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On the Generalizations of Some Factors Theorems for Infinite Series and Fourier Series

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Abstract. Quite recently, Bor [Quaest. Math. (doi.org/10.2989/16073606.2019.1578836, in press)] has proved a new result on weighted arithmetic mean summability factors of non decreasing sequences and application on Fourier series. In this paper, we establish a general theorem dealing with absolute matrix summability by using an almost increasing sequence and normal matrices in place of a positive non-decreasing sequence and weighted mean matrices, respectively. So, we extend his result to more general cases.

1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by u_n^α the nth Cesàro mean of order α, with $\alpha > -1$, of the sequence (s_n), that is (see [15])

$$u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v, \tag{1}$$

where

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)....(\alpha+n)}{n!} = O(n^\alpha), \quad A_{-n}^\alpha = 0 \quad \texttt{for} \quad n > 0. \tag{2}$$

Let $a_1, a_2, ..., a_n$ be n arbitrary real numbers; their arithmetic mean A is defined to be

$$A = \frac{a_1 + a_2 + ... + a_n}{n}. \tag{3}$$

A series $\sum a_n$ is said to be summable $|C, \alpha|_k$, $k \geq 1$, if (see [17])

$$\sum_{n=1}^{\infty} n^{k-1} |u_n^\alpha - u_{n-1}^\alpha|^k < \infty. \tag{4}$$

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If we take $\alpha = 1$, then we have $|C,1|_k$ summability. Let (p_n) be a sequence of positive numbers such that $P_n = \sum_{v=0}^{n} p_v \to \infty$ as $n \to \infty$, $(P_{-i} = p_{-i} = 0, \quad i \geq 1)$. The sequence-to-sequence transformation

$$w_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{5}$$

defines the sequence (w_n) of the weighted arithmetic mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [18]). The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k$, $k \geq 1$, if (see [4])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |w_n - w_{n-1}|^k < \infty. \tag{6}$$

In the special case when $p_n = 1$ for all n (respect. $k = 1$), then $|\bar{N}, p_n|_k$ summability is the same as $|C,1|_k$ (respect. $|\bar{N}, p_n|$ (see [23]) summability. Also if we take $p_n = \frac{1}{n+1}$ and $k = 1$, then we obtain $|R, \log n, 1|$ summability (see [3]).

Let $\sum a_n$ be a given series with partial sums (s_n). Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines a sequence-to-sequence transformation, mapping of the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, ... \tag{7}$$

A series $\sum a_n$ is said to be summable $|A, \theta_n|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} \theta_n^{k-1} |A_n(s) - A_{n-1}(s)|^k < \infty, \tag{8}$$

where (θ_n) is any sequence of positive constants (see [20] and [21]). If we put $\theta_n = \frac{P_n}{p_n}$, we have $\left|A, p_n\right|_k$ summability (see [22]). When A is the matrix of weighted mean $(\bar{N}, p_n)$, and $\theta_n = \frac{P_n}{p_n}$, for all n, then $|A, \theta_n|_k$ summability reduces to $\left|\bar{N}, p_n\right|_k$, $k \geq 1$ summability. Further, If $\theta_n = n$ for $n \geq 1$ and A is the matrix of Cesàro mean (C, α), then it is the same as summability $|C, \alpha|_k$ in Flett's notation. By a weighted mean matrix we state

$$a_{nv} = \begin{cases} \frac{p_v}{P_n}, & 0 \leq v \leq n \\ 0 & v > n, \end{cases}$$

where (p_n) is a sequence of positive numbers with $P_n = p_0 + p_1 + p_2 + ... + p_n \to \infty$ as $n \to \infty$.

A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (z_n) and two positive constants A and B such that $Az_n \leq b_n \leq Bz_n$ (see [2]). It is known that every increasing sequences is an almost increasing sequence but the converse need not be true. Many papers concerning almost increasing sequences have been done (see [7]-[14], [24]-[28]). Quite recently, Bor has proved the following theorems concerning on summability factors of the absolute weighted mean using a positive non-decreasing sequence. In Section 2 we give the main results of paper and we generalize Theorem 1.2 for more general matrix summability method by using almost increasing sequences in place of positive non-decreasing sequence. So, we extend Theorem 1.2 to more general cases. In Section 3 we give a theorem dealing with application of absolute matrix summability to Fourier series.

Theorem 1.1. ([6]) *Let (X_n) be a positive non-decreasing sequence and suppose that there exists sequences (β_n) and*

(λ_n) such that

$$|\Delta\lambda_n| \leq \beta_n, \tag{9}$$

$$\beta_n \to 0 \quad as \quad n \to \infty \tag{10}$$

$$\sum_{n=1}^{\infty} n|\Delta\beta_n|X_n < \infty, \tag{11}$$

$$|\lambda_n|X_n = O(1). \tag{12}$$

If

$$\sum_{n=1}^{m} \frac{|t_n|^k}{n} = O(X_m) \quad as \quad m \to \infty, \tag{13}$$

and (p_n) is a sequence that

$$P_n = O(np_n), \tag{14}$$

$$P_n\Delta p_n = O(p_n p_{n+1}), \tag{15}$$

then the series $\sum_{n=1}^{\infty} a_n \frac{P_n\lambda_n}{np_n}$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

Later on, Bor has proved the following theorem under weaker conditions.

Theorem 1.2. ([14]) *Let (X_n) be a positive non-decreasing sequence. If the sequences (X_n) , (β_n), (λ_n), and (p_n) satisfy the conditions (9)-(12), (14)-(15), and*

$$\sum_{n=1}^{m} \frac{|t_n|^k}{nX_n^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{16}$$

then the series $\sum_{n=1}^{\infty} a_n \frac{P_n\lambda_n}{np_n}$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

2. Main Results

Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, ... \quad \bar{\Delta}a_{nv} = a_{nv} - a_{n-1,v}, \quad a_{-1,0} = 0 \tag{17}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{\Delta}\bar{a}_{nv}, \quad n = 1, 2, ... \tag{18}$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv}s_v = \sum_{v=0}^{n} \bar{a}_{nv}a_v \tag{19}$$

and

$$\bar{\Delta}A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv}a_v. \tag{20}$$

With this notation we have the following theorem.

Theorem 2.1. *Let $A = (a_{nv})$ be a positive normal matrix such that*

$$\bar{a}_{n0} = 1,\ n = 0, 1, ..., \tag{21}$$

$$a_{n-1,v} \geq a_{nv},\ \text{for}\ \ n \geq v + 1, \tag{22}$$

$$a_{nn} = O(\frac{p_n}{P_n}), \tag{23}$$

$$na_{nn} = O(1), \tag{24}$$

$$\hat{a}_{n,v+1} = O(v|\Delta_v \hat{a}_{nv}|). \tag{25}$$

Let (X_n) be an almost increasing sequence and $(\theta_n a_{nn})$ be a non-increasing sequence. If the sequences (X_n), (β_n), (λ_n), and (p_n) satisfy the conditions (9)-(12) and (14)-(15) of Theorem 1.1, and the condition

$$\sum_{n=1}^{m} (\theta_n a_{nn})^{k-1} \frac{|t_n|^k}{nX_n^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{26}$$

is satisfied, then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{np_n}$ is summable $|A, \theta_n|_k$, $k \geq 1$.

We need the following lemmas for the proof of Theorem 2.1

Lemma 2.2. ([19]) *Under conditions on (X_n), (β_n), and (λ_n) as expressed in the statement of Theorem 1.1, we have the following:*

$$nX_n\beta_n = O(1), \tag{27}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{28}$$

Lemma 2.3. ([6]) *If the conditions (14) and (15) of Theorem 1.1 are satisfied, then $\Delta\left(\frac{P_n}{n^2 p_n}\right) = O\left(\frac{1}{n^2}\right)$.*

Remark 2.4. Under the conditions on the sequence (λ_n) of Theorem 1.1, we have that (λ_n) is bounded and $\Delta\lambda_n = O(1/n)$ (see [5]).

Proof of Theorem 2.1. Let (V_n) denotes the A-transform of the series $\sum a_n \frac{P_n \lambda_n}{np_n}$. Then, by the definition, we have that

$$V_n - V_{n-1} = \sum_{v=1}^{n} \hat{a}_{nv} a_v \frac{P_v \lambda_v}{vp_v}.$$

Applying Abel's transformation to this sum, we have that

$$\bar{\Delta} V_n = \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv} P_v \lambda_v}{v^2 p_v} \right) \sum_{r=1}^{v} r a_r + \frac{\hat{a}_{nn} P_n \lambda_n}{n^2 p_n} \sum_{r=1}^{n} r a_r$$

$$\bar{\Delta} V_n = \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv} P_v \lambda_v}{v^2 p_v} \right) (v+1) t_v + \frac{\hat{a}_{nn} P_n \lambda_n}{n^2 p_n} (n+1) t_n,$$

by the formula for the difference of products of sequences (see [18]) we have

$$\bar{\Delta} V_n = \frac{a_{nn} P_n \lambda_n}{n^2 p_n} (n+1) t_n + \sum_{v=1}^{n-1} \frac{P_v \lambda_v}{v^2 p_v} \Delta_v(\hat{a}_{nv}) t_v (v+1) + \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \lambda_{v+1} \Delta \left(\frac{P_v}{v^2 p_v} \right) (v+1) t_v$$
$$+ \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \frac{P_v}{v^2 p_v} \Delta\lambda_v t_v (v+1)$$
$$\bar{\Delta} V_n = V_{n,1} + V_{n,2} + V_{n,3} + V_{n,4}.$$

To complete the proof of Theorem 2.1, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \mid V_{n,r} \mid^k < \infty, \quad \text{for} \quad r = 1,2,3,4. \tag{29}$$

Firstly, by applying Abel's transformation and in view of the hypotheses of Theorem 2.1 we have

$$\sum_{n=1}^{m} \theta_n^{k-1} \mid V_{n,1} \mid^k \leq \sum_{n=1}^{m} \theta_n^{k-1} a_{nn}^k \left(\frac{P_n}{p_n}\right)^k \left(\frac{n+1}{n}\right)^k |\lambda_n|^k \frac{|t_n|^k}{n^k}$$
$$= O(1) \sum_{n=1}^{m} (\theta_n a_{nn})^{k-1} |\lambda_n||\lambda_n|^{k-1} \frac{|t_n|^k}{n^k} a_{nn} \left(\frac{P_n}{p_n}\right)^k$$
$$= O(1) \sum_{n=1}^{m} (\theta_n a_{nn})^{k-1} |\lambda_n||\lambda_n|^{k-1} \frac{|t_n|^k}{n^k} \left(\frac{p_n}{P_n}\right)\left(\frac{P_n}{p_n}\right)^k$$
$$= O(1) \sum_{n=1}^{m} (\theta_n a_{nn})^{k-1} |\lambda_n||\lambda_n|^{k-1} \frac{|t_n|^k}{n^k} n^{k-1}$$
$$= O(1) \sum_{n=1}^{m} (\theta_n a_{nn})^{k-1} |\lambda_n| \frac{1}{X_n^{k-1}} \frac{|t_n|^k}{n}$$
$$= O(1) \sum_{n=1}^{m-1} \Delta|\lambda_n| \sum_{v=1}^{n} (\theta_v a_{vv})^{k-1} \frac{|t_v|^k}{v X_v^{k-1}} + O(1)|\lambda_m| \sum_{n=1}^{m} (\theta_n a_{nn})^{k-1} \frac{|t_n|^k}{n X_n^{k-1}}$$
$$= O(1) \sum_{n=1}^{m-1} |\Delta\lambda_n| X_n + O(1)|\lambda_m| X_m = O(1) \sum_{n=1}^{m-1} \beta_n X_n + O(1)|\lambda_m| X_m = O(1) \quad \text{as} \quad m \to \infty.$$

By applying Hölder's inequality with indices k and k', where $k > 1$ and $\frac{1}{k} + \frac{1}{k'} = 1$ and as in $V_{n,1}$, we have that

$$\sum_{n=2}^{m+1} \theta_n^{k-1} \mid V_{n,2} \mid^k = \sum_{n=2}^{m+1} \theta_n^{k-1} \left| \sum_{v=1}^{n-1} \frac{P_v \lambda_v}{v^2 p_v} \Delta_v(\hat{a}_{nv})(v+1) t_v \right|^k$$
$$= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left| \sum_{v=1}^{n-1} \Delta_v(\hat{a}_{nv}) \frac{\lambda_v}{v} t_v \frac{P_v}{p_v} \right|^k$$
$$= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})||\lambda_v||t_v| \right)^k$$
$$= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})||\lambda_v|^k |t_v|^k \right) \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \right)^{k-1}$$
$$= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} a_{nn}^{k-1} \sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})||\lambda_v|^k |t_v|^k$$
$$= O(1) \sum_{v=1}^{m} |\lambda_v|^k |t_v|^k \sum_{n=v+1}^{m+1} (\theta_n a_{nn})^{k-1} |\Delta_v(\hat{a}_{nv})|$$
$$= O(1) \sum_{v=1}^{m} (\theta_v a_{vv})^{k-1} |\lambda_v|^k |t_v|^k \sum_{n=v+1}^{m+1} |\Delta_v(\hat{a}_{nv})|$$

$$= O(1) \sum_{v=1}^{m} (\theta_v a_{vv})^{k-1} \, v a_{vv} \frac{|\lambda_v|^k}{v} |t_v|^k$$
$$= O(1) \sum_{v=1}^{m} (\theta_v a_{vv})^{k-1} \, \frac{1}{X_v^{k-1}} |t_v|^k |\lambda_v| \frac{1}{v} = O(1) \quad \text{as} \quad m \to \infty,$$

by virtue of the hypotheses of Theorem 2.1. Also, since $\Delta\left(\frac{P_v}{v^2 p_v}\right) = O\left(\frac{1}{v^2}\right)$, by Lemma 2.3, we have

$$\sum_{n=2}^{m+1} \theta_n^{k-1} \mid V_{n,3} \mid^k = \sum_{n=2}^{m+1} \theta_n^{k-1} \left| \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \Delta\left(\frac{P_v}{v^2 p_v}\right) \lambda_{v+1} t_v (v+1) \right|^k$$
$$= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| |\lambda_{v+1}| |t_v| \frac{1}{v} \right)^k$$
$$= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| |\lambda_{v+1}| |t_v| \right)^k$$
$$= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| |\lambda_{v+1}|^k |t_v|^k \right) \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \right)^{k-1}$$
$$= O(1) \sum_{n=2}^{m+1} (\theta_n a_{nn})^{k-1} \sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| |\lambda_{v+1}|^k |t_v|^k$$
$$= O(1) \sum_{v=1}^{m} |\lambda_{v+1}|^k |t_v|^k \sum_{n=v+1}^{m+1} (\theta_n a_{nn})^{k-1} \, |\Delta_v(\hat{a}_{nv})|$$
$$= O(1) \sum_{v=1}^{m} (\theta_v a_{vv})^{k-1} \, |\lambda_{v+1}|^k |t_v|^k \sum_{n=v+1}^{m+1} |\Delta_v(\hat{a}_{nv})|$$
$$= O(1) \sum_{v=1}^{m} (\theta_v a_{vv})^{k-1} \, |\lambda_{v+1}|^k |t_v|^k a_{vv}$$
$$= O(1) \sum_{v=1}^{m} (\theta_v a_{vv})^{k-1} \, \frac{1}{v X_v^{k-1}} |\lambda_{v+1}| |t_v|^k = O(1) \quad \text{as} \quad m \to \infty,$$

by virtue of the hypotheses of Theorem 2.1. Finally, by virtue of the hypotheses of Theorem 2.1, by Lemma 2.2, we have $v\beta_v = O(\frac{1}{X_v})$, then

$$\sum_{n=2}^{m+1} \theta_n^{k-1} \mid V_{n,4} \mid^k = \sum_{n=2}^{m+1} \theta_n^{k-1} \left| \sum_{v=1}^{n-1} \hat{a}_{n,v+1} (v+1) \frac{P_v}{v^2 p_v} \Delta\lambda_v t_v \right|^k$$
$$= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| |\Delta\lambda_v| |t_v| \right)^k$$
$$= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\sum_{v=1}^{n-1} v |\Delta_v(\hat{a}_{nv})| |\Delta\lambda_v| |t_v| \right)^k$$
$$= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\sum_{v=1}^{n-1} (v\beta_v)^k |t_v|^k |\Delta_v \hat{a}_{nv}| \right) \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \right)^{k-1}$$

$$= O(1)\sum_{n=2}^{m+1}(\theta_n a_{nn})^{k-1}\left(\sum_{v=1}^{n-1}(v\beta_v)^k|t_v|^k|\Delta_v\hat{a}_{nv}|\right)$$
$$= O(1)\sum_{v=1}^{m}(v\beta_v)(v\beta_v)^{k-1}|t_v|^k\sum_{n=v+1}^{m+1}(\theta_n a_{nn})^{k-1}|\Delta_v(\hat{a}_{nv})|$$
$$= O(1)\sum_{v=1}^{m}(\theta_v a_{vv})^{k-1}(v\beta_v)(v\beta_v)^{k-1}|t_v|^k\sum_{n=v+1}^{m+1}|\Delta_v(\hat{a}_{nv})|$$
$$= O(1)\sum_{v=1}^{m}(\theta_v a_{vv})^{k-1}(v\beta_v)(v\beta_v)^{k-1}|t_v|^k a_{vv}$$
$$= O(1)\sum_{v=1}^{m}(\theta_v a_{vv})^{k-1}\frac{1}{X_v^{k-1}}\beta_v|t_v|^k\frac{v}{v}$$
$$= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v}(\theta_r a_{rr})^{k-1}\frac{|t_r|^k}{rX_r^{k-1}} + O(1)m\beta_m\sum_{v=1}^{m}(\theta_v a_{vv})^{k-1}\frac{|t_v|^k}{vX_v^{k-1}}$$
$$= O(1)\sum_{v=1}^{m-1}v|\Delta\beta_v|X_v + O(1)\sum_{v=1}^{m-1}X_v\beta_v + O(1)m\beta_m X_m = O(1) \quad \text{as} \quad m\to\infty,$$

This completes the proof of Theorem 2.1. □

If we take (X_n) as a positive non-decreasing sequence, $\theta_n = \frac{P_n}{p_n}$, then we have a result concerning the $|A, p_n|_k$ summability factors (see [1]).

3. An Application of Absolute Matrix Summability to Fourier Series

Let f be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$. Without any loss of generality the constant term in the Fourier series of f can be taken to be zero, so that

$$f(t) \sim \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} C_n(t). \tag{30}$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt.$$

We write

$$\phi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\}, \tag{31}$$

$$\phi_\alpha(t) = \frac{\alpha}{t^\alpha}\int_0^t (t-u)^{\alpha-1}\phi(u)\,du, \quad (\alpha > 0). \tag{32}$$

It is well known that if $\phi(t) \in \mathcal{BV}(0,\pi)$, then $t_n(x) = O(1)$, where $t_n(x)$ is the $(C,1)$ mean of the sequence $(nC_n(x))$ (see [16]).

The Fourier series play an important role in many areas of applied mathematics and mechanics. Using these series, Bor has obtained the following result.

Theorem 3.1. ([14]) *Let (X_n) be a positive non-increasing sequence. If $\phi_1(t) \in \mathcal{BV}(0,\pi)$, and the sequences (p_n), (λ_n), (β_n) and (X_n) satisfy the conditions of Theorem 1.2, then the series $\sum \frac{C_n(x)P_n\lambda_n}{np_n}$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.*

Similarly to Theorem 2.1 we can prove the following result.

Theorem 3.2. *Let A be a positive normal matrix satisfying the conditions of Theorem 2.1 Let Let* (X_n) *be an almost increasing sequence. If* $\phi_1(t) \in \mathcal{BV}(0,\pi)$*, and the sequences* (p_n)*,* (λ_n)*,* (β_n)*, and* (X_n) *satisfy the conditions of Theorem 1.2, then the series* $\sum \frac{C_n(x)P_n\lambda_n}{np_n}$ *is summable* $|A,\theta_n|_k$*,* $k \geq 1$*.*

We now apply the above theorems to the weighted mean in which $A = (a_{nv})$ is defined as $a_{nv} = \frac{p_v}{P_n}$ when $0 \leq v \leq n$, where $P_n = p_0 + p_1 + ... + p_n$. Therefore, it is well known that

$$\bar{a}_{nv} = \frac{P_n - P_{v-1}}{P_n} \quad \text{and} \quad \hat{a}_{n,v+1} = \frac{p_n P_v}{P_n P_{n-1}}.$$

If we take $\theta_n = \frac{P_n}{p_n}$ in Theorem 3.2, then we have a result concerning the $|A, p_n|_k$ summability factors of the trigonometric Fourier series, and if we take $a_{nv} = \frac{p_v}{P_n}$ Theorem 3.2, then we have another result dealing with $\left|\bar{N}, p_n, \theta_n\right|_k$ summability factors of the trigonometric Fourier series. Also, if we put $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all n in Theorem 3.2, then we obtain a result concerning $|C,1,\theta_n|_k$ summability factors of the trigonometric Fourier series. Moreover, if we take $\theta_n = \frac{P_n}{p_n}$, $k = 1$ and $a_{nv} = \frac{p_v}{P_n}$ in Theorem 3.2, then we have a result dealing with $\left|\bar{N}, p_n\right|$ summability factors of the trigonometric Fourier series ,and if we take $\theta_n = n$, $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all n in Theorem 3.2, then we obtain a result concerning the $|C,1|_k$ summability factors of the trigonometric Fourier series.

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