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Relative Cyclic Subgroup Commutativity Degrees of Finite Groups

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Abstract. In this paper we introduce and study the relative cyclic subgroup commutativity degrees of a finite group. We show that there is a finite group with n such degrees for all $n \in \mathbb{N}^* \setminus \{2\}$ and we indicate some classes of finite groups with few relative cyclic subgroup commutativity degrees. In the end, we prove that the set containing all relative cyclic subgroup commutativity degrees of finite groups is dense in $[0, 1]$.

1. Introduction

The starting points of this paper are some probabilistic aspects associated with finite groups that were introduced in [24] and [25]. More exactly, for a finite group G and a subgroup H of G the quantities

$$sd(G) = \frac{1}{|L(G)|^2} |\{(H, K) \in L(G)^2 \mid HK = KH\}|$$

and

$$sd(H, G) = \frac{1}{|L(H)||L(G)|} |\{(H_1, G_1) \in L(H) \times L(G) \mid H_1G_1 = G_1H_1\}|$$

are called the subgroup commutativity degree of G and the relative subgroup commutativity degree of the subgroup H of G, respectively. Above, $L(G)$ denotes the subgroup lattice of G. Also, in [25], the author suggests (see Problem 3.6) to study the restriction of the function

$$sd : L(G) \times L(G) \longrightarrow [0, 1]$$

to $L_1(G) \times L_1(G)$, where $L_1(G)$ denotes the poset of cyclic subgroups of G. Our aim is to provide some answers to this open problem. The idea of working only with cyclic subgroups instead of taking into account all subgroups of a finite group G was applied in [28], where the cyclic subgroup commutativity degree of G, defined as

$$csd(G) = \frac{1}{|L_1(G)|^2} |\{(H, K) \in L_1(G)^2 \mid HK = KH\}|,$$

was studied. This concept led to some fruitful results and this is an additional argument to introduce and study the restriction of $sd : L(G) \times L(G) \longrightarrow [0, 1]$ to $L_1(G) \times L_1(G)$.

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The paper is organized as follows. We introduce the relative cyclic subgroup commutativity degree of a subgroup H of G and we point out some of its general properties in Section 2. In Section 3, we study the existence of a finite group with n (relative) cyclic subgroup commutativity degrees, where n is a positive integer. Also, we indicate some classes of finite groups with few relative cyclic subgroup commutativity degrees. In Section 4, we discuss about the density of the set containing all relative cyclic subgroup commutativity degrees of finite groups. In the last Section we indicate some further research directions.

For more examples of probabilistic aspects associated with finite groups, such as the (relative) commutativity degree, subgroup S-commutativity degree and strong subgroup commutativity degree, and information on their related properties and applications, we refer the reader to [3, 5], [8]-[12], [16]-[18], [20]. Also, one may be interested in the connection between the (cyclic) subgroup commutativity degree and the (cyclic) factorization number of finite groups. This aspect was studied in [6, 19, 30]. Most of our notation is standard and will usually not be repeated here. Elementary notions and results on groups can be found in [22]. For subgroup lattice concepts we refer the reader to [21, 23, 26].

2. General properties of relative cyclic subgroup commutativity degrees

Let G be a finite group and H be one of its subgroups. We define the relative cyclic subgroup commutativity degree of H to be the quantity

$$csd(H,G)=\frac{1}{|L_1(H)||L_1(G)|}|\{(H_1,G_1)\in L_1(H)\times L_1(G)\mid H_1G_1=G_1H_1\}|.$$

It is obvious that

$$csd(G,G)=csd(G)\ \text{ and }\ 0<csd(H,G)\leq 1,\ \forall\, H\in L(G).$$

Also, for a subgroup H of G we have $csd(H,G)=1$ if and only if all cyclic subgroups of H permute with all cyclic subgroups of G. But the permutability of all cyclic subgroups of H with all cyclic subgroups of G is equivalent with the permutability of all subgroups of H with all subgroups of G (see Consequence (9) on the page 202 of [21]). Therefore,

$$csd(H,G)=1\Longleftrightarrow sd(H,G)=1.$$

We will denote by $N(G)$ the normal subgroup lattice of G and by $[H]$ the conjugacy class of a subgroup H of G. We can express the relative cyclic subgroup commutativity degree of a subgroup H of G as

$$csd(H,G)=\frac{1}{|L_1(H)||L_1(G)|}\sum_{H_1\in L_1(H)}|C_1(H_1)|,$$

where $C_1(H_1)=\{G_1\in L_1(G)\mid H_1G_1=G_1H_1\}$. Since $N(G)\cap L_1(G)\subseteq C_1(H_1)$, for all $H_1\in L_1(H)$, so

$$csd(H,G)\geq\frac{|L_1(H)||N(G)\cap L_1(G)|}{|L_1(H)||L_1(G)|}\Longleftrightarrow csd(H,G)\geq\frac{|N(G)\cap L_1(G)|}{|L_1(G)|}.$$

Also, if $H\neq G$, we can establish a connection between $csd(H,G)$ and $csd(H)$. More exactly, we have

$$\{(H_1,G_1)\in L_1(H)^2\mid H_1G_1=G_1H_1\}\subseteq\{(H_1,G_1)\in L_1(H)\times L_1(G)\mid H_1G_1=G_1H_1\},$$

and this leads us to

$$csd(H,G)\geq\frac{|L_1(H)|^2csd(H)}{|L_1(H)||L_1(G)|}\Longleftrightarrow csd(H,G)\geq\frac{|L_1(H)|}{|L_1(G)|}csd(H).$$

Let $G=\bigtimes_{i=1}^{k}G_i$ be a direct product of some finite groups G_i having coprime orders. In this case the subgroup lattice of G is decomposable, so any subgroup H of G can be written as $H=\bigtimes_{i=1}^{k}H_i$, where

$H_i \in L(G_i)$, for all $i \in \{1, 2, \ldots, k\}$. Moreover, a cyclic subgroup $H_1 = \bigtimes_{i=1}^{k} H_i^1$ of H permutes with a cyclic subgroup $G_1 = \bigtimes_{i=1}^{k} G_i^1$ of G if and only if $H_i^1 G_i^1 = G_i^1 H_i^1$, for all $i \in \{1, 2, \ldots, k\}$. Consequently, the function $csd : L(G) \times L(G) \longrightarrow [0, 1]$ is multiplicative in both arguments. Hence, we deduce the following two results.

Proposition 2.1. *Let $G = \bigtimes_{i=1}^{k} G_i$ be a finite group, where $(G_i)_{i=\overline{1,k}}$ is a family of finite groups having coprime orders. Then*

$$csd(H, G) = \prod_{i=1}^{k} csd(H_i, G_i).$$

Corollary 2.2. *Let G be a finite nilpotent group and $(P_i)_{i=\overline{1,k}}$ its Sylow subgroups. Then*

$$csd(H, G) = \prod_{i=1}^{k} csd(H_i, P_i).$$

Finally, since a subgroup H of G is isomorphic to any of its conjugates $H^g \in [H]$, we have

$$csd(H, G) = csd(H^g, G), \ \forall\, H^g \in [H].$$

As a consequence, we obtain a useful result that will be used in the next Section.

Proposition 2.3. *Let G be a finite group. The function $csd(-, G) : L(G) \longrightarrow [0, 1]$ is constant on the conjugacy classes of subgroups of G.*

3. On the number of relative cyclic subgroup commutativity degrees of finite groups

The class of finite groups with two relative subgroup commutativity degrees was investigated in [7]. In this Section, we study a similar problem involving the number of relative cyclic subgroup commutativity degrees which is equal to $|Im\ f_1|$, where

$$f_1 : L(G) \longrightarrow [0, 1] \text{ is given by } f_1(H) = csd(H, G), \ \forall\, H \in L(G).$$

For a finite group G, it is easy to see that $|Im\ f_1| = 1$ if and only if all cyclic subgroups of G are permutable. Hence,

$$|Im\ f_1| = 1 \Longleftrightarrow csd(G) = 1 \Longleftrightarrow G \text{ is an Iwasawa group (i.e. a nilpotent modular group)},$$

the second equivalence being indicated in [28]. Our next result is a criterion which indicates a sufficient condition such that a finite group G has at least 3 relative cyclic subgroup commutativity degrees.

Proposition 3.1. *Let G be a finite group such that the inequality $csd(G) < \frac{1}{2} + \frac{|N(G) \cap L_1(G)|}{2|L_1(G)|}$ holds. Then $|Im\ f_1| > 2$.*

Proof. If $|Im\ f_1| = 1$, then G is an Iwasawa group and our hypothesis is not satisfied. Also, if we assume that $|Im\ f_1| = 2$, then there is a minimal subgroup H of G such that $csd(H, G) = csd(G)$. Moreover, the minimality of H implies that $csd(K,G) = 1$ for all subgroups K of G which are strictly contained in H. Therefore,

$$\begin{aligned} csd(H,G) &= \frac{1}{|L_1(H)||L_1(G)|}\sum_{K\in L_1(H)}|C_1(K)| = \frac{1}{|L_1(H)||L_1(G)|}\Big(\sum_{\substack{K\in L_1(H)\\ K\neq H}}|C_1(K)| + |C_1(H)|\Big)\\ &\geq \frac{(|L_1(H)|-1)|L_1(G)| + |N(G)\cap L_1(G)|}{|L_1(H)||L_1(G)|} = 1 - \frac{|L_1(G)| - |N(G)\cap L_1(G)|}{|L_1(H)||L_1(G)|}\\ &\geq 1 - \frac{|L_1(G)| - |N(G)\cap L_1(G)|}{2|L_1(G)|} = \frac{1}{2} + \frac{|N(G)\cap L_1(G)|}{2|L_1(G)|}. \end{aligned}$$

Since $csd(G) = csd(H,G)$, our previous reasoning contradicts the hypothesis. Hence, our proof is complete. □

Some well known classes of non-abelian finite 2-groups are the following ones:

– the dihedral groups
$$D_{2^n} = \langle x, y \mid x^{2^n} = y^2 = 1, yxy = x^{-1}\rangle, n \geq 3,$$

– the generalized quaternion groups
$$Q_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^4 = 1, yxy^{-1} = x^{2^{n-1}-1}\rangle, n \geq 3,$$

– the quasidihedral groups
$$S_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^2 = 1, y^{-1}xy = x^{2^{n-2}-1}\rangle, n \geq 4.$$

As an application of the criterion provided by Proposition 3.1, we prove that there is no finite group with two relative cyclic subgroup commutativity degrees contained in any of the above remarkable classes of groups.

Corollary 3.2. *Let G be a finite group isomorphic to a group that is contained in any of the families $\{D_{2^n}\}_{n\geq 3}$, $\{Q_{2^n}\}_{n\geq 3}$ and $\{S_{2^n}\}_{n\geq 4}$. Then $|Im\ f_1| \neq 2$.*

Proof. Explicit formulas for computing the cyclic subgroup commutativity degree as well as the structure of the poset of cyclic subgroups for each class of finite groups in which we are interested in are provided by [28]. Also, information concerning the number of normal subgroups contained in such groups can be found in [27]. We recall that

$$csd(D_{2^n}) = \frac{n^2 + (n+1)2^n}{(n+2^{n-1})^2},\ csd(Q_{2^n}) = \frac{n^2 + (n+1)2^{n-1}}{(n+2^{n-2})^2},\ csd(S_{2^n}) = \frac{n^2 + 3n\cdot 2^{n-2} + 5\cdot 2^{n-3}}{(n+3\cdot 2^{n-3})^2}.$$

Let $G \cong D_{2^n}$, with $n \geq 3$. Then, according to Subsection 3.2 of [28], we have

$$N(G) = L(\langle x\rangle) \cup \{\langle x^2, y\rangle, \langle x^2, xy\rangle, G\},$$

and this implies that $|N(G)\cap L_1(G)| = n$. Then the hypothesis of Proposition 3.1 is rewritten as

$$\frac{n^2 + (n+1)2^n}{(n+2^{n-1})^2} < \frac{1}{2} + \frac{n}{2(n+2^{n-1})}.$$

This inequality holds for $n \geq 5$. Consequently, $|Im\ f_1| > 2$ for $G \cong D_{2^n}$, with $n \geq 5$. Inspecting the subgroup lattices of D_8 and D_{16}, we obtain $|Im\ f_1| = 3$ and $|Im\ f_1| = 4$, respectively.

Since $csd(Q_8) = 1$, Q_8 is an Iwasawa group and this leads to $|Im\ f_1| = 1$. Let $n \geq 4$ and let G be isomorphic to Q_{2^n} or S_{2^n}. In [27], it is indicated that G has an unique minimal normal subgroup, this being the center $Z(G) = \langle x^{2^{n-2}} \rangle$. Moreover, the following isomorphism holds

$$\frac{G}{Z(G)} \cong D_{2^{n-1}}.$$

Then $|N(G) \cap L_1(G)| = 1 + |N(D_{2^{n-1}}) \cap L_1(D_{2^{n-1}})| = n$. Hence, if $G \cong Q_{2^n}$, the hypothesis of Proposition 3.1 becomes

$$\frac{n^2 + (n+1)2^{n-1}}{(n + 2^{n-2})^2} < \frac{1}{2} + \frac{n}{2(n + 2^{n-2})},$$

and if $G \cong S_{2^n}$, the same relation is rewritten as

$$\frac{n^2 + 3n \cdot 2^{n-2} + 5 \cdot 2^{n-3}}{(n + 3 \cdot 2^{n-3})^2} < \frac{1}{2} + \frac{n}{2(n + 3 \cdot 2^{n-3})}.$$

The first inequality holds for $n \geq 6$, while the second one is true for $n \geq 5$. By direct computation, one can check that $|Im\ f_1| = 4$ if $G \cong Q_{16}$, $|Im\ f_1| = 5$ if $G \cong Q_{32}$ and $|Im\ f_1| = 6$ if $G \cong S_{16}$. Finally, we remark that all groups that were studied are not having 2 relative cyclic subgroup commutativity degrees, as we have stated. □

The possible values of $|Im\ f_1|$ that were obtained in our last proof for the class of generalized quaternion groups, indicate our next result.

Theorem 3.3. *Let G be a finite group isomorphic to Q_{2^n}, where $n \geq 3$. Then*

$$|Im\ f_1| = \begin{cases} 1, & \text{if } n = 3 \\ n, & \text{if } n \geq 4 \end{cases}.$$

Proof. As we observed in the proof of Corollary 3.2, Q_8 is an Iwasawa group, so $|Im\ f_1| = 1$. Let $n \geq 4$ be a positive integer. The cyclic normal subgroups of Q_{2^n} are the subgroups H of the maximal subgroup $\langle x \rangle \cong \mathbb{Z}_{2^{n-1}}$. Therefore, we have

$$f_1(H) = 1,\ \forall\ H \in L(\langle x \rangle).$$

All other conjugacy classes of subgroups of Q_{2^n} are $[H_i]$, where H_i is a subgroup of Q_{2^n} isomorphic to $\mathbb{Z}_4$, if $i = 2$, or to Q_{2^i}, if $i \in \{3, 4, \ldots, n\}$. We denote by r_i the quantity $csd(H_i, Q_{2^n})$, where $i \in \{2, 3, \ldots, n\}$. Then, we have

$$r_i = \frac{i|L_1(Q_{2^n})| + 2^{i-2}(|N(Q_{2^n}) \cap L_1(Q_{2^n})| + 2)}{|L_1(H_i)||L_1(Q_{2^n})|} = \frac{i(n + 2^{n-2}) + 2^{i-2}(n+2)}{(i + 2^{i-2})(n + 2^{n-2})},\ \forall\ i \in \{2, 3, \ldots, n\}.$$

Assume that there is a pair $(i, j) \in \{2, 3, \ldots, n\} \times \{2, 3, \ldots, n\}$, with $i \neq j$, such that $r_i = r_j$. This equality leads to

$$\frac{i(n + 2^{n-2}) + 2^{i-2}(n+2)}{i + 2^{i-2}} = \frac{j(n + 2^{n-2}) + 2^{j-2}(n+2)}{j + 2^{j-2}}.$$

Consider the function $g : [2, \infty) \longrightarrow \mathbb{R}$ defined by $g(x) = \frac{x(n+2^{n-2})+2^{x-2}(n+2)}{x+2^{x-2}}$, where $n \geq 4$ is a fixed positive integer. This function is derivable since it is an elementary function. Then, the derivative of g exists and satisfies the inequality

$$g'(x) = -\frac{2^x(x \ln 2 - 1)(2^n - 8)}{(4x + 2^x)^2} < 0,\ \forall\ x \in [2, \infty),$$

which implies that g is a strictly decreasing function. Hence, g is one-to-one and the equality $r_i = r_j$, that may be rewritten as $g(i) = g(j)$, leads to $i = j$, a contradiction. Therefore, $r_i \neq r_j$, for all $i, j \in \{2, 3, \ldots, n\}$, with $i \neq j$. Finally, assume that $r_n = 1$. Then $n^2 + (n+1)2^{n-1} = n + 2^{n-2}$, which is not true since $n^2 > n$ and

$2^{n-1} > 2^{n-2}$ for any positive integer $n \geq 4$. Therefore, if $n \geq 4$, the generalized quaternion group Q_{2^n} has n relative cyclic subgroup commutativity degrees since $r_2 > r_3 > \ldots > r_n > 1$. $\square$

A direct consequence of Theorem 3.3 is related to the existence of a finite group G with a predetermined number of relative cyclic subgroup commutativity degrees.

Corollary 3.4. *Let n be a positive integer such that $n \neq 2$. Then there exists a finite group G with n relative cyclic subgroup commutativity degrees.*

We move our attention on indicating some classes of finite groups with few relative cyclic subgroup commutativity degrees. Considering a finite group G, since we want to obtain a small value for $|Im\ f_1|$, it is natural to choose G to be isomorphic to a group with $\nu(G) = 1$ or $\nu(G) = 2$, where $\nu(G)$ denotes the number of conjugacy classes of non-normal subgroups. The main advantage of choosing such groups is that they were completely classified in [1] and [13], respectively. Therefore, we start by recalling these two classifications.

Theorem 3.5. *Let G be a finite group. Then $\nu(G) = 1$ if and only if G is isomorphic to one of the following groups:*

(1) $N \rtimes P$, where $N \cong \mathbb{Z}_p, P \cong \mathbb{Z}_{q^n}, [N, \Phi(P)] = 1$ and p, q are primes such that $q | p-1$;

(2) $M(p^n) = \langle x, y \mid x^{p^{n-1}} = y^p = 1, x^y = x^{1+p^{n-2}} \rangle$, where p is a prime and $n \geq 3$ if $p \geq 3$ or $n \geq 4$ if $p = 2$.

Theorem 3.6. *Let G be a finite group. Then $\nu(G) = 2$ if and only if G is isomorphic to one of the following groups:*

(1) A_4;

(2) $\langle x, y \mid x^q = y^{p^n} = 1, x^y = x^k \rangle$, where p, q are prime numbers such that $p^2 | q-1$, $n > 1$ and $k^{p^2} \equiv 1\ (mod\ q)$ with $k \neq 1$;

(3) $\langle x, y, z \mid x^r = y^{p^n} = z^q = [x, z] = [y, z] = 1, x^y = x^k \rangle$, where p, q, r are prime numbers such that $p \neq q, q \neq r, p | r-1$ and $k^p \equiv 1\ (mod\ r)$ with $k \neq 1$; [1)]

(4) $\langle x, y \mid x^{q^2} = y^{p^n} = 1, x^y = x^k \rangle$, where p, q are primes such that $p | q-1$ and $k^p \equiv 1\ (mod\ q^2)$ with $k \neq 1$;

(5) $M(p^n) \times \mathbb{Z}_q$, where p, q are primes such that $p \neq q$ and $n \geq 3$ if $p \geq 3$ or $n \geq 4$ if $p = 2$;

(6) $\mathbb{Z}_4 \rtimes \mathbb{Z}_4$;

(7) Q_{16};

(8) $\langle x, y \mid x^4 = y^{2^n} = 1, y^x = y^{1+2^{n-1}} \rangle$, where $n \geq 3$;

(9) D_8.

Another argument for working with the above listed finite groups is that we can provide explicit formulas for computing different relative cyclic subgroups commutativity degrees, as we will see in our next two proofs. For convenience, we will denote by G_i the finite group of type (i) from each of the above classifications. The following two results indicate the value of $|Im\ f_1|$ for each group G_i.

Theorem 3.7. *Let G be a finite group such that $\nu(G) = 1$. Then*

$$|Im\ f_1| = \begin{cases} 3, & if\ G \cong G_1 \\ 1, & if\ G \cong G_2 \end{cases}.$$

[1)]Note that the condition $q \neq r$ must be added to the group (3) in Theorem I of [13], as shows the example $G = S_3 \times \mathbb{Z}_3$; in this case we have $q = r = 3$, but $\nu(G) = 3$.

Proof. According to Theorem 3.5, $G_1 \cong \mathbb{Z}_p \rtimes \mathbb{Z}_{q^n}$, where p, q are primes such that $q|p-1$ and n is a positive integer. The conjugacy class of non-normal subgroups of this group is $[H]$, where $H \cong \mathbb{Z}_{q^n}$. It is clear that its size is $|[H]| = p$. Moreover, two distinct non-normal subgroups cannot permute since this would imply the existence of a subgroup of order q^{n+1} of G_1. Also, we remark that all other proper subgroups of G_1 are cyclic and normal. Hence,

$$csd(\mathbb{Z}_{q^n}, G_1) = \frac{|L_1(\mathbb{Z}_{q^{n-1}})||L_1(G_1)| + |N(G_1) \cap L_1(G_1)| + 1}{|L_1(\mathbb{Z}_{q^n})||L_1(G_1)|} = \frac{n(2n+p) + 2n + 1}{(n+1)(2n+p)},$$

$$csd(G_1) = \frac{|N(G_1) \cap L_1(G_1)||L_1(G_1)| + p(|N(G_1) \cap L_1(G_1)| + 1)}{|L_1(G_1)|^2} = \frac{2n(2n+p) + p(2n+1)}{(2n+p)^2}.$$

If we assume that these two relative cyclic commutativity degrees are equal, we get $p = 1$ or $p = 2$. The second equality further implies $q = 1$, so, in both situations, we arrive at a contradiction. Therefore, $|Im\ f_1| = 3$.

Since G_2 is an Iwasawa group, we have $|Im\ f_1| = 1$, and our proof is complete. □

Theorem 3.8. *Let G be a finite group such that $\nu(G) = 2$. Then*

$$|Im\ f_1| = \begin{cases} 5, & \text{if } G \text{ is isomorphic to one of the groups } G_1, G_2 \\ 3, & \text{if } G \text{ is isomorphic to one of the groups } G_3, G_6, G_9 \\ 4, & \text{if } G \text{ is isomorphic to one of the groups } G_4, G_7 \\ 1, & \text{if } G \text{ is isomorphic to one of the groups } G_5, G_8 \end{cases}.$$

Proof. We start by finding the number of relative cyclic subgroup commutativity degrees of the nilpotent groups listed in Theorem 3.6. As we indicated in the proof of Corollary 3.2 and in the statement of Theorem 3.3, we have $|Im\ f_1| = 3$, if $G \cong G_9$, and $|Im\ f_1| = 4$, if $G \cong G_7$. Also, by inspecting the subgroup lattice of G_6, it is easy to see that $|Im\ f_1| = 3$. Going further, since p and q are distinct primes, we have $(p^n, q) = 1$, and we can apply Proposition 2.3 of [28] to deduce that

$$csd(G_5) = csd(M(p^n) \times \mathbb{Z}_q) = csd(M(p^n))csd(\mathbb{Z}_q) = 1.$$

Then G_5 is an Iwasawa group and this leads to $|Im\ f_1| = 1$. Finally, we remark that G_8 is a non-hamiltonian 2-group having an abelian normal subgroup $N \cong \mathbb{Z}_{2^n}$. Clearly $\frac{G_8}{N}$ is cyclic and there are $x \in G_8$ and an integer $m = n-1 \geq 2$ such that $G_8 = \langle N, x\rangle$ and $g^x = g^{1+2^m}$, for all $g \in N$. Consequently, G_8 is a modular group, as Theorem 9 of [23] points out. Then G_8 is an Iwasawa group, so $|Im\ f_1| = 1$.

We move our attention to the non-nilpotent groups described by Theorem 3.6. It is easy to see that $|Im\ f_1| = 5$ for the alternating group $A_4 \cong G_1$. The group G_2 is isomorphic to $\mathbb{Z}_q \rtimes \mathbb{Z}_{p^n}$, where p, q are primes such that $p^2|q-1$ and $n > 1$. The conjugacy classes of non-normal subgroups are $[H_1]$ and $[H_2]$, where $H_1 \cong \mathbb{Z}_{p^{n-1}}$ and $H_2 \cong \mathbb{Z}_{p^n}$. The size of these two conjugacy classes is q. We remark that two distinct subgroups from $[H_1]$ cannot permute since we would obtain a subgroup of order p^n. But the only subgroup of order p^n contained in G_2 are isomorphic to $\mathbb{Z}_{p^n}$. This would imply that $\mathbb{Z}_{p^n}$ has two distinct subgroups isomorphic to $\mathbb{Z}_{p^{n-1}}$, a contradicition. Also, two distinct subgroups from $[H_2]$ cannot permute since there is no subgroup of order p^{n+2} contained in G_2. We add that the only conjugacy class of proper normal non-cyclic subgroups is $[H]$, where $H \cong \mathbb{Z}_q \rtimes \mathbb{Z}_{p^{n-1}}$. Moreover, all q conjugates of H_1 are contained in H. Hence, we have

$$csd(\mathbb{Z}_{p^{n-1}}, G_2) = \frac{|L_1(\mathbb{Z}_{p^{n-2}})||L_1(G_2)| + |N(G_2) \cap L_1(G_2)| + 2}{|L_1(\mathbb{Z}_{p^{n-1}})||L_1(G_2)|} = \frac{(n-1)(n+q-1) + n}{n(n+q-1)},$$

$$csd(\mathbb{Z}_{p^n}, G_2) = \frac{|L_1(\mathbb{Z}_{p^{n-2}})||L_1(G_2)| + 2(|N(G_2) \cap L_1(G_2)| + 2)}{|L_1(\mathbb{Z}_{p^n})||L_1(G_2)|} = \frac{(n-1)(n+q-1) + 2n}{(n+1)(n+q-1)},$$

$$csd(\mathbb{Z}_q \rtimes \mathbb{Z}_{p^n}, G_2) = \frac{|L_1(\mathbb{Z}_{p^{n-2}q})||L_1(G_2)| + q(|N(G_2) \cap L_1(G_2)| + 2)}{|L_1(\mathbb{Z}_q \rtimes \mathbb{Z}_{p^n})||L_1(G_2)|} = \frac{2(n-1)(n+q-1)+nq}{(2n+q-2)(n+q-1)},$$

$$csd(G_2) = \frac{|N(G_2) \cap L_1(G_2)||L_1(G_2)| + 2q(|N(G_2) \cap L_1(G_2)| + 2)}{|L_1(G_2)|^2} = \frac{(n-1)(n+q-1)+nq}{(n+q-1)^2}.$$

One can easily check that if we assume that any two of the above quantities are equal, we always contradict the fact that p and q are primes. Consequently, G_2 has 5 relative cyclic subgroup commutativity degrees.

The group G_3 is isomorphic to $(\mathbb{Z}_r \rtimes \mathbb{Z}_{p^n}) \times \mathbb{Z}_q$, where p, q, r are primes such that $p \neq q$, $q \neq r$, $p|r-1$ and n is a positive integer. Remark that the first component of the direct product actually is the group of type (1) which appears in Theorem 3.5. Since $(rp^n, q) = 1$ and $[\mathbb{Z}_{p^n}]$ is the unique non-normal conjugacy class of $\mathbb{Z}_r \rtimes \mathbb{Z}_{p^n}$, the non-normal conjugacy classes of subgroups of G_3 are $[\mathbb{Z}_{p^n} \times H]$, where H is a subgroup of $\mathbb{Z}_q$. Applying Proposition 2.1 and using some results that were found during the proof of Theorem 3.7, we get

$$csd(\mathbb{Z}_{p^n} \times \{1\}, G_3) = csd(\mathbb{Z}_{p^n}, \mathbb{Z}_r \rtimes \mathbb{Z}_{p^n})csd(\{1\}, \mathbb{Z}_q) = \frac{n(2n+r)+2n+1}{(n+1)(2n+r)},$$

$$csd(\mathbb{Z}_{p^n} \times \mathbb{Z}_q, G_3) = csd(\mathbb{Z}_{p^n}, \mathbb{Z}_r \rtimes \mathbb{Z}_{p^n})csd(\mathbb{Z}_q) = \frac{n(2n+r)+2n+1}{(n+1)(2n+r)},$$

$$csd(\mathbb{Z}_r \rtimes \mathbb{Z}_{p^n}, G_3) = csd(\mathbb{Z}_r \rtimes \mathbb{Z}_{p^n})csd(\{1\}, \mathbb{Z}_q) = \frac{2n(2n+r)+r(2n+1)}{(2n+r)^2},$$

$$csd(G_3) = csd(\mathbb{Z}_r \rtimes \mathbb{Z}_{p^n})csd(\mathbb{Z}_q) = \frac{2n(2n+r)+r(2n+1)}{(2n+r)^2}.$$

As we saw in the proof of Theorem 3.7, the above quantities are different. Hence, G_3 has 3 relative cyclic subgroup commutativity degrees.

Finally, the group G_4 is isomorphic to $\mathbb{Z}_{q^2} \rtimes \mathbb{Z}_{p^n}$, where p, q are primes such that $p|q-1$ and n is a positive integer. The two conjugacy classes of non-normal subgroups are $[H_1]$ and $[H_2]$, where $H_1 \cong \mathbb{Z}_{p^n}$ and $H_2 \cong \mathbb{Z}_q \rtimes \mathbb{Z}_{p^n}$. Each of the q conjugates of H_2 contains q conjugates of H_1. Again, it is easy to check that 2 distinct subgroups from $[H_1]$ cannot permute, and this is also true if we work with 2 distinct subgroups from $[H_2]$. All other proper subgroups of G_4 are normal and cyclic. Consequently, we have

$$csd(\mathbb{Z}_{p^n}, G_4) = \frac{|L_1(\mathbb{Z}_{p^{n-1}})||L_1(G_4)| + |N(G_4) \cap L_1(G_4)| + 1}{|L_1(\mathbb{Z}_{p^n})||L_1(G_4)|} = \frac{n(3n+q^2)+3n+1}{(n+1)(3n+q^2)},$$

$$csd(\mathbb{Z}_q \rtimes \mathbb{Z}_{p^n}, G_4) = \frac{|L_1(\mathbb{Z}_{p^{n-1}q})||L_1(G_4)| + q(|N(G_4) \cap L_1(G_4)| + 1)}{|L_1(\mathbb{Z}_q \rtimes \mathbb{Z}_{p^n})||L_1(G_4)|} = \frac{2n(3n+q^2)+q(3n+1)}{(2n+q)(3n+q^2)},$$

$$csd(G_4) = \frac{|N(G_4) \cap L_1(G_4)||L_1(G_4)| + q^2(|N(G) \cap L_1(G_4)| + 1)}{|L_1(G_4)|^2} = \frac{3n(3n+q^2)+q^2(3n+1)}{(3n+q)^2}.$$

Once again, one arrives at a contradiction if assumes that any two of the above 3 quantities are equal. Then $|Im\ f_1| = 4$ and our proof is finished. □

We recall that a Frobenius group is a finite group G which contains a proper subgroup H such that $H \cap H^g = \{1\}$, for all $g \in G \setminus H$. The previous equality shows that there is a connection between this class of finite groups and the quantity $\nu(G)$. More exactly, it is clear that $\nu(G) \geq 1$ for any Frobenius group G. Note that some of the groups classified in Theorems 3.5 and 3.6 are Frobenius groups. Hence, it would be interesting to find out how the number of relative cyclic subgroup commutativity degrees is related to such groups. A well-known class of Frobenius groups is formed of the dihedral groups

$$D_{2n} = \langle x, y \mid x^n = y^2 = 1, yxy = x^{-1} \rangle,$$

where $n \geq 3$ is an odd number. Further, we show that there is no dihedral group having two relative cyclic subgroup commutativity degrees. In particular, the result holds for the above mentioned class of Frobenius groups.

Theorem 3.9. *Let $G \cong D_{2n}$, where $n \geq 3$. Then $|Im\ f_1| \neq 2$.*

Proof. Since G is not an Iwasawa group, it is clear that $|Im\ f_1| \geq 2$. By following Subsection 3.2 of [28], we have

$$L_1(G) = L(\langle x \rangle) \cup \{\langle x^{i-1}y \rangle \mid i \in \{1, 2, \dots, n\}\}.$$

Also, if we denote by $C_1(H)$ the set containing all cyclic subgroups of G that permute with a subgroup H of G, we get

$$|C_1(H)| = \tau(n) + n,\ \forall\ H \in L(\langle x \rangle) \text{ and } |C_1(\langle x^{i-1}y \rangle)| = \begin{cases} \tau(n) + 1, & \text{if } n \equiv 1\ (mod\ 2) \\ \tau(n) + 2, & \text{if } n \equiv 0\ (mod\ 2) \end{cases}.$$

According to Corollary 3.2, if $n = 2^k$, where $k \geq 2$, the conclusion follows. So, it is sufficient to analyse the cases $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ and $n = 2^l p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, where $p_1, p_2, \dots, p_k$ are odd primes, $k \geq 1$ and $l \geq 1$. We consider the subgroups $H = \langle y \rangle \cong \mathbb{Z}_2$ and $K = \langle x^{\frac{n}{p_1}}, y \rangle \cong D_{2p_1}$ of G. Note that $H \subset K$ and H does not permute with all cyclic subgroups of G. Hence, $csd(H, G)$ and $csd(K, G)$ are different from 1.

In the first case, we have

$$csd(H, G) = \frac{2\tau(n) + n + 1}{2(\tau(n) + n)} \text{ and } csd(K, G) = \frac{(p_1 + 2)\tau(n) + 2n + p_1}{(p_1 + 2)(\tau(n) + n)}.$$

If we suppose that $csd(H, G) = csd(K, G)$, we get $n = 1$, which is false.

In the second case, we have

$$csd(H, G) = \frac{2\tau(n) + n + 2}{2(\tau(n) + n)} \text{ and } csd(K, G) = \frac{(p_1 + 2)\tau(n) + 2n + 2p_1}{(p_1 + 2)(\tau(n) + n)}.$$

The equality $csd(H, G) = csd(K, G)$ implies that $n = 2$, which is false.

Hence, $csd(H, G) \neq csd(K, G)$, so, in both cases, we have $|Im\ f_1| \geq 3$. □

Some of the results that were proved in this Section indicate that there are no finite groups with two relative cyclic subgroup commutativity degrees. The arguments are that we deduced that the function $f_1 : L(G) \longrightarrow [0, 1]$ takes n values for all $n \in \mathbb{N}^* \setminus \{2\}$, and, the fact that $|Im\ f_1| \neq 2$ for all groups with $\nu(G) = 1$ or $\nu(G) = 2$. One may expect that an increase of $\nu(G)$ leads to an increase of $|Im\ f_1|$. This is not necessarily true since $D_8 \times \mathbb{Z}_{3^{n-1}}$ has $2n$ conjugacy classes of non-normal subgroups, but $|Im\ f_1| = 3$, for all $n \geq 2$. Therefore, we formulate the following conjecture.

Conjecture 3.10. *There are no finite groups with two relative cyclic subgroup commutativity degrees.*

However, if we study the cardinality of the set $Im\ g_1$, where

$$g_1 : L(G) \longrightarrow [0, 1] \text{ is given by } g_1(H) = csd(H),\ \forall\ H \in L(G),$$

we can prove that there is a finite group G with n cyclic subgroup commutativity degrees for any positive integer n. Remark that g_1 is constant on the conjugacy classes of subgroups of a finite group G, a property that was also satisfied by f_1. Our previous statement concerning the existence of a finite group with a predetermined number of cyclic subgroup commutativity degrees is a consequence of the following result which ends this Section.

Theorem 3.11. *Let G be a finite group isomorphic to $Q_{2^{n+2}}$, where $n \geq 1$. Then $|Im\ g_1| = n$.*

Proof. If $G \cong Q_8$, then G is an Iwasawa group and $|Im\ g_1| = 1$. Now, let G be a finite group isomorphic to $Q_{2^{n+2}}$, where $n \geq 2$. The conjugacy classes of non-Iwasawa subgroups of G are $[H_i]$, where $H_i \cong Q_{2^i}$, for $i \in \{4, 5, \ldots, n+2\}$. We have

$$csd(H_i) = \frac{i^2 + (i+1)2^{i-1}}{(i + 2^{i-2})^2},\ \forall\ i \in \{4, 5, \ldots, n+2\}.$$

The function $h : [4, \infty) \longrightarrow \mathbb{R}$ given by $h(x) = \frac{x^2+(x+1)2^{x-1}}{(x+2^{x-1})^2}$ is strictly decreasing since

$$h'(x) = -\frac{2\{2^{x+1}x^2 \ln 2 + 2^{x+2} + 2^x x[(2^x - 2)\ln 2 - 2] + 2^{2x}(\ln 2 - 1)\}}{(2x + 2^x)^3} < 0,\ \forall\ x \in [4, \infty).$$

Then h is one-to-one and this leads to $csd(H_i) \neq csd(H_j)$, for all $i, j \in \{4, 5, \ldots, n+2\}$, with $i \neq j$. Consequently, for $n \geq 2$, we have $|Im\ g_1| = n$. Hence, it is true that the generalized quaternion group $Q_{2^{n+2}}$, with $n \geq 1$, has n cyclic subgroup commutativity degrees. □

4. The density of the set containing all relative cyclic subgroup commutativity degrees

In [7], it was proved that the set containing all relative subgroup commutativity degrees of finite groups is dense in $[0, 1]$. In this Section, we investigate a similar problem. More exactly, we study the density of the set

$$R = \{csd(H, G) \mid G \text{ is a finite group}, H \in L(G)\}$$

in $[0, 1]$.

Theorem 4.1. *The set R is dense in* $[0, 1]$.

Proof. We must prove that for all $\alpha \in [0, 1]$, there is a sequence $(H_n, G_n)_{n\in\mathbb{N}}$, with $H_n \in L(G_n)$, for all $n \in \mathbb{N}$, such that $\lim\limits_{n\to\infty} csd(H_n, G_n) = \alpha$. For $\alpha = 1$, we choose the constant sequence $(G_n, G_n)_{n\in\mathbb{N}}$, where G_n is isomorphic to an Iwasawa finite group G for all $n \in \mathbb{N}$. If $\alpha = 0$, we select the sequence $(Q_{2^n}, Q_{2^n})_{n\geq 3}$ since

$$\lim_{n\to\infty} csd(Q_{2^n}, Q_{2^n}) = \lim_{n\to\infty} csd(Q_{2^n}) = \lim_{n\to\infty} \frac{n^2 + (n+1)2^{n-1}}{(n + 2^{n-2})^2} = 0.$$

Let $\alpha = \frac{a}{b} \in (0, 1) \cap \mathbb{Q}$, where a and b are some positive integers such that $a < b$. We recall that during the proof of Theorem 3.7, for a finite group $G \cong \mathbb{Z}_p \rtimes \mathbb{Z}_{q^n}$, where p, q are primes such that $q|p-1$ and n is a positive integer, we deduced that

$$csd(\mathbb{Z}_{q^n}, \mathbb{Z}_p \rtimes \mathbb{Z}_{q^n}) = \frac{n(2n + p) + 2n + 1}{(n+1)(2n + p)}.$$

Consequently, we have

$$\lim_{p\to\infty} csd(\mathbb{Z}_{q^n}, \mathbb{Z}_p \rtimes \mathbb{Z}_{q^n}) = \lim_{p\to\infty} \frac{n(2n + p) + 2n + 1}{(n+1)(2n + p)} = \frac{n}{n+1}.$$

Further, we consider the sequence $(q_i)_{i\in\mathbb{N}}$, where, for each $i \in \mathbb{N}$, q_i is a prime number of the form $4k + 3$, with $k \in \mathbb{N}$. Since $(4q_i, 1) = 1$, for all $i \in \mathbb{N}$, there is a sequence of primes $(p_i)_{i\in\mathbb{N}}$ such that $p_i = 4h_i q_i + 1$, for all $i \in \mathbb{N}$. In this way, for each prime q_i, we find a prime p_i such that $q_i|p_i - 1$, for all $i \in \mathbb{N}$. Moreover the sequences $(p_i)_{n\in\mathbb{N}}, (q_i)_{n\in\mathbb{N}}$ are strictly increasing and $p_i \neq q_j$, for all $i, j \in \mathbb{N}$.

Let $(k_n^1), (k_n^2), \ldots, (k_n^{b-a})$ some strictly increasing and disjoint subsequences of $\mathbb{N}$. Our reasoning that involved prime numbers indicates that we can select the sequences

$$(H_j^n, G_j^n)_{n\in\mathbb{N}} = (\mathbb{Z}_{q_{k_n^j}^{a+j-1}}, \mathbb{Z}_{p_{k_n^j}} \rtimes \mathbb{Z}_{q_{k_n^j}^{a+j-1}})_{n\in\mathbb{N}}, \text{ where } j \in \{1, 2, \ldots, b-a\}.$$

Then

$$\lim_{n\to\infty} csd(H_j^n, G_j^n) = \frac{a+j-1}{a+j}, \ \forall\, j \in \{1, 2, \ldots, b-a\}.$$

Finally, one can build the sequence $\left(\bigtimes_{j=1}^{b-a} H_j^n, \bigtimes_{j=1}^{b-a} G_j^n\right)_{n\in\mathbb{N}}$. The remarkable properties of the sequences $(p_i)_{i\in\mathbb{N}}$ and $(q_i)_{i\in\mathbb{N}}$ imply that the subgroup lattice of $\bigtimes_{j=1}^{b-a} G_j^n$ is decomposable for all $n \in \mathbb{N}$. Hence, according to Proposition 2.1, we have

$$\lim_{n\to\infty} csd\Big(\bigtimes_{j=1}^{b-a} H_j^n, \bigtimes_{j=1}^{b-a} G_j^n\Big) = \prod_{j=1}^{b-a} \lim_{n\to\infty} csd(H_j^n, G_j^n) = \prod_{j=1}^{b-a} \frac{a+j-1}{a+j} = \frac{a}{b} = \alpha.$$

Hence $[0,1] \cap \mathbb{Q} \subseteq \overline{R}$. But $R \subseteq [0,1]$, so $\overline{R} \subseteq [0,1]$ since $[0,1]$ is a closed set. Then $[0,1] \cap \mathbb{Q} \subseteq \overline{R} \subseteq [0,1]$, which further leads to $\overline{[0,1] \cap \mathbb{Q}} \subseteq \overline{R} \subseteq [0,1]$. We deduce that $\overline{R} = [0,1]$ since it is well known that the closure of $[0,1] \cap \mathbb{Q}$ is $[0,1]$. Therefore, the set R is dense in $[0,1]$. □

5. Further research

The number of (cyclic) subgroup commutativity degrees and the number of relative (cyclic) subgroup commutativity degrees of a finite group are some quantities that may be further analysed. Besides Conjecture 3.10, we point out three additional open problems.

Problem 5.1. *Determine all finite groups with two cyclic subgroup commutativity degrees. In other words, find all finite groups such that* $|Im\ g_1| = 2$.

A starting point for this research direction is suggested in [29], where all minimal non-Iwasawa finite groups were classified.

Problem 5.2. *Determine all non-Iwasawa finite groups having the same number of relative subgroup commutativity degrees and relative cyclic subgroup commutativity degrees.*

Some examples are the alternating group A_4 and the dihedral group D_8.

One may go further and study the connections between the number of relative cyclic subgroup commutativity degrees and finite groups with 3 or 4 non-normal conjugacy classes of subgroups. For the classification of finite groups G with $\nu(G) = 3$, we refer the reader to [14, 15]. For the classification of finite groups G with $\nu(G) = 4$, details can be found in [2, 4]. However, we note that such a study has its difficulties. For instance, if we examine the classification of finite groups G with $\nu(G) = 3$, one may easily compute the relative cyclic subgroup commutativity degrees for some groups of small fixed order and, consequently, determine the quantity $|Im\ f_1|$. But, there are other groups for which this task is more complex. Some of them are the small order central products $D_8 * \mathbb{Z}_4$ and $D_8 * \mathbb{Z}_8$. The main difficulty concerning such groups is the fact that they have 3 non-normal conjugacy classes of subgroups and the representatives of these classes are isomorphic to $\mathbb{Z}_2$. This means that, if we are to draw the poset of conjugacy classes of subgroups, these 3 conjugacy classes of subgroups are located right above the conjugacy class associated with the trivial subgroup. This leads to computing a significant number of relative cyclic subgroup commutativity degrees and a lot of counting arguments should be carried out. Also, the entire reasoning must be generalized to a central product of type $D_8 * \mathbb{Z}_{2^n}$, for all $n \geq 2$, which is a difficult task.

Another group with 3 non-normal conjugacy classes of subgroups is $G \cong \mathbb{Z}_r \rtimes \mathbb{Z}_{pq}$, where p, q, r are distinct primes such that $pq|r-1$. We note that G is a Frobenius group. In this case, to determine $|Im\ f_1|$, one needs to find $csd(G), csd(\mathbb{Z}_{pq}, G), csd(\mathbb{Z}_p, G)$, $csd(\mathbb{Z}_q, G), csd(\mathbb{Z}_r \rtimes \mathbb{Z}_q, G)$ and $csd(\mathbb{Z}_r \rtimes \mathbb{Z}_p, G)$. Than, one must compare these relative cyclic subgroup commutativity degrees to check how the primes p, q, r are affecting the quantity $|Im\ f_1|$ (a similar reasoning was done in the proof of Theorem 3.8, where we determined the number of relative cyclic subgroup commutativity degrees for the groups G_2, G_3 and G_4). Again, this implies a significant number of counting arguments that may be conducted in another paper. Hence, we suggest a final research direction.

Problem 5.3. *Let G be a finite group with 3 or 4 non-normal conjugacy classes of subgroups. Study the relative cyclic subgroup commutativity degrees of G. What can be said about* $|Im\ f_1|$*?*

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