plane. Let $f(z)$ be an analytic in $\overline{\mathbb{C}} - \{z_0\}$ with $\sigma_{[p,q]}(f,z_0) = \sigma \in (0,\infty)$. Then its $[p,q]$-type is defined by

$$\tau_{[p,q],M}(f,z_0) = \limsup_{r\to 0} \frac{\log_p^+ M_{z_0}(r,f)}{(\log_{q-1} \frac{1}{r})^\sigma}. \tag{4}$$

Here, we study the growth of solutions of (1) by using the concepts of $[p,q]$-order and $[p,q]$-type.

Theorem 1.2. *Let $A_0(z), A_1(z), \ldots, A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$ satisfying $\max\{\sigma_{[p,q]}(A_j,z_0) : j \ne 0\} < \sigma_{[p,q]}(A_0,z_0) < \infty$. Then, every nontrivial solution $f(z)$ of (1), that is analytic in $\overline{\mathbb{C}} - \{z_0\}$, satisfies $\sigma_{[p+1,q]}(f,z_0) = \sigma_{[p,q]}(A_0,z_0)$.*

The following example shows that the Theorem 1.2 is sharp. $f(z) = e^{e^{\frac{1}{(z-z_0)^n}}}$ solves the following equation

$$f'' + A_1(z)f' + A_0(z)f = 0, \tag{5}$$

where $A_1(z) = -\frac{n}{(z_0-z)^{n+1}} - \frac{n+1}{z_0-z}$, $A_0(z) = \frac{n^2}{(z_0-z)^{2n+2}} e^{\frac{2}{(z_0-z)^n}}$. Then $\sigma_{[1,1]}(A_1,z_0) = 0 < n = \sigma_{[1,1]}(A_0,z_0)$ and $\sigma_{[2,1]}(f,z_0) = n$.

In Theorem 1.2, we know that the coefficient $A_0(z)$ is a dominant coefficient in terms of $[p,q]$-order. The following result shows that the coefficient $A_0(z)$ is a dominant coefficient in terms of $[p,q]$-type.

Theorem 1.3. *Let $A_0(z), A_1(z), \ldots, A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$ satisfying the following conditions:*

(i) $\max\{\sigma_{[p,q]}(A_j, z_0) : j \neq 0\} \leq \sigma_{[p,q]}(A_0, z_0) < \infty$;
(ii) $\max\{\tau_{[p,q],M}(A_j, z_0) : \sigma_{[p,q]}(A_j, z_0) = \sigma_{[p,q]}(A_0, z_0)\} < \tau_{[p,q],M}(A_0, z_0)$.

Then, every nontrivial solution $f(z)$ of (1), *that is analytic in $\overline{\mathbb{C}} - \{z_0\}$, satisfies $\sigma_{[p+1,q]}(f, z_0) = \sigma_{[p,q]}(A_0, z_0)$.*

We get two results above concerning the growth of solutions of equation (1) when the coefficient $A_0(z)$ is a dominant coefficient. A natural question is: what can we say about the growth of solutions of equation (1) when the coefficient $A_s(z)$ is a dominant coefficient, where $s \neq 0$. Next we study also this question, and prove the following result.

Theorem 1.4. *Let $A_0(z), A_1(z), \ldots, A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$ satisfying* $\max\{\sigma_{[p,q]}(A_j, z_0) : j \neq s\} < \sigma_{[p,q]}(A_s, z_0) < \infty$. *Then, every nontrivial solution $f(z)$ of* (1), *that is analytic in $\overline{\mathbb{C}} - \{z_0\}$, satisfies $\sigma_{[p+1,q]}(f, z_0) \leq \sigma_{[p,q]}(A_s, z_0) \leq \sigma_{[p,q]}(f, z_0)$.*

In our results, we suppose always that $f(z)$ is analytic in $\overline{\mathbb{C}} - \{z_0\}$, the following example shows there exists a solution $f(z)$ of equation (1) such that $f(z)$ is not analytic in $\overline{\mathbb{C}} - \{z_0\}$ provided all coefficients $A_j(z)$ of (1) are analytic in $\overline{\mathbb{C}} - \{z_0\}$. We consider the equation (5) again, where $A_1(z) = e^{\frac{1}{(z_0-z)}}, A_0(z) = \frac{1}{(z_0-z)} e^{\frac{1}{(z_0-z)}}$. The function $f(z) = z_0 - z$ solves (5), and $f(z)$ is not analytic in $\overline{\mathbb{C}} - \{z_0\}$.

2. Preliminary results

In order to prove our results, the following preliminary results are needed. Firstly, we denote the logarithmic measure of a set $E \subset (0, 1)$ by $m_l(E) = \int_E \frac{1}{t} dt$, denote the central index of an analytic function $g(z)$ in $\mathbb{C}$ by $V(r, g)$ which can be found in [10, p. 50], and denote the central index of an analytic function $f(z)$ in $\overline{\mathbb{C}} - \{z_0\}$ by $V_{z_0}(r, f)$ which can be found in [5, p. 996].

We get the first lemma which the [p, q]-order of an analytic function $f(z)$ in $\overline{\mathbb{C}} - \{z_0\}$ is described by its central index $V_{z_0}(r, f)$.

Lemma 2.1. *Let $f(z)$ be a nonconstant analytic function in $\overline{\mathbb{C}} - \{z_0\}$. Then*

$$\limsup_{r \to 0} \frac{\log_p^+ V_{z_0}(r, f)}{\log_q \frac{1}{r}} = \sigma_{[p,q]}(f, z_0).$$

Proof. Set $g(\omega) = f(z_0 - \frac{1}{\omega})$ and $\sigma_{[p,q]}(g) = \limsup_{R \to \infty} \frac{\log_{p+1}^+ M(R, g)}{\log_q R}$. By [5, Remark 7], then g is an entire function in $\mathbb{C}$ and

$$V_{z_0}(r, f) = V(R, g), \quad R = \frac{1}{r}. \tag{6}$$

From [8, p. 57], we get

$$\limsup_{R \to \infty} \frac{\log_p^+ V(R, g)}{\log_q R} = \sigma_{[p,q]}(g). \tag{7}$$

It follows from [3, Lemma 2.2] that $T(R, g) = T_{z_0}(r, f)$, and then

$$\sigma_{[p,q]}(g) = \sigma_{[p,q]}(f, z_0).$$

Combining (6) and (7), we get that the conclusion holds. □

The following two lemmas plays an important role in the proof of our results.

Lemma 2.2. *Let $f(z)$ be a nonconstant analytic function in $\overline{\mathbb{C}} - \{z_0\}$ with $\sigma_{[p,q]}(f, z_0) = \sigma$. Then there exists a set $E \subset (0, 1)$ having infinite logarithmic measure such that for all $|z - z_0| = r \in E$,*

$$\lim_{r\to 0} \frac{\log_{p+1} M_{z_0}(r, f)}{\log_q \frac{1}{r}} = \sigma.$$

Proof. By (3), then there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to 0 satisfying $r_{n+1} < \frac{n}{n+1} r_n$ and $\lim_{n\to\infty} \frac{\log_{p+1} M_{z_0}(r_n,f)}{\log_q \frac{1}{r_n}} = \sigma$. Therefore, there exists an $n_0 \in N^+$ such that for all $n > n_0$ and for any $r \in [\frac{n}{n+1} r_n, r_n]$, we get

$$\frac{\log_{p+1} M_{z_0}(r_n, f)}{\log_q \frac{1}{\frac{n}{n+1} r_n}} \le \frac{\log_{p+1} M_{z_0}(r, f)}{\log_q \frac{1}{r}} \le \frac{\log_{p+1} M_{z_0}(\frac{n}{n+1} r_n, f)}{\log_q \frac{1}{r_n}}.$$

Since $\lim_{n\to\infty} \frac{\log_{p+1} M_{z_0}(r_n,f)}{\log_q \frac{1}{\frac{n}{n+1} r_n}} = \sigma,\quad \lim_{n\to\infty} \frac{\log_{p+1} M_{z_0}(\frac{n}{n+1} r_n,f)}{\log_q \frac{1}{r_n}} = \sigma$, then for any $r \in \left[\frac{n}{n+1} r_n, r_n\right]$, we get

$$\lim_{r\to 0} \frac{\log_{p+1} M_{z_0}(r, f)}{\log_q \frac{1}{r}} = \sigma.$$

Set $E = \bigcup_{n=n_0}^{\infty} \left[\frac{n}{n+1} r_n, r_n\right]$. Then

$$\mathrm{m_l}(E) = \int_E \frac{1}{t} dt = \sum_{n=n_0}^{\infty} \int_{\frac{n}{n+1} r_n}^{r_n} \frac{1}{t} dt = \sum_{n=n_0}^{\infty} \log\left(1 + \frac{1}{n}\right) = \infty.$$

□

Remark 2.3. *If $f(z)$ is a nonconstant meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$ with $\sigma_{[p,q]}(f, z_0) = \sigma$, then by* (2) *and using similar way as in the proof of Lemma 2.2, we can easily get that there exists a set $E \subset (0, 1)$ having infinite logarithmic measure such that for all $|z - z_0| = r \in E$,*

$$\lim_{r\to 0} \frac{\log_p T_{z_0}(r, f)}{\log_q \frac{1}{r}} = \sigma.$$

Lemma 2.4. *Let $f(z)$ be a nonconstant analytic function in $\overline{\mathbb{C}} - \{z_0\}$ with $\sigma_{[p,q]}(f, z_0) = \sigma \in (0, \infty)$ and $\tau_{[p,q],M}(f, z_0) = \tau \in (0, \infty)$. Then, for any giving $\beta \in (0, \tau)$, there exists a set $F \subset (0, 1)$ of infinite logarithmic measure such that for all $|z - z_0| = r \in F$,*

$$\log M_{z_0}(r, f) \ge \exp_{p-1}\left(\beta(\log_{q-1} \frac{1}{r})^{\sigma}\right).$$

Proof. By using similar method as in the proof of Lemma 2.2, the conclusion is hold. Here we omit the details. □

In order to prove Lemma 2.6, the following Lemma 2.5 is needed.

Lemma 2.5. *Let $g : (0, 1) \to R$, $h : (0, 1) \to R$ be monotone decreasing functions such that $g(r) \ge h(r)$ possibly outside an exceptional set $E \subset (0, 1)$ that has finite logarithmic measure ($\int_E \frac{1}{t} dt < \infty$). Then for any given $\beta > 1$, there exists a constant $0 < r_0 < 1$, such that for all $r \in (0, r_0)$, we have $g(r^\beta) \ge h(r)$.*

Proof. Set $\alpha = \int_E \frac{1}{t} dt < \infty$, and choose $r_0 = \exp(\frac{\alpha}{1-\beta}) \in (0, 1)$. For any $0 < r < r_0$, the interval $I_r = [r^\beta, r]$ meets the complement of E, since

$$\int_{I_r} \frac{1}{t} dt = \int_{r^\beta}^{r} \frac{1}{t} dt = \log r - \log r^\beta = (1 - \beta) \log r > (1 - \beta) \log r_0 = \alpha.$$

Thus, by the monotonicity of g and h, there exists $t \in I_r$, we have

$$g(r^\beta) \geq g(t) \geq h(t) \geq h(r).$$

□

Now, we get the upper bound of the growth of solutions of equation (1).

Lemma 2.6. *Let $A_j(z)$ be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$ satisfying $\sigma_{[p,q]}(A_j, z_0) \leq \sigma < \infty$, $j = 0, 1, \ldots, k-1$. If $f(z)$ is a solution of* (1) *that is analytic in $\overline{\mathbb{C}} - \{z_0\}$, then $\sigma_{[p+1,q]}(f, z_0) \leq \sigma$.*

Proof. By (1), we have

$$\left|\frac{f^{(k)}}{f}\right| \leq |A_{k-1}(z)| \cdot \left|\frac{f^{(k-1)}}{f}\right| + \cdots + |A_s(z)| \cdot \left|\frac{f^{(s)}}{f}\right| + \cdots + |A_0(z)|. \tag{8}$$

Since $\sigma_{[p,q]}(A_j, z_0) \leq \sigma$ $(j = 0, \ldots, k-1)$, then for any given $\varepsilon > 0$, there exists $r_0 \in (0,1)$ such that for all $|z_0 - z| = r \in (0, r_0)$,

$$|A_j(z)| < \exp_p\left(\log_{q-1}\frac{1}{r}\right)^{\sigma+\varepsilon}, \quad j = 0, 1, \ldots, k-1. \tag{9}$$

By [5, Theorem 8], there exists a set $E \subset (0,1)$ that has finite logarithmic measure, such that for all $j = 0, 1, \ldots, k$ and $r \notin E$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| = |1 + o(1)| \cdot \left(\frac{V_{z_0}(r, f)}{r}\right)^j, \quad r \to 0, \tag{10}$$

where z is a point in the circle $|z_0 - z| = r$ that satisfies $|f(z)| = \max\limits_{|z_0 - z| = r} |f(z)|$.

Combining (8), (9) and (10), for all $|z - z_0| = r \in (0, r_0) \backslash E$ and $|f(z)| = M_{z_0}(r, f)$, we get

$$V_{z_0}(r) \leq kr \exp_p\left(\log_{q-1}\frac{1}{r}\right)^{\sigma+\varepsilon} |1 + o(1)|. \tag{11}$$

It follows from Lemma 2.1, Lemma 2.5 and (11), we get this conclusion. □

We need the following Lammas 2.7-2.8 to prove Theorem 1.4.

Lemma 2.7. *Let $f(z)$ be a nonconstant meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$. Then the following statements hold.*

(i) $T_{z_0}(r, \frac{1}{f}) = T_{z_0}(r, f) + O(1)$;

(ii) $T_{z_0}(r, f') < O\left(T_{z_0}(r, f) + \log\frac{1}{r}\right)$, $\quad r \in (0, r_0] \backslash E$, *where* $E \subset (0, r_0]$ *with* $\mathrm{m_l}(E) < \infty$.

Proof. (i) Set $\frac{1}{g(\omega)} = \frac{1}{f\left(z_0 - \frac{1}{\omega}\right)}$, by using similar reason as in the proof of [3, Lemma 2.2], we get $T\left(R, \frac{1}{g}\right) = T_{z_0}\left(\frac{1}{R}, \frac{1}{f}\right)$, combining [3, Lemma 2.2] and the first main theorem in Nevanlinna theory, we get

$$T_{z_0}\left(r, \frac{1}{f}\right) = T_{z_0}(r, f) + O(1), \quad r = \frac{1}{R}.$$

(ii) Since $T_{z_0}(r, f') = m_{z_0}(r, f') + N_{z_0}(r, f') \leq 2T_{z_0}(r, f) + m_{z_0}\left(r, \frac{f'}{f}\right)$. It follows from this and [3, Lemma 2.4] that there exists a set $E \subset (0, r_0]$ that has finite logarithmic measure such that for all $|z_0 - z| = r \in (0, r_0] \backslash E$,

$$T_{z_0}(r, f') \leq O\left(T_{z_0}(r, f) + \log\frac{1}{r}\right).$$

□

Lemma 2.8. *Let f_1 be an analytic function in $\overline{\mathbb{C}} - \{z_0\}$ satisfying $\sigma_{[p,q]}(f_1, z_0) = \sigma_1 > 0$ and f_2 be an analytic function in $\overline{\mathbb{C}} - \{z_0\}$ satisfying $\sigma_{[p,q]}(f_2, z_0) = \sigma_2 < \infty$. If $\sigma_2 < \sigma_1$, then there exists a set $E \subset (0, 1)$ having infinite logarithmic measure such that for all $|z - z_0| = r \in E$, $\lim\limits_{r\to 0} \frac{T_{z_0}(r,f_2)}{T_{z_0}(r,f_1)} = 0$.*

Proof. By (2), for any given $\varepsilon \in (0, \frac{\sigma_1-\sigma_2}{2})$, there exists $r_0 \in (0, 1)$ such that for all $|z - z_0| = r \in (0, r_0)$,

$$T_{z_0}(r, f_2) \le \exp_p((\sigma_2 + \varepsilon) \log_q \frac{1}{r}). \tag{12}$$

By Remark 2.3, there exists a set $E \subset (0, r_0)$ of infinite logarithmic measure such that for all $|z - z_0| = r \in E$,

$$T_{z_0}(r, f_1) \ge \exp_p((\sigma_1 - \varepsilon) \log_q \frac{1}{r}). \tag{13}$$

It follows from (12) and (13) that for all $r \in E \bigcap (0, r_0)$, we get

$$0 \le \frac{T_{z_0}(r, f_2)}{T_{z_0}(r, f_1)} \le \frac{\exp_p((\sigma_2 + \varepsilon) \log_q \frac{1}{r})}{\exp_p((\sigma_1 - \varepsilon) \log_q \frac{1}{r})} \to 0, \ as\ r \to 0.$$

This implies the conclusion holds. □

3. Proof of Theorem 1.2

Set $\sigma_{[p,q]}(A_0, z_0) = \sigma$. Let α and β be constants with $\max\{\sigma_{[p,q]}(A_j, z_0) : j \ne 0\} < \beta < \alpha < \sigma$. By (3), for any given $\varepsilon \in \left(0, \min(\frac{\alpha-\beta}{2}, \frac{\sigma-\alpha}{2})\right)$, there exists r_1 such that for all $|z_0 - z| = r \in (0, r_1)$,

$$|A_j(z)| < \exp_p\left(\log_{q-1} \frac{1}{r}\right)^{\beta+\varepsilon}, \quad j = 1, 2, \ldots, k-1. \tag{14}$$

Applying Lemma 2.2 to $A_0(z)$, for ε given above, there exist a r_2 and a set $E_1 \subset (0, 1)$ with infinite logarithmic measure such that for all $|z - z_0| = r \in (0, r_2] \cap E_1$ and $|A_0(z)| = M_{z_0}(r, A_0)$,

$$|A_0(z)| > \exp_p\left(\log_{q-1} \frac{1}{r}\right)^{\sigma-\varepsilon}. \tag{15}$$

Set $r_0 = \min(r_1, r_2)$, $\gamma > 1$ is constant. By [3, Lemma 2.4], there exists a set $E_2 \subset (0, r_0]$ that has finite logarithmic measure, and a constant λ that depends on γ such that for $|z - z_0| = r \in (0, r_0]\backslash E_2$,

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \le \lambda \left(\frac{1}{r^2} T_{z_0}\left(\frac{r}{\gamma}, f\right) \log T_{z_0}\left(\frac{r}{\gamma}, f\right)\right)^j, \quad j = 0, 1, \ldots, k. \tag{16}$$

By (1), we get

$$|A_0(z)| \le \left|\frac{f^{(k)}}{f}\right| + \cdots + \left|A_j(z)\right| \cdot \left|\frac{f^{(j)}}{f}\right| + \cdots + |A_1(z)| \cdot \left|\frac{f'}{f}\right|. \tag{17}$$

Set $E_0 = (0, r_0] \cap E_1\backslash E_2$, obviously E_0 has infinite logarithmic measure. Combining (14), (15), (16) and (17), for $|z - z_0| = r \in E_0$,

$$\exp_p\left(\log_{q-1} \frac{1}{r}\right)^{\sigma-\varepsilon} \le \lambda \left(\frac{1}{r} T_{z_0}\left(\frac{r}{\gamma}, f\right)\right)^{2k} \exp_p\left(\log_{q-1} \frac{1}{r}\right)^{\beta+\varepsilon}.$$

This implies that $\sigma_{[p+1,q]}(f, z_0) \ge \sigma$. It follows from this and Lemma 2.6 that $\sigma_{[p+1,q]}(f, z_0) = \sigma_{[p,q]}(A_0, z_0)$.

4. Proof of Theorem 1.3

Set $\sigma_{[p,q]}(A_0, z_0) = \sigma$, $\tau_{[p,q],M}(A_0, z_0) = \tau$. Let β_1 and β_2 be constants with $\max\{\tau_{[p,q],M}(A_j, z_0) : \sigma_{[p,q]}(A_j, z_0) = \sigma_{[p,q]}(A_0, z_0)\} < \beta_1 < \beta_2 < \tau$, $\gamma > 1$ is constant. By (4), there exists $r_0 \in (0, 1)$ such that for all $|z - z_0| = r \in (0, r_0)$,

$$|A_j(z)| \leq \exp_p\left(\beta_1 \left(\log_{q-1} \frac{1}{r}\right)^{\sigma}\right), \quad j = 1, 2, \ldots, k. \tag{18}$$

By [3, Lemma 2.4], there exists a set $E_1 \subset (0, r_0]$ having finite logarithmic measure and a constant $\lambda > 0$ that depends only on γ such that for all $|z - z_0| = r \notin E_1$, we have (16) holds. By Lemma 2.4, there exists a set $E_2 \subset (0, 1)$ of infinite logarithmic measure such that for all $|z - z_0| = r \in E_2$,

$$M_{z_0}(r, A_0) \geq \exp_p\left(\beta_2 \left(\log_{q-1} \frac{1}{r}\right)^{\sigma}\right). \tag{19}$$

Set $E_0 = E_2 \backslash E_1$, obviously, $\mathrm{m_l}(E_0) = \infty$. Applying (16), (18), (19) to (17), for all z satisfying $|z - z_0| = r \in E_0$ and $|A_0(z)| = M_{z_0}(r, f)$, we get

$$\exp_p\left(\beta_2 \left(\log_{q-1} \frac{1}{r}\right)^{\sigma}\right) \leq k\lambda \left(\frac{1}{r} T_{z_0}(\alpha r, f)\right)^{2k} \exp_p\left(\beta_1 \left(\log_{q-1} \frac{1}{r}\right)^{\sigma}\right).$$

This implies that $\sigma_{[p+1,q]}(f, z_0) \geq \sigma$, and by Lemma 2.6, the conclusion holds.

5. Proof of Theorem 1.4

By (1), we get

$$m_{z_0}(r, A_s) \leq \sum_{j \neq s} m_{z_0}\left(r, \frac{f^{(j)}}{f^{(s)}}\right) + \sum_{j \neq s} m_{z_0}(r, A_j) + \log k. \tag{20}$$

By Lemma 2.7, for constant $r_0 \in (0, 1)$, there is a set $E_1 \subset (0, r_0]$ that has finite logarithmic measure such that for all $|z_0 - z| = r \in (0, r_0] \backslash E_1$,

$$\sum_{j \neq s} m_{z_0}\left(r, \frac{f^{(j)}}{f^{(s)}}\right) \leq O\left\{T_{z_0}(r, f) + \log \frac{1}{r}\right\}. \tag{21}$$

By Lemma 2.8, for any given $\varepsilon \in \left(0, \frac{1}{2(k-1)}\right)$, there exists a set $E_2 \subset (0, r_0)$ with infinite logarithmic measure such that for sufficiently small $|z - z_0| = r \in E_2$,

$$m_{z_0}(r, A_j) \leq \varepsilon \cdot m_{z_0}(r, A_s), \quad j \neq s. \tag{22}$$

Combining (20), (21) and (22), for all $|z_0 - z| = r \in E_2 \backslash E_1$,

$$\frac{1}{2} m_{z_0}(r, A_s) \leq O\{T_{z_0}(r, f) + \log \frac{1}{r}\} + O(1).$$

This implies that

$$\sigma_{[p,q]}(A_s, z_0) \leq \sigma_{[p,q]}(f, z_0).$$

Combining Lemma 2.6, the conclusion can be deduced.

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