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New Generalizations for Convex Functions via Conformable Fractional Integrals

Alper Ekinci[a], Nazlıcan Eroğlu[b]

[a]*Bandırma Vocational High School, Bandırma Onyedi Eylul University, Balıkesir, Turkey*
[b]*Ağrı İbrahim Çeçen University, Ağrı, Turkey*

Abstract. The main objective of this study is to obtain conformable analogs of some inequalities and to examine some integral inequalities for functions whose modulus of first derivatives are convex and concave. We obtain generalizations including conformable fractional integrals by separating $[a, b]$ interval to s equal subintervals.

1. Introduction

Since inequalities and convex functions are active study topics in most of the fields of mathematics, there is a great deal of interest in this area in recent years. There has been a high increase in the number of studies due to this interest. The concept of convexity encountered in the numerical applications of branches such as industry, business fields, medicine, and art is seen in every aspect of life. For example, we can talk about convexity, even in the balance of chance games. We will start with this important definition: A function $f : [a, b] \to \mathbb{R}$ is said to be convex, if we have

$$f(\alpha x + (1-\alpha) y) \le \alpha f(x) + (1-\alpha) f(y)$$

for all $x, y \in [a, b]$ and $\alpha \in [0, 1]$.

There are many applications of direct or indirect convex functions in mathematical analysis, applied mathematics, probability theory and various other fields of mathematics. However, convex functions are closely related to the inequality theory and many important inequalities are the result of the application of convex functions. For example; General inequalities such as Hölder and Minkowski inequalities are the result of Jensen inequality for convex functions. In this context, it can be said that inequalities have an important place in the theory of convex functions.

Many inequalities were found in the late 19th and early 20th centuries. Some of these inequalities have become fundamental written for the class of convex functions. The inequality, which is expressed by Hermite in 1881 and called Hermite-Hadamard in many sources today, is one of them and can be given as following (See [11], [10]): Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+(b)}{2}$$

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Communicated by Miodrag Spalević
Email addresses: alperekinci@hotmail.com (Alper Ekinci), nazlicaneroglu@hotmail.com (Nazlıcan Eroğlu)

is known as the Hermite-Hadamard inequality.

A motivating inequality of Hadamard type has been proved by Latif and Dragomir in [1] as following:

Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a,b]$ then the following inequality holds:*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \left(\frac{b-a}{96}\right)\left[\left|f'(a)\right|+4\left|f'\left(\frac{3a+b}{4}\right)\right| +2\left|f'\left(\frac{a+b}{2}\right)\right|+4\left|f'\left(\frac{a+3b}{4}\right)\right|+\left|f'(b)\right|\right].$$

In [2], Özdemir et al. presented the following generalization:

Theorem 1.2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° where $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a,b]$ then the following inequality holds:*

$$\left|\sum_{k=0}^{\frac{n-1}{2}} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)-\frac{n+1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{b-a}{6(n+1)}\sum_{k=0}^{\frac{n-1}{2}}\left(4\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right| +\left|f'\left(\frac{a(n-2k+1)+b(2k)}{n+1}\right)\right|+\left|f'\left(\frac{a(n-2k-1)+b(2k+2)}{n+1}\right)\right|\right)$$

where n is an odd number.

In [12], Khalil et al. gave a new definition called "conformable fractional derivative". They not only proved further properties of this definitons but also gave the differences with the other fractional derivatives. Besides, another considerable study have presented by Abdeljawad to discuss the basic concepts of fractional calculus. Scientists stated that these definitions of this new fractional derivative and integral are an understandable, feasible and effective definitions. In [3], Abdeljawad gave the following definitions of right-left conformable fractional integrals:

Definition 1.3. *Let $\alpha \in (n, n+1]$, $n = 0, 1, 2, \ldots$ and set $\beta = \alpha - n$. Then the left and right conformable fractional integral of any order $\alpha > 0$ is defined by respectively*

$$\left(I_\alpha^a f\right)(t) = \frac{1}{n!}\int_a^t (t-x)^n (x-a)^{\beta-1} f(x)\,dx,$$

and

$$\left({}^bI_\alpha f\right)(t) = \frac{1}{n!}\int_t^b (x-t)^n (b-x)^{\beta-1} f(x)\,dx.$$

Let us recall the Beta function defined as follows:

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt, \qquad a, b > 0,$$

where $\Gamma(\alpha)$ is Gamma function. The Incomplete Beta function is defined by

$$B_x(a,b)=\int_0^x t^{a-1}(1-t)^{b-1}\,dt.$$

Based on the above definition, Set and Çelik presented the following identity in [4]:

Lemma 1.4. *Assume that $a,b\in\mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ is a differentiable function on (a,b). If $f'\in L[a,b]$ then the following equality holds:*

$$\begin{aligned}
&\Psi_\alpha(a,b)\\
=\;&\frac{-(b-a)\alpha}{16}\left[\int_0^1 B_t(n+1,\alpha-n)f'\left(ta+(1-t)\frac{3a+b}{4}\right)dt\right.\\
&-\int_0^1 B_{1-t}(\alpha-n,n+1)f'\left(t\frac{3a+b}{4}+(1-t)\frac{a+b}{2}\right)dt\\
&+\int_0^1 B_t(n+1,\alpha-n)f'\left(t\frac{a+b}{2}+(1-t)\frac{a+3b}{4}\right)dt\\
&\left.-\int_0^1 B_{1-t}(\alpha-n,n+1)f'\left(t\frac{a+3b}{4}+(1-t)b\right)dt\right]
\end{aligned}$$

for $\alpha\in(n,n+1]$, $n=0,1,2,...$ where $B_t(.,.)$ is incompleted beta function and

$$\begin{aligned}
&\Psi_\alpha(a,b)\\
=\;&\frac{\alpha}{4}\left[B(n+1,\alpha-n)\left(f(a)+f\left(\frac{a+b}{2}\right)\right)\right.\\
&\left.+B(\alpha-n,n+1)\left(f\left(\frac{a+b}{2}\right)+f(b)\right)\right]-\frac{\alpha 4^{\alpha-1}n!}{(b-a)^\alpha}\\
&\times\left[\left(I_\alpha^a f\right)\left(\frac{3a+b}{4}\right)+\left(I_\alpha^{\frac{3a+b}{4}}f\right)\left(\frac{a+b}{2}\right)+\left(I_\alpha^{\frac{a+b}{2}}f\right)\left(\frac{a+3b}{4}\right)+\left(I_\alpha^{\frac{a+3b}{4}}f\right)(b)\right].
\end{aligned}$$

For the recent studies of inequalities including conformable fractional integrals, we can refer the papers [5–9]. The main aim of this paper is to prove a generalization of Lemma 1.4 and establish some more general integral inequalities for convex functions by using conformable fractional integral operators.

2. Main Results

In order to prove the main results, we need the following integral identity that involve conformable fractional integral operator.

Lemma 2.1. *Let $f:[a,b]\to\mathbb{R}$ is a differentiable mapping on (a,b) where $a,b\in\mathbb{R}$ with $a<b$. If $f'\in L[a,b]$, then the following identity holds:*

$$\begin{aligned}
&\sum_{k=0}^{j-1}\int_0^1\left[B_t(n+1,\alpha-n)f'\left[t\lambda(k+1)+(1-t)\lambda(k)\right]\right]dt\\
=\;&\frac{j}{b-a}\sum_{k=0}^{j-1}\left\{B(n+1,\alpha-n)f[\lambda(k+1)]-n!\left(\frac{j}{b-a}\right)^\alpha\left({}^{\lambda(k+1)}I_\alpha f\right)(\lambda(k))\right\}
\end{aligned}$$

for $\alpha \in (n, n+1]$, $n = 0, 1, 2, ..$*where* $j \in \mathbb{Z}^+$ *and for* $k \in \mathbb{Z}$, $\lambda(k) = \frac{k}{j}(b-a) + a$.

Proof. By using the fact that

$$\lambda(k+1) - \lambda(k) = \frac{b-a}{j},$$

and by integrating by parts, we obtain

$$\begin{aligned}
&\int_0^1 \left[B_t(n+1, \alpha-n) f'[t\lambda(k+1) + (1-t)\lambda(k)]\right] dt \\
= &\frac{j}{b-a} B_t(n+1, \alpha-n) f[t\lambda(k+1) + (1-t)\lambda(k)]\Big|_0^1 \\
&-\frac{j}{b-a}\int_0^1 t^n (1-t)^{\alpha-n-1} f[t\lambda(k+1) + (1-t)\lambda(k)] dt \\
= &\frac{j}{b-a} B(n+1, \alpha-n) f[\lambda(k+1)] \\
&-\frac{j}{b-a}\int_0^1 t^n (1-t)^{\alpha-n-1} f[t\lambda(k+1) + (1-t)\lambda(k)] dt.
\end{aligned}$$

If we make use of the substitution $x = t\lambda(k+1) + (1-t)\lambda(k)$ for the above integral, we get

$$\begin{aligned}
&\int_0^1 t^n (1-t)^{\alpha-n-1} f[t\lambda(k+1) + (1-t)\lambda(k)] dt \\
= &\int_{\lambda(k)}^{\lambda(k+1)} (x-\lambda(k))^n (\lambda(k+1) - x)^{\alpha-n-1} \left(\frac{j}{b-a}\right)^{\alpha} f(x) dx \\
= &n! \left(\frac{j}{b-a}\right)^{\alpha} \left({}^{\lambda(k+1)}I_{\alpha} f\right)(\lambda(k))
\end{aligned}$$

which completes the proof. □

Theorem 2.2. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be differentiable on* I° *such that* $f' \in L[a,b]$ *with* $a, b \in I$, $a < b$ *and* $\alpha > 0$. *If* $|f'|$ *is convex on* $[a,b]$ *then the following inequality holds for conformable fractional integrals:*

$$\begin{aligned}
&\left|\frac{j}{b-a}\sum_{k=0}^{j-1}\left\{B(n+1, \alpha-n) f[\lambda(k+1)] - n!\left(\frac{j}{b-a}\right)^{\alpha}\left({}^{\lambda(k+1)}I_{\alpha} f\right)(\lambda(k))\right\}\right| \\
\leq &\sum_{k=0}^{j-1}\left\{\frac{B(n+1, \alpha-n) - B(n+3, \alpha-n)}{2}\left|f'(\lambda(k+1))\right| + \frac{B(n+1, \alpha-n+2)}{2}\left|f'(\lambda(k))\right|\right\},
\end{aligned}$$

where $j \in \mathbb{Z}^+$, $\alpha \epsilon (n, n+1]$, $n = 0, 1, 2, ...$ *and for* $k \in \mathbb{Z}$, $\lambda(k) = \frac{k}{j}(b-a) + a$.

Proof. Using Lemma 2.1 and property of modulus, we can write

$$\begin{aligned}
&\left|\frac{j}{b-a}\sum_{k=0}^{j-1}\left\{B(n+1,\alpha-n)\,f[\lambda(k+1)]-n!\left(\frac{j}{b-a}\right)^{\alpha}\left({}^{\lambda(k+1)}I_{\alpha}f\right)(\lambda(k))\right\}\right| \\
\leq\ & \sum_{k=0}^{j-1}\int_0^1 B_t(n+1,\alpha-n)\left|f'[t\lambda(k+1)+(1-t)\lambda(k)]\right|dt.
\end{aligned}$$

Since $\left|f'\right|$ is convex on $[a,b]$, then we have

$$\begin{aligned}
&\int_0^1\left[B_t(n+1,\alpha-n)\left|f'[t\lambda(k+1)+(1-t)\lambda(k)]\right|\right]dt \\
\leq\ & \int_0^1 B_t(n+1,\alpha-n)\left[t\left|f'(\lambda(k+1))\right|+(1-t)\left|f'(\lambda(k))\right|\right]dt
\end{aligned}$$

Using the properties of incomplete beta functions and integrating by parts, we obtain;

$$\begin{aligned}
\int_0^1 B_t(n+1,\alpha-n)\,tdt &= B_t(n+1,\alpha-n)\left.\frac{t^2}{2}\right|_0^1-\frac{1}{2}\int_0^1 t^{n+2}(1-t)^{\alpha-n-1}dt \\
&= \frac{B(n+1,\alpha-n)-B(n+3,\alpha-n)}{2}
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 B_t(n+1,\alpha-n)(1-t)\,dt \\
=\ & B_t(n+1,\alpha-n)\left.\left(t-\frac{t^2}{2}\right)\right|_0^1-\int_0^1 t^n(1-t)^{\alpha-n-1}\left(t-\frac{t^2}{2}\right)dt \\
=\ & \frac{B(n+1,\alpha-n)-2B(n+2,\alpha-n)+B(n+3,\alpha-n)}{2} \\
=\ & \frac{B(n+1,\alpha-n+2)}{2}.
\end{aligned}$$

We get the desired result. □

Corollary 2.3. *Under the conditions of Theorem 2.2, if we choose* $j=2,3$ *and* 4, *respectively, we have*

i) Case for $j=2$

$$\begin{aligned}
&\left|\frac{2}{b-a}\left\{B(n+1,\alpha-n)\left[f\left(\frac{a+b}{2}\right)+f(b)\right]\right.\right. \\
&\left.\left.-n!\left(\frac{2}{b-a}\right)^{\alpha}\left[\left({}^{\frac{a+b}{2}}I_{\alpha}f\right)(a)+\left({}^{b}I_{\alpha}f\right)\left(\frac{a+b}{2}\right)\right]\right\}\right| \\
\leq\ & \frac{B(n+1,\alpha-n)-B(n+3,\alpha-n)}{2}\left[\left|f'\left(\frac{a+b}{2}\right)\right|+\left|f'(b)\right|\right] \\
&+\frac{B(n+1,\alpha-n+2)}{2}\left[\left|f'(a)\right|+\left|f'\left(\frac{a+b}{2}\right)\right|\right].
\end{aligned}$$

ii) Case for $j = 3$

$$\begin{aligned}
&\left|\frac{3}{b-a}\left\{B(n+1,\alpha-n)\left[f\left(\frac{2a+b}{3}\right)+f\left(\frac{2b+a}{3}\right)+f(b)\right]\right.\right.\\
&\left.\left.-n!\left(\frac{3}{b-a}\right)^{\alpha}\left[\left({}^{\frac{2a+b}{3}}I_{\alpha}f\right)(a)+\left({}^{\frac{2b+a}{3}}I_{\alpha}f\right)\left(\frac{2a+b}{3}\right)+\left({}^{b}I_{\alpha}f\right)\left(\frac{2b+a}{3}\right)\right]\right\}\right|\\
\leq\ &\frac{B(n+1,\alpha-n)-B(n+3,\alpha-n)}{2}\left[\left|f'\left(\frac{2a+b}{3}\right)\right|+\left|f'\left(\frac{2b+a}{3}\right)\right|+\left|f'(b)\right|\right]\\
&+\frac{B(n+1,\alpha-n+2)}{2}\left[\left|f'(a)\right|+\left|f'\left(\frac{2a+b}{3}\right)\right|+\left|f'\left(\frac{2b+a}{3}\right)\right|\right].
\end{aligned}$$

iii) Case for $j = 4$

$$\begin{aligned}
&\left|\frac{4}{b-a}\left\{B(n+1,\alpha-n)\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+b}{2}\right)+f\left(\frac{3b+a}{4}\right)+f(b)\right]\right.\right.\\
&\left.\left.-n!\left(\frac{4}{b-a}\right)^{\alpha}\left[\left({}^{\frac{3a+b}{4}}I_{\alpha}f\right)(a)+\left({}^{\frac{a+b}{2}}I_{\alpha}f\right)\left(\frac{3a+b}{4}\right)+\left({}^{\frac{3b+a}{4}}I_{\alpha}f\right)\left(\frac{a+b}{2}\right)+\left({}^{b}I_{\alpha}f\right)\left(\frac{3b+a}{4}\right)\right]\right\}\right|\\
\leq\ &\frac{B(n+1,\alpha-n)-B(n+3,\alpha-n)}{2}\left[\left|f'\left(\frac{3a+b}{4}\right)\right|+\left|f'\left(\frac{a+b}{2}\right)\right|+\left|f'\left(\frac{3b+a}{4}\right)\right|+\left|f'(b)\right|\right]\\
&+\frac{B(n+1,\alpha-n+2)}{2}\left[\left|f'(a)\right|+\left|f'\left(\frac{3a+b}{4}\right)\right|+\left|f'\left(\frac{a+b}{2}\right)\right|+\left|f'\left(\frac{3b+a}{4}\right)\right|\right].
\end{aligned}$$

Corollary 2.4. *In Theorem 2.2, if we set $\alpha = 1$ and $n = 0$, one can obtain*

$$\begin{aligned}
&\left|\frac{j}{b-a}\sum_{k=0}^{j-1}\left\{f\left[\lambda(k+1)\right]-\frac{j}{b-a}\int_{\lambda(k)}^{\lambda(k+1)}f(x)\,dx\right\}\right|\\
\leq\ &\sum_{k=0}^{j-1}\left\{\frac{1}{3}\left|f'(\lambda(k+1))\right|+\frac{1}{6}\left|f'(\lambda(k))\right|\right\}.
\end{aligned}$$

Theorem 2.5. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable on I° such that $f' \in L[a,b]$ with $a, b \in I$, $a < b$ and $\alpha > 0$. If $\left|f'\right|^{q}$ is convex on $[a,b]$ and $q > 1$ then the following inequality holds for conformable fractional integrals:*

$$\begin{aligned}
&\left|\frac{j}{b-a}\sum_{k=0}^{j-1}\left\{B(n+1,\alpha-n)\,f\left[\lambda(k+1)\right]-n!\left(\frac{j}{b-a}\right)^{\alpha}\left({}^{\lambda(k+1)}I_{\alpha}f\right)(\lambda(k))\right\}\right|\\
\leq\ &\left(\int_{0}^{1}\left|B_t(n+1,\alpha-n)\right|^{p}dt\right)^{\frac{1}{p}}2^{-\frac{1}{q}}\sum_{k=0}^{j-1}\left\{\left|f'(\lambda(k+1))\right|^{q}+\left|f'(\lambda(k))\right|^{q}\right\}^{\frac{1}{q}},
\end{aligned}$$

where $j \in \mathbb{Z}^{+}$, $\frac{1}{p}+\frac{1}{q}=1$, $\alpha\epsilon(n,n+1]$, $n = 0,1,2,\ldots$ and for $k \in \mathbb{Z}$, $\lambda(k) = \frac{k}{j}(b-a)+a$.

Proof. By using Lemma 2.1 and Hölder inequality, we obtain

$$\begin{aligned}
&\left|\frac{j}{b-a}\sum_{k=0}^{j-1}\left\{B(n+1,\alpha-n)f[\lambda(k+1)]-n!\left(\frac{j}{b-a}\right)^{\alpha}\left({}^{\lambda(k+1)}I_{\alpha}f\right)(\lambda(k))\right\}\right| \qquad (1)\\
\leq\ & \sum_{k=0}^{j-1}\int_0^1 B_t(n+1,\alpha-n)\left|f'[t\lambda(k+1)+(1-t)\lambda(k)]\right|dt\\
\leq\ & \left(\int_0^1 |B_t(n+1,\alpha-n)|^p dt\right)^{\frac{1}{p}}\sum_{k=0}^{j-1}\left(\int_0^1\left|f'[t\lambda(k+1)+(1-t)\lambda(k)]\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $\left|f'\right|^q$ is convex on $[a,b]$, we can write

$$\begin{aligned}
&\int_0^1\left|f'[t\lambda(k+1)+(1-t)\lambda(k)]\right|^q dt\\
\leq\ & \int_0^1\left[t\left|f'(\lambda(k+1))\right|^q+(1-t)\left|f'(\lambda(k))\right|^q\right]dt\\
=\ & \frac{1}{2}\left\{\left|f'(\lambda(k+1))\right|^q+\left|f'(\lambda(k))\right|^q\right\}.
\end{aligned}$$

Writing these expressions in (1) and simplifying the inequality, we get the desired result. □

Corollary 2.6. *Under the conditions of Theorem 2.5, if we choose $j=2$, we have*

$$\begin{aligned}
&\left|\frac{2}{b-a}\left\{B(n+1,\alpha-n)\left[f\left(\frac{a+b}{2}\right)+f(b)\right]\right.\right.\\
&\left.\left.-n!\left(\frac{2}{b-a}\right)^{\alpha}\left[\left({}^{\frac{a+b}{2}}I_{\alpha}f\right)(a)+\left({}^{b}I_{\alpha}f\right)\left(\frac{a+b}{2}\right)\right]\right\}\right|\\
\leq\ & \left(\int_0^1 |B_t(n+1,\alpha-n)|^p dt\right)^{\frac{1}{p}}\\
\times\ & 2^{-\frac{1}{q}}\left\{\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(a)\right|^q\right]^{\frac{1}{q}}+\left[\left|f'(b)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Corollary 2.7. *In Theorem 2.5, if we set $\alpha=1$ and $n=0$, we obtain the following inequality;*

$$\begin{aligned}
&\left|\frac{j}{b-a}\sum_{k=0}^{j-1}\left\{f[\lambda(k+1)]-\frac{j}{b-a}\int_{\lambda(k)}^{\lambda(k+1)}f(x)\,dx\right\}\right|\\
\leq\ & \left(\frac{1}{p+1}\right)^{\frac{1}{p}}(2)^{-\frac{1}{q}}\sum_{k=0}^{j-1}\left\{\left|f'(\lambda(k+1))\right|^q+\left|f'(\lambda(k))\right|^q\right\}^{\frac{1}{q}}.
\end{aligned}$$

Theorem 2.8. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable on I° such that $f' \in L[a,b]$ with $a,b \in I$, $a < b$ and $\alpha > 0$. If $|f'|^q$ is convex on $[a,b]$ and $q \geq 1$ then the following inequality holds for conformable fractional integrals:*

$$\begin{aligned}
&\left|\frac{j}{b-a}\sum_{k=0}^{j-1}\left\{B(n+1,\alpha-n)\,f[\lambda(k+1)] - n!\left(\frac{j}{b-a}\right)^{\alpha}\left({}^{\lambda(k+1)}I_\alpha f\right)(\lambda(k))\right\}\right| \\
\leq\;& (B(n+1,\alpha-n+1))^{1-\frac{1}{q}}\sum_{k=0}^{j-1}\left\{\frac{B(n+1,\alpha-n)-B(n+3,\alpha-n)}{2}\left|f'(\lambda(k+1))\right|^q\right. \\
&\left.+\frac{B(n+1,\alpha-n+2)}{2}\left|f'(\lambda(k))\right|^q\right\}^{\frac{1}{q}},
\end{aligned}$$

where $j \in \mathbb{Z}^+$, $\alpha \epsilon (n, n+1]$, $n = 0,1,2,...$ and for $k \in \mathbb{Z}$, $\lambda(k) = \frac{k}{j}(b-a)+a$.

Proof. By using Lemma 2.1 and power mean inequality, we have

$$\begin{aligned}
&\left|\frac{j}{b-a}\sum_{k=0}^{j-1}\left\{B(n+1,\alpha-n)\,f[\lambda(k+1)] - n!\left(\frac{j}{b-a}\right)^{\alpha}\left({}^{\lambda(k+1)}I_\alpha f\right)(\lambda(k))\right\}\right| \\
\leq\;& \sum_{k=0}^{j-1}\int_0^1 B_t(n+1,\alpha-n)\left|f'[t\lambda(k+1)+(1-t)\lambda(k)]\right|dt \\
\leq\;& \left(\int_0^1 B_t(n+1,\alpha-n)\,dt\right)^{1-\frac{1}{q}}\sum_{k=0}^{j-1}\left(\int_0^1 B_t(n+1,\alpha-n)\left|f'[t\lambda(k+1)+(1-t)\lambda(k)]\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

By using integrating by parts, we get

$$\begin{aligned}
\int_0^1 B_t(n+1,\alpha-n)\,dt &= B_t(n+1,\alpha-n)\,t|_0^1 - \int_0^1 t^{n+1}(1-t)^{\alpha-n-1}dt \\
&= B(n+1,\alpha-n) - B(n+2,\alpha-n) \\
&= B(n+1,\alpha-n+1)
\end{aligned}$$

Since $|f'|^q$ is convex on $[a,b]$, then we have

$$\begin{aligned}
&\int_0^1 B_t(n+1,\alpha-n)\left|f'[t\lambda(k+1)+(1-t)\lambda(k)]\right|^q dt \\
\leq\;& \int_0^1 B_t(n+1,\alpha-n)\left[t\left|f'(\lambda(k+1))\right|^q + (1-t)\left|f'(\lambda(k))\right|^q\right]dt \\
=\;& \frac{B(n+1,\alpha-n)-B(n+3,\alpha-n)}{2}\left|f'(\lambda(k+1))\right|^q + \frac{B(n+1,\alpha-n+2)}{2}\left|f'(\lambda(k))\right|^q.
\end{aligned}$$

Combining these results completes the proof. □

Corollary 2.9. *Under the conditions of Theorem 2.8, if we choose $j = 2$, we have*

$$\begin{aligned}
&\left|\frac{2}{b-a}\left\{B(n+1,\alpha-n)\left[f\left(\frac{a+b}{2}\right)+f(b)\right]\right.\right.\\
&\left.\left.-n!\left(\frac{2}{b-a}\right)^{\alpha}\left[\left({}^{\frac{a+b}{2}}I_{\alpha}f\right)(a)+\left({}^{b}I_{\alpha}f\right)\left(\frac{a+b}{2}\right)\right]\right\}\right|\\
\leq\ &(B(n+1,\alpha-n+1))^{1-\frac{1}{q}}\\
&\times\left\{\left[\frac{B(n+1,\alpha-n)-B(n+3,\alpha-n)}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^{q}+\frac{B(n+1,\alpha-n+2)}{2}\left|f'(a)\right|^{q}\right]^{\frac{1}{q}}\right.\\
&\left.+\left[\frac{B(n+1,\alpha-n)-B(n+3,\alpha-n)}{2}\left|f'(b)\right|^{q}+\frac{B(n+1,\alpha-n+2)}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^{q}\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Corollary 2.10. *In Theorem 2.8, if we take $\alpha=1$ and $n=0$, we have*

$$\begin{aligned}
&\left|\frac{j}{b-a}\sum_{k=0}^{j-1}\left\{f[\lambda(k+1)]-\frac{j}{b-a}\int_{\lambda(k)}^{\lambda(k+1)}f(x)\,dx\right\}\right|\\
\leq\ &\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\sum_{k=0}^{j-1}\left\{\frac{1}{3}\left|f'(\lambda(k+1))\right|^{q}+\frac{1}{6}\left|f'(\lambda(k))\right|^{q}\right\}^{\frac{1}{q}}.
\end{aligned}$$

Theorem 2.11. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be differentiable on I° such that $f'\in L[a,b]$ with $a,b\in I$, $a<b$ and $\alpha>0$. If $\left|f'\right|^{q}$ is concave on $[a,b]$ and $q>1$ then the following inequality holds for conformable fractional integrals:*

$$\begin{aligned}
&\left|\frac{j}{b-a}\sum_{k=0}^{j-1}\left\{B(n+1,\alpha-n)f[\lambda(k+1)]-n!\left(\frac{j}{b-a}\right)^{\alpha}\left({}^{\lambda(k+1)}I_{\alpha}f\right)(\lambda(k))\right\}\right|\\
\leq\ &\left(\int_0^1\left|B_t(n+1,\alpha-n)\right|^{\frac{q}{q-1}}dt\right)^{\frac{q-1}{q}}\sum_{k=0}^{j-1}\left|f'\left(\frac{\lambda(k+1)+\lambda(k)}{2}\right)\right|
\end{aligned}$$

where $j\in\mathbb{Z}^{+}$, $\frac{1}{p}+\frac{1}{q}=1$, $\alpha\epsilon(n,n+1]$, $n=0,1,2,\ldots$ and for $k\in\mathbb{Z}$, $\lambda(k)=\frac{k}{j}(b-a)+a$.

Proof. From Lemma 2.1 and by using the Hölder integral inequality for $q>1$ and $p=\frac{q}{q-1}$, we have

$$\begin{aligned}
&\left|\frac{j}{b-a}\sum_{k=0}^{j-1}\left\{B(n+1,\alpha-n)f[\lambda(k+1)]-n!\left(\frac{j}{b-a}\right)^{\alpha}\left({}^{\lambda(k+1)}I_{\alpha}f\right)(\lambda(k))\right\}\right|\\
\leq\ &\left(\int_0^1\left|B_t(n+1,\alpha-n)\right|^{\frac{q}{q-1}}dt\right)^{\frac{q-1}{q}}\sum_{k=0}^{j-1}\left(\int_0^1\left|f'[t\lambda(k+1)+(1-t)\lambda(k)]\right|^{q}dt\right)^{\frac{1}{q}}
\end{aligned}$$

Since $\left|f'\right|^{q}$ is concave on $[a,b]$, by using the Hadamard inequality for concave functions, we have

$$\begin{aligned}
&\int_0^1\left|f'[t\lambda(k+1)+(1-t)\lambda(k)]\right|^{q}dt\\
\leq\ &\left|f'\left(\frac{\lambda(k+1)+\lambda(k)}{2}\right)\right|^{q}
\end{aligned}$$

By combining these inequalities we get the desired result. □

Corollary 2.12. *Under the conditions of Theorem 2.11, if we choose $j = 2$ and $j = 3$, we have*

i) Case for $j = 2$

$$\left|\frac{2}{b-a}\left\{B(n+1,\alpha-n)\left[f\left(\frac{a+b}{2}\right)+f(b)\right]\right.\right.$$
$$\left.\left.-n!\left(\frac{2}{b-a}\right)^{\alpha}\left[\left({}^{\frac{a+b}{2}}I_{\alpha}f\right)(a)+\left({}^{b}I_{\alpha}f\right)\left(\frac{a+b}{2}\right)\right]\right\}\right|$$
$$\leq \left(\int_0^1 |B_t(n+1,\alpha-n)|^{\frac{q}{q-1}}dt\right)^{\frac{q-1}{q}}\left[\left|f'\left(\frac{3a+b}{4}\right)\right|+\left|f'\left(\frac{3b+a}{4}\right)\right|\right].$$

ii) Case for $j = 3$

$$\left|\frac{3}{b-a}\left\{B(n+1,\alpha-n)\left[f\left(\frac{2a+b}{3}\right)+f\left(\frac{2b+a}{3}\right)+f(b)\right]\right.\right.$$
$$\left.\left.-n!\left(\frac{3}{b-a}\right)^{\alpha}\left[\left({}^{\frac{2a+b}{3}}I_{\alpha}f\right)(a)+\left({}^{\frac{2b+a}{3}}I_{\alpha}f\right)\left(\frac{2a+b}{3}\right)+\left({}^{b}I_{\alpha}f\right)\left(\frac{2b+a}{3}\right)\right]\right\}\right|$$
$$\leq \left(\int_0^1 |B_t(n+1,\alpha-n)|^{\frac{q}{q-1}}dt\right)^{\frac{q-1}{q}}\left[\left|f'\left(\frac{5a+b}{6}\right)\right|+\left|f'\left(\frac{a+b}{2}\right)\right|+\left|f'\left(\frac{5b+a}{6}\right)\right|\right].$$

Corollary 2.13. *In Theorem 2.11 if we take $\alpha = 1$ and $n = 0$, we have*

$$\left|\frac{j}{b-a}\sum_{k=0}^{j-1}\left\{f[\lambda(k+1)]-\frac{j}{b-a}\int_{\lambda(k)}^{\lambda(k+1)}f(x)\,dx\right\}\right|$$
$$\leq \left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}}\sum_{k=0}^{j-1}\left|f'\left(\frac{\lambda(k+1)+\lambda(k)}{2}\right)\right|.$$

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