Filomat 33:14 (2019), 4519–4524
https://doi.org/10.2298/FIL1914519B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Improved Inequalities for the Extension of Euclidean Numerical Radius

Akram Babri Bajmaeh[a], Mohsen Erfanian Omidvar[a]

[a] *Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad, Iran.*

Abstract. This paper aims to discuss inequalities involving extension of Euclidean numerical radius. We obtain a refinement of the inequality shown by Sattari et al. We give an improvement of the inequality presented by Kittaneh for the numerical radius. In fact we show that if $T \in \mathcal{B}(\mathcal{H})$, then

$$\omega^2(T) \leq \frac{1}{2}\|T^*T + TT^*\| - \inf_{\|x\|=1} \phi(x),$$

where $\phi(x) = \left\langle \left(\left| |T| - \langle |T|x, x\rangle \right|^2 + \left| |T^*| - \langle |T^*|x, x\rangle \right|^2 \right) x, x \right\rangle$.

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$. For $T \in \mathcal{B}(\mathcal{H})$, let $\omega(T) = \sup_{\|x\|=1} |\langle Tx, x\rangle|$ and $\|T\| = \sup_{\|x\|=1} \|Tx\|$, denote the numerical radius, the usual operator norm of T, respectively. It is well know that $\omega(\cdot)$ is a norm on $\mathcal{B}(\mathcal{H})$, and that for all $T \in \mathcal{B}(\mathcal{H})$,

$$\frac{1}{2}\|T\| \leq \omega(T) \leq \|T\|. \tag{1.1}$$

For other results concerning the numerical radius of bounded linear operator on a Hilbert space, see [1], [2], [10].
In [6], Kittaneh has improved (1.1) in the following manner:

$$\frac{1}{4}\|T^*T + TT^*\| \leq \omega^2(T) \leq \frac{1}{2}\|T^*T + TT^*\|. \tag{1.2}$$

Let $(T_1, \ldots, T_n) \in \mathcal{B}(\mathcal{H})^{(n)}$, the Euclidean numerical radius of $(T_1, \ldots, T_n)$ is defined in [9] by

$$\omega_2(T_1, \ldots, T_n) = \sup_{\|x\|=1} \left(\sum_{i=1}^{n} |\langle T_i x, x\rangle|^2\right)^{\frac{1}{2}}.$$

2010 *Mathematics Subject Classification*. Primary 47A12; secondary 47A30, 47A63.
Keywords. bounded linear operator, numerical radius, Euclidean numerical radius, positive operator.
Received: 09 March 2019; Revised: 15 June 2019; Accepted: 02 July 2019
Communicated by Fuad Kittaneh
Email addresses: babri6387@yahoo.com (Akram Babri Bajmaeh), math.erfanian@gmail.com (Mohsen Erfanian Omidvar)

In the following Sattari et al. [8] considered the generalized Euclidean numerical radius of a n-tuple of bounded linear operators on a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$, as follows

$$\omega_p(T_1,\ldots,T_n) = \sup_{\|x\|=1}(\sum_{i=1}^{n}|\langle T_i x, x\rangle|^p)^{\frac{1}{p}}.$$

It is worth to mention here that $\omega_p : \mathcal{B}^n(\mathcal{H}) \longrightarrow [0,\infty)$ is a norm and if $(T_1,\ldots,T_n), (A_1,\ldots,A_n), (B_1,\ldots,B_n) \in \mathcal{B}^n(\mathcal{H})$, $p \geq 1$ and $r \geq 2$, then

$$\omega_p^{rp}(A_1^*T_1B_1,\ldots,A_n^*T_nB_n) \leq \frac{1}{2}\left\|\sum_{i=1}^{n}([B_i^* f^2(|T_i|)B_i]^{rp} + [A_i^* g^2(|T_i^*|)A_i]^{rp})\right\|. \tag{1.3}$$

In this paper we establish inequalities that refine the inequality (1.3) and second inequality in (1.2). We also compare our result with the corresponding inequality obtained in [8]. Some applications of these inequalities are considered as well.

2. Main Results

To prove our generalized numerical radius inequality, we need several well known lemmas. The first lemma is known as the generalized mixed Schwartz inequality, which has been proved in [5].

Lemma 2.1. *Let $T \in \mathcal{B}(\mathcal{H})$, then*

$$|\langle Tx, y\rangle|^2 \leq \langle |T|^{2\alpha}x, x\rangle\langle |T^*|^{2(1-\alpha)}y, y\rangle,$$

for all $x, y \in \mathcal{H}$ and for all α with $0 \leq \alpha \leq 1$.

The second lemma is a simple consequence of the classical Jensen inequality concerning the convexity or concavity of certain power function. This is a special case of Schlomilchs inequality for weighted means of non negative real numbers. For generalization of this lemma, we refer to [3].

Lemma 2.2. *For $a, b \geq 0$, $0 < \alpha < 1$, and $r \neq 0$, let $M_r(a,b,\alpha) = (\alpha a^r + (1-\alpha)b^r)^{\frac{1}{r}}$ and let $M_0(a,b,\alpha) = a^\alpha b^{1-\alpha}$. Then*

$$M_r(a,b,\alpha) \leq M_s(a,b,\alpha). \qquad r \leq s$$

The third lemma, which is called Hölder-McCarthy inequality, is a well-known result that follows from the spectral theorem for positive operators and Jensen's inequality (see [7]).

Lemma 2.3. *Let $T \in \mathcal{B}(\mathcal{H})$ be positive, and let $x \in \mathcal{H}$ be any unit vector. Then*

(i) $\langle Tx, x\rangle^r \leq \langle T^r x, x\rangle, \qquad r \geq 1,$

(ii) $\langle T^r x, x\rangle \leq \langle Tx, x\rangle^r, \qquad 0 < r \leq 1.$

The forth lemma is an immediate consequence of the spectral theorem for self-adjoint operators. For generalization of this lemma, we refer to [5].

Lemma 2.4. *Let $T \in \mathcal{B}(\mathcal{H})$ and f and g be non-negative continuous function on $[0,\infty)$ satisfying $f(t)g(t) = t$. Then*

$$|\langle Tx, y\rangle| \leq \|f(|T|)x\|\,\|g(|T^*|)y\|,$$

for all $x, y \in \mathcal{H}$.

In the following lemma proved by Kian for positive operators and $r \geq 2$ (see[4]).

Lemma 2.5. *Let $T \in \mathcal{B}(\mathcal{H})$ be a positive operator, and $x \in \mathcal{H}$ be any unit vector. Then*

$$\langle Tx, x\rangle^r \leq \langle T^r x, x\rangle - \langle |T - \langle Tx, x\rangle|^r x, x\rangle,$$

for $r \geq 2$.

Now we are ready to state the main result of this section and generalization of inequalities.

Theorem 2.6. *Let $(T_1, \ldots, T_n), (A_1, \ldots, A_n), (B_1, \ldots, B_n) \in \mathcal{B}(\mathcal{H})^{(n)}$ and let f, g be non-negative continuous on $[0, \infty)$ satisfying $f(t)g(t) = t$, then*

$$\omega_p^{rp}(A_1^* T_1 B_1, \ldots, A_n^* T_n B_n) \leq \frac{n^{r-1}}{2} \left\| \sum_{i=1}^n ([B_i^* f^2(|T_i|) B_i]^{rp} + [A_i^* g^2(|T_i^*|) A_i]^{rp}) \right\| - \inf_{\|x\|=1} \phi(x),$$

for $p \geq 1$ and $r \geq 2$, where

$$\phi(x) = \frac{n^{r-1}}{2} \left\langle \sum_{i=1}^n \left(\left| (B_i^* f^2(|T_i|) B_i)^p - \langle (B_i^* f^2(|T_i|) B_i)^p x, x\rangle \right|^r + \left| (A_i^* g^2(|T_i^*|) A_i)^p - \langle (A_i^* g^2(|T_i^*|) B_i)^p x, x\rangle \right|^r \right) x, x \right\rangle.$$

Proof. For every unit vector $x \in \mathcal{H}$, we have

$$\begin{aligned}
\sum_{i=1}^n |\langle A_i^* T_i B_i x, x\rangle|^p &= \sum_{i=1}^n |\langle T_i B_i x, A_i x\rangle|^p \\
&\leq \sum_{i=1}^n \|f(|T_i|) B_i x\|^p \|g(|T_i^*|) A_i x\|^p \qquad \text{(by Lemma 2.4)} \\
&= \sum_{i=1}^n \langle B_i^* f^2(|T_i|) B_i x, x\rangle^{\frac{p}{2}} \langle A_i^* g^2(|T_i^*|) A_i x, x\rangle^{\frac{p}{2}} \\
&\leq \sum_{i=1}^n \langle (B_i^* f^2(|T_i|) B_i)^p x, x\rangle^{\frac{1}{2}} \langle (A_i^* g^2(|T_i^*|) A_i)^p x, x\rangle^{\frac{1}{2}} \qquad \text{(by Lemma 2.3)} \\
&\leq \sum_{i=1}^n \left(\frac{1}{2} \langle (B_i^* f^2(|T_i|) B_i)^p x, x\rangle + \langle (A_i^* g^2(|T_i^*|) A_i)^p x, x\rangle \right) \\
&\qquad \text{(by arithmetic-geometric mean inequality)} \\
&\leq \sum_{i=1}^n \left(\frac{1}{2} \langle (B_i^* f^2(|T_i|) B_i)^p x, x\rangle^r + \langle (A_i^* g^2(|T_i^*|) A_i)^p x, x\rangle^r \right)^{\frac{1}{r}} \qquad \text{(by Lemma 2.2)} \\
&\leq (\frac{1}{2})^{\frac{1}{r}} \sum_{i=1}^n \Bigg[\left(\left\langle (B_i^* f^2(|T_i|) B_i)^{rp} x, x \right\rangle - \left\langle \left| (B_i^* f^2(|T_i|) B_i)^p - \langle (B_i^* f^2(|T_i|) B_i)^p x, x\rangle \right|^r x, x \right\rangle \right) \\
&\qquad + \left(\left\langle (A_i^* g^2(|T_i^*|) A_i)^{rp} x, x \right\rangle - \left\langle \left| (A_i^* g^2(|T_i^*|) A_i)^p - \langle (A_i^* g^2(|T_i^*|) A_i)^p x, x\rangle \right|^r x, x \right\rangle \right) \Bigg]^{\frac{1}{r}} \\
&\qquad \text{(by Lemma 2.5)}
\end{aligned}$$

$$
\begin{aligned}
&= (\frac{1}{2})^{\frac{1}{r}} \sum_{i=1}^{n} \Big(\Big\langle \Big((B_i^* f^2(|T_i|)B_i)^{rp} + (A_i^* g^2(|T_i^*|)A_i)^{rp} \Big) x, x \Big\rangle \\
&\quad - \Big\langle \Big| (B_i^* f^2(|T_i|)B_i)^p - \langle (B_i^* f^2(|T_i|)B_i)^p x, x\rangle \Big|^r x, x \Big\rangle \\
&\quad - \Big\langle \Big| (A_i^* g^2(|T_i^*|)A_i)^p - \langle (A_i^* g^2(|T_i^*|)A_i)^p x, x\rangle \Big|^r x, x \Big\rangle \Big)^{\frac{1}{r}} \\
&= (\frac{1}{2})^{\frac{1}{r}} \sum_{i=1}^{n} \Big(\Big\langle \Big((B_i^* f^2(|T_i|)B_i)^{rp} + (A_i^* g^2(|T_i^*|)A_i)^{rp} \Big) x, x \Big\rangle \\
&\quad - \Big\langle \Big(\Big| (B_i^* f^2(|T_i|)B_i)^p - \langle (B_i^* f^2(|T_i|)B_i)^p x, x\rangle \Big|^r \\
&\qquad + \Big(\Big| (A_i^* g^2(|T_i^*|)A_i)^p - \langle (A_i^* g^2(|T_i^*|)A_i)^p x, x\rangle \Big|^r \Big) x, x \Big\rangle \Big)^{\frac{1}{r}} \\
&\le n^{1-\frac{1}{r}} (\frac{1}{2})^{\frac{1}{r}} \Big[\Big\langle \sum_{i=1}^{n} \Big((B_i^* f^2(|T_i|)B_i)^{rp} + (A_i^* g^2(|T_i^*|)A_i)^{rp} \Big) x, x \Big\rangle \\
&\quad - \Big\langle \sum_{i=1}^{n} \Big(\Big| (B_i^* f^2(|T_i|)B_i)^p - \langle (B_i^* f^2(|T_i|)B_i)^p x, x\rangle \Big|^r \\
&\qquad + \Big| (A_i^* g^2(|T_i^*|)A_i)^p - \langle (A_i^* g^2(|T_i^*|)B_i)^p x, x\rangle \Big|^r \Big) x, x \Big\rangle \Big]^{\frac{1}{r}} \\
&\qquad\qquad \text{(by the concavity of the } f(t) = t^{1/r}\text{).}
\end{aligned}
$$

Thus

$$
\begin{aligned}
\left(\sum_{i=1}^{n} |\langle A_i^* T_i B_i x, x\rangle|^p \right)^r &\le \frac{n^{r-1}}{2} \Big\langle \sum_{i=1}^{n} \Big((B_i^* f^2(|T_i|)B_i)^{rp} + (A_i^* g^2(|T_i^*|)A_i)^{rp} \Big) x, x \Big\rangle \\
&\quad - \frac{n^{r-1}}{2} \Big\langle \sum_{i=1}^{n} \Big(\Big| (B_i^* f^2(|T_i|)B_i)^p - \langle (B_i^* f^2(|T_i|)B_i)^p x, x\rangle \Big|^r \\
&\qquad + \Big| (A_i^* g^2(|T_i^*|)A_i)^p - \langle (A_i^* g^2(|T_i^*|)B_i)^p x, x\rangle \Big|^r \Big) x, x \Big\rangle \\
&\le \frac{n^{r-1}}{2} \Big\langle \sum_{i=1}^{n} \Big((B_i^* f^2(|T_i|)B_i)^{rp} + (A_i^* g^2(|T_i^*|)A_i)^{rp} \Big) x, x \Big\rangle - \inf_{\|x\|=1} \phi(x),
\end{aligned}
$$

where

$$
\begin{aligned}
\phi(x) = \frac{n^{r-1}}{2} \Big\langle \sum_{i=1}^{n} \Big(&\Big| (B_i^* f^2(|T_i|)B_i)^p - \langle (B_i^* f^2(|T_i|)B_i)^p x, x\rangle \Big|^r \\
&+ \Big| (A_i^* g^2(|T_i^*|)A_i)^p - \langle (A_i^* g^2(|T_i^*|)B_i)^p x, x\rangle \Big|^r \Big) x, x \Big\rangle.
\end{aligned}
$$

Now the result following by taking the supremum over all unit vector in $\mathcal{H}$. □

Remark 2.7. *In Theorem* 2.6, $\inf_{\|x\|=1} \phi(x) = 0$ *if and only if*

$$
0 \in \bigcap_{i=1}^{n} \overline{W\Big(\Big| (B_i^* f^2(|T_i|)B_i)^p - \langle (B_i^* f^2(|T_i|)B_i)^p x, x\rangle \Big|^r + \Big| (A_i^* g^2(|T_i^*|)A_i)^p - \langle (A_i^* g^2(|T_i^*|)A_i)^p x, x\rangle \Big|^r \Big)}.
$$

An important special case of Theorem 2.6, which leads to sharper of inequality (1.3) for $\phi(x) > 0$. Choosing $A_i = B_i = I$ we get.

Corollary 2.8. *Let $(T_1, \dots, T_n) \in \mathcal{B}(\mathcal{H})^{(n)}$ and let f, g be non-negative continuous on $[0, \infty)$ satisfying $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then*

$$\omega_p^{rp}(T_1, \dots, T_n) \le \frac{n^{r-1}}{2} \left\| \sum_{i=1}^{n} \left(f^{2rp}(|T_i|) + g^{2rp}(|T_i^*|) \right) \right\| - \inf_{\|x\|=1} \phi(x),$$

for $p \ge 1$ and $r \ge 2$, where

$$\phi(x) = \frac{n^{r-1}}{2} \left\langle \sum_{i=1}^{n} \left(\left| f^{2p}(|T_i|) - \langle f^{2p}(|T_i|)x, x\rangle \right|^r + \left| g^{2p}(|T_i^*|) + \langle g^{2p}(|T_i^*|)x, x\rangle \right|^r \right) x, x \right\rangle.$$

Corollary 2.9. *Let $T \in \mathcal{B}(\mathcal{H})$, then*

$$\omega^2(T) \le \frac{1}{2} \|T^*T + TT^*\| - \inf_{\|x\|=1} \phi(x),$$

where

$$\phi(x) = \left\langle \left(\left| |T| - \langle |T|x, x\rangle \right|^2 + \left| |T^*| - \langle |T^*|x, x\rangle \right|^2 \right) x, x \right\rangle.$$

Proof. *The result follows by letting $T_1 = T$ and $f(t) = g(t) = t^{\frac{1}{2}}$, $r = 2$, $p = 1$, $n = 1$ in Corollary 2.8.* □

Remark 2.10. *Corollary 2.9 gives a sharper inequality than (1.2).*

Corollary 2.11. *Let $(T_1, \dots, T_n), (A_1, \dots, A_n), (B_1, \dots, B_n) \in \mathcal{B}(\mathcal{H})^{(n)}$. Then*

$$\omega_p^{rp}(A_1^* T_1 B_1, \dots, A_n^* T_n B_n) \le \frac{n^{r-1}}{2} \left\| \sum_{i=1}^{n} \left((B_i^* |T_i| B_i)^{rp} + (A_i^* |T_i^*| A_i)^{rp} \right) \right\| - \inf_{\|x\|=1} \phi(x),$$

for $p \ge 1$ and $r \ge 2$, where

$$\phi(x) = \frac{n^{r-1}}{2} \left\langle \sum_{i=1}^{n} \left(\left| (B_i^* |T_i| B_i)^p - \langle (B_i^* |T_i| B_i)^p x, x\rangle \right|^r + \left| (A_i^* |T_i^*| A_i)^p - \langle (A_i^* |T_i^*| A_i)^p x, x\rangle \right|^r \right) x, x \right\rangle.$$

Corollary 2.12. *Let $(A_1, \dots, A_n), (B_1, \dots, B_n) \in \mathcal{B}(\mathcal{H})^{(n)}$. Then*

$$\omega_p^{rp}(A_1^* B_1, \dots, A_n^* B_n) \le \frac{n^{r-1}}{2} \left\| \sum_{i=1}^{n} \left(|B_i|^{2rp} + |A_i|^{2rp} \right) \right\| - \inf_{\|x\|=1} \phi(x),$$

for $p \ge 1$ and $r \ge 2$, where

$$\phi(x) = \frac{n^{r-1}}{2} \left\langle \sum_{i=1}^{n} \left(\left| |B_i|^{2p} - \langle |B_i|^{2p} x, x\rangle \right|^r + \left| |A_i|^{2p} - \langle |A_i|^{2p} x, x\rangle \right|^r \right) x, x \right\rangle.$$

Theorem 2.13. *Let $(T_1, \dots, T_n) \in \mathcal{B}(\mathcal{H})^{(n)}$. Then*

$$\omega_p^p(T_1, \dots, T_n) \le \frac{1}{2^p} \left\| \sum_{i=1}^{n} \left(|T_i|^{2\alpha} + |T_i^*|^{2(1-\alpha)} \right)^p \right\| - \inf_{\|x\|=1} \phi(x),$$

for $0 \le \alpha \le 1$, and $p \ge 2$, where

$$\phi(x) = \frac{1}{2^p} \left\langle \sum_{i=1}^{n} \left| |T_i|^{2\alpha} + |T_i^*|^{2(1-\alpha)} - \langle (|T_i|^{2\alpha} + |T_i^*|^{2(1-\alpha)}) x, x\rangle \right|^p x, x \right\rangle.$$

Proof. By using the arithmetic- geometric mean for any unit vector $x \in \mathcal{H}$. We have

$$\begin{aligned}
\sum_{i=1}^{n} |\langle T_i x, x\rangle|^p &\le \sum_{i=1}^{n} \left(\langle |T_i|^{2\alpha} x, x\rangle^{\frac{1}{2}} \langle |T_i^*|^{2(1-\alpha)} x, x\rangle^{\frac{1}{2}} \right)^p \qquad \text{(by Lemma 2.1)} \\
&\le \frac{1}{2^p} \sum_{i=1}^{n} \Big(\langle |T_i|^{2\alpha} x, x\rangle + \langle |T_i^*|^{2(1-\alpha)} x, x\rangle \Big)^p \\
&= \frac{1}{2^p} \sum_{i=1}^{n} \Big\langle (|T_i|^{2\alpha} + |T_i^*|^{2(1-\alpha)}) x, x \Big\rangle^p \\
&\le \frac{1}{2^p} \Bigg(\Big\langle \sum_{i=1}^{n} (|T_i|^{2\alpha} + |T_i^*|^{2(1-\alpha)})^p x, x \Big\rangle \\
&\qquad - \Big\langle \sum_{i=1}^{n} \Big| |T_i|^{2\alpha} + |T_i^*|^{2(1-\alpha)} - \langle (|T_i|^{2\alpha} + |T_i^*|^{2(1-\alpha)}) x, x\rangle \Big|^p x, x \Big\rangle \Bigg) \\
&\qquad\qquad \text{(by Lemma 2.5)} \\
&\le \frac{1}{2^p} \Big\langle \sum_{i=1}^{n} (|T_i|^{2\alpha} + |T_i^*|^{2(1-\alpha)})^p x, x \Big\rangle - \inf_{\|x\|=1} \phi(x),
\end{aligned}$$

where

$$\phi(x) = \frac{1}{2^p} \Big\langle \sum_{i=1}^{n} \Big| |T_i|^{2\alpha} + |T_i^*|^{2(1-\alpha)} - \langle (|T_i|^{2\alpha} + |T_i^*|^{2(1-\alpha)}) x, x\rangle \Big|^p x, x \Big\rangle,$$

now the result following by taking the supremum over all unit vector in $\mathcal{H}$. □

Corollary 2.14. *Let $T = C + iD$ be the Cartesian decomposition of T, then*

$$\omega^2(T) \le \frac{1}{4} \left\| \Big(|C|^{2\alpha} + |C|^{2(1-\alpha)} \Big)^2 + \Big(|D|^{2\alpha} + |D|^{2(1-\alpha)} \Big)^2 \right\| - \inf_{\|x\|=1} \phi(x),$$

for $0 \le \alpha \le 1$, where

$$\phi(x) = \frac{1}{4} \Big\langle \Big(\Big| |C|^{2\alpha} + |C|^{2(1-\alpha)} - \langle (|C|^{2\alpha} + |C|^{2(1-\alpha)}) x, x\rangle \Big|^2 + \Big| |D|^{2\alpha} + |D|^{2(1-\alpha)} - \langle (|D|^{2\alpha} + |D|^{2(1-\alpha)}) x, x\rangle \Big|^2 \Big) x, x \Big\rangle.$$

Proof. We observe that if $T = C + iD$ is the Cartesian decomposition of T, then $\omega_2^2(C, D) = \omega^2(T)$. By choosing $n = 2$, $p = 2$, $T_1 = C$ and $T_2 = D$ in Theorem 2.13, we conclude that the desired inequality. □

References

[1] S.S. Dragomir, A survey of some recent inequalities for the norm and numerical radius of operators in Hilbert spaces, Banach J. Math. Anal. 1 (2007), no. 2, 154-175.
[2] K.E. Gustafson, D.K.M. Rao, Numerical Range, Springer-Verlag, New York, 1997.
[3] G.H. Hardy, J.E. Littlewood, G. Polya, Inequalities, 2nd ed. Cambridge Univ. Press, Cambridge, (1988).
[4] M. Kian , Operator Jensen inequality for superquadratic functions, Linear algebra Appl. 456 (2014), 82-87.
[5] F. Kittaneh, Notes on some inequalities for Hilbert space operators, Pub1. Res. Inst. Math. Sci. 24(1988), 283-293.
[6] F. Kittaneh, Numerical radius inequalities for Hilbert space operators, Pub1. Studia Math. 168(1) (2005), 73-80.
[7] C. A. McCarthy, C_p. Isr. J. Math. 5 (1967), 249–271 .
[8] M. Sattari, M.S. Moslehian, K. Shebrawi, Extension of Euclidean operator radius inequalities, Math. Scand. 120 (2017), no. 1, 129–144.
[9] P.G. Popescu, Unitary invariants in multivariable operator theory, Mem. Amer. Math. Soc. 200 (2009), no. 941.
[10] T. Yamazaki, On upper and lower bounds for the numerical radius and an equality condition, Studia Math. 178 (2007), no. 1, 83-89.