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On the Mixed-Type Generalized Inverses of the Products of Two Operators

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Abstract. Let A, B and AB be closed range operators. The explicit matrix expressions for various generalized inverses are obtained by using block operator matrix methods. Some subtle relationships between the properties of sub-blocks in operator matrices A, B and their range relations are built. New necessary and sufficient conditions for the equivalent relations, inclusion relations and mixed-type generalized inverses relations are presented. Some recent mixed-type reverse-order laws results are covered and many new mixed-type generalized inverses relations are established by using this block-operator matrix technique.

1. Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces. We denote the set of all bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$ by $\mathcal{B}(\mathcal{H},\mathcal{K})$. For $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$, let A^*, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ be the adjoint, the range and the null space of A, respectively. $I_{\mathcal{M}}$ denotes the identity onto $\mathcal{M}$ or I if there is no confusion. A generalized inverse of A is an operator $G \in \mathcal{B}(\mathcal{K},\mathcal{H})$ which satisfies some of the following four equations, which are said to be the Penrose conditions:

$$(1)\ AGA = A, \quad (2)\ GAG = G, \quad (3)\ (AG)^* = AG, \quad (4)\ (GA)^* = GA.$$

For a subset $\{K\} \subseteq \{1,2,3,4\}$, we say that G is a $\{K\}$-inverse of A if G satisfies the Moore-Penrose equation (j) for each $j \in K$. We use $A\{K\}$ for the collection of all K-inverses of A. The unique $\{1,2,3,4\}$-inverse of A is denoted by $A^\dagger$, which is called the Moore-Penrose inverse of A.

In the 1960s, Greville was the first to study it by considering the reverse order law for the Moore-Penrose inverse and gave a classical result

$$(AB)^\dagger = B^\dagger A^\dagger \Longleftrightarrow \mathcal{R}(A^*AB) \subseteq \mathcal{R}(B), \quad \mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$$

for complex matrices A and B in [15]. This result was extended for linear bounded operators on Hilbert spaces by Bouldin [1] and Izumino [16]. Many scholars have considered the mixed-type generalized inverses in different settings (matrix, operator algebras, C^*-algebras, rings etc). For two matrices cases, the

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necessary and sufficient conditions for $B\{1,2,3\}A\{1,2,3\} \subseteq (AB)\{1,2,3\}$ and $B\{1,2,4\}A\{1,2,4\} \subseteq (AB)\{1,2,4\}$ are presented in [26] by using the expressions for maximal and minimal ranks of the generalized Schur-complement. The necessary and sufficient conditions for $(AB)\{1,3\} \subseteq B\{1,3\}A\{1,3\}$ and $(AB)\{1,4\} \subseteq B\{1,4\}A\{1,4\}$ are presented in [3]. See [20, 22, 24, 25, 27, 28] for more matrix cases. X. Liu, S. Huang and D.S. Cvetković-Ilić in [17], J. Wang, H. Zhang and G. Ji in [21] considered the necessary and sufficient conditions for reverse order law in the case of bounded linear operators on Hilbert spaces. D.S. Cvetković-Ilić and Harte [4] offered purely algebraic necessary and sufficient conditions for reverse order law $B\{1,3\}A\{1,3\} \subseteq (AB)\{1,3\}$ for generalized inverses in C*-algebras, extending rank conditions for matrices and range conditions for Hilbert space operators. Some more contributors in this area can be seen in [5–12, 23].

In this paper, by the block operator matrix technique, we obtain the necessary and sufficient conditions for which the equivalent relations or the inclusion relations among some mixed-type generalized inverses hold. Specifically, we discuss the mixed-type $\{K\}$-generalized inverses and relations among $A\{K\}$, $B\{K\}$ and $(AB)\{K\}$ when $\{K\} \in \{\{1\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{1,2,3,4\}\}$, present their detailed matrix expressions and build their relations. Many mixed-type generalized inverses relations are established by using this block-operator matrix technique.

2. The matrix representations of two operators A and B

Throughout this paper, we suppose that $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and $B \in \mathcal{B}(\mathcal{L},\mathcal{H})$ with $\mathcal{R}(A)$, $\mathcal{R}(B)$ and $\mathcal{R}(AB)$ being closed. Also, suppose that $AB \neq 0$. It is well known that A, as an operator from $\mathcal{H} = \mathcal{R}(A^*) \oplus \mathcal{N}(A)$ into $\mathcal{K} = \mathcal{R}(A) \oplus \mathcal{N}(A^*)$, has the diagonal matrix form $A = A_1 \oplus 0$, where $A_1 \in \mathcal{B}(\mathcal{R}(A^*), \mathcal{R}(A))$ is invertible. In this case, the Moore-Penrose inverse $A^\dagger$ of A can be represented by $A^\dagger = A_1^{-1} \oplus 0$.

We begin with the following auxiliary notations [2, 17–19, 21]. Denote by

$$\begin{cases} \mathcal{H}_1 &= \mathcal{R}(A^*) \ominus \mathcal{H}_2, \\ \mathcal{H}_2 &= \mathcal{R}(A^*) \cap \mathcal{N}(B^*), \\ \mathcal{H}_3 &= \mathcal{N}(A) \ominus \mathcal{H}_4, \\ \mathcal{H}_4 &= \mathcal{N}(A) \cap \mathcal{N}(B^*), \end{cases} \quad \begin{cases} \mathcal{K}_1 &= \mathcal{R}(A) \ominus \mathcal{K}_2, \\ \mathcal{K}_2 &= (A^*)^\dagger \mathcal{H}_2, \\ \mathcal{K}_3 &= \mathcal{N}(A^*), \end{cases} \quad \begin{cases} \mathcal{L}_1 &= \mathcal{R}(B^*A^*), \\ \mathcal{L}_2 &= \mathcal{R}(B^*) \ominus \mathcal{L}_1, \\ \mathcal{L}_3 &= \mathcal{N}(B). \end{cases} \tag{1}$$

Note that, if $\mathcal{M}$ and $\mathcal{N}$ are two closed subspaces of $\mathcal{H}$, the orthogonal different is defined by $\mathcal{M} \ominus \mathcal{N} = \mathcal{M} \cap \mathcal{N}^\perp$. The space decomposition forms in (1) first appeared in the paper [21, Theorem 1]. It was also used in the papers [17, 18, 29, 30]. Then

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4, \quad \mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \mathcal{K}_3, \quad \mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$$

and

$$A = \begin{pmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{R}(A^*) \ominus \mathcal{H}_2 \\ \mathcal{R}(A^*) \cap \mathcal{N}(B^*) \\ \mathcal{N}(A) \ominus \mathcal{H}_4 \\ \mathcal{N}(A) \cap \mathcal{N}(B^*) \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(A) \ominus \mathcal{K}_2 \\ (A^*)^\dagger \mathcal{H}_2 \\ \mathcal{N}(A^*) \end{pmatrix},$$

$$B = \begin{pmatrix} B_{11} & B_{12} & 0 \\ 0 & 0 & 0 \\ B_{31} & B_{32} & 0 \\ 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{R}(B^*A^*) \\ \mathcal{R}(B^*) \ominus \mathcal{L}_1 \\ \mathcal{N}(B) \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(A^*) \ominus \mathcal{H}_2 \\ \mathcal{R}(A^*) \cap \mathcal{N}(B^*) \\ \mathcal{N}(A) \ominus \mathcal{H}_4 \\ \mathcal{N}(A) \cap \mathcal{N}(B^*) \end{pmatrix},$$

where $A_{ij} \in \mathcal{B}(\mathcal{H}_j, \mathcal{K}_i)$ and $B_{ij} \in \mathcal{B}(\mathcal{L}_j, \mathcal{H}_i)$. Taking $*$-operation,

$$A^* = \begin{pmatrix} A_{11}^* & A_{21}^* & 0 \\ A_{12}^* & A_{22}^* & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{K}_3 \end{pmatrix} \to \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{pmatrix}.$$

Since

$$A^*\mathcal{K}_2 = A^*(A^*)^\dagger\mathcal{H}_2 = A^\dagger A\mathcal{H}_2 = \mathcal{H}_2 = \mathcal{R}(A^*) \cap \mathcal{N}(B^*),$$

we get that $A^*_{21} = 0$ and A^*_{22} is surjective. Since $\mathcal{K}_2 \subseteq \mathcal{R}(A) = \mathcal{N}(A^*)^\perp$, we get that A^*_{22} is injective. Hence, A^*_{22} is invertible. Since A^*, as an operator from $\mathcal{K} = \mathcal{R}(A) \oplus \mathcal{N}(A^*)$ into $\mathcal{H} = \mathcal{R}(A^*) \oplus \mathcal{N}(A)$ has the diagonal matrix form $A^* = A^*_1 \oplus 0$ with $A^*_1 =: \begin{pmatrix} A^*_{11} & 0 \\ A^*_{12} & A^*_{22} \end{pmatrix} \in \mathcal{B}(\mathcal{K}_1 \oplus \mathcal{K}_2, \mathcal{H}_1 \oplus \mathcal{H}_2)$ being invertible, the invertibility of A^*_1 and A^*_{22} imply that A^*_{11} is invertible. Hence,

$$A = \begin{pmatrix} A_{11} & A_{12} & 0 & 0 \\ 0 & A_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{pmatrix} \to \begin{pmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{K}_3 \end{pmatrix}, \quad \text{where } A_{11}, A_{22} \text{ are invertible.} \tag{2}$$

Similarly, from

$$B^* = \begin{pmatrix} B^*_{11} & 0 & B^*_{31} & 0 \\ B^*_{12} & 0 & B^*_{32} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{R}(A^*) \ominus \mathcal{H}_2 \\ \mathcal{R}(A^*) \cap \mathcal{N}(B^*) \\ \mathcal{N}(A) \ominus \mathcal{H}_4 \\ \mathcal{N}(A) \cap \mathcal{N}(B^*) \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(B^*A^*) \\ \mathcal{R}(B^*) \ominus \mathcal{L}_1 \\ \mathcal{N}(B) \end{pmatrix},$$

it is obvious that $B^*_{12} = 0$, $\mathcal{R}(B^*_{11}) = \mathcal{L}_1$ and B^*_{11} is invertible. Since $\mathcal{L}_1 \oplus \mathcal{L}_2 = \mathcal{R}(B^*) = \mathcal{R}(B^*_{11}) \oplus \mathcal{R}(B^*_{32})$, we get B^*_{32} is surjective and a closed range operator, i.e., $B^\dagger_{32}B_{32} = I$. Hence,

$$B = \begin{pmatrix} B_{11} & 0 & 0 \\ 0 & 0 & 0 \\ B_{31} & B_{32} & 0 \\ 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{L}_3 \end{pmatrix} \to \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{pmatrix}, \quad \text{where } B_{11} \text{ is invertible and } B^\dagger_{32}B_{32} = I. \tag{3}$$

Throughout this paper, we assume that A and B have the matrix forms (2) and (3), respectively and denote by

$$\Delta =: \left[B^*_{11}B_{11} + B^*_{31}(I - B_{32}B^\dagger_{32})B_{31}\right]^{-1}. \tag{4}$$

Next, we present the explicit matrix expressions for K-generalized inverses $A\{K\}$, $B\{K\}$ and $(AB)\{K\}$ when $\{K\} \in \{\{1\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{1,2,3,4\}\}$. The following results are elementary but useful.

Theorem 2.1. *Let A and B be denoted as the matrix forms (2) and (3), respectively. Then the following results hold.*
(i) *The generalized inverses $A\{K\}$ have the representations:*

$$A\{1\} = \begin{pmatrix} A^{-1}_{11} & -A^{-1}_{11}A_{12}A^{-1}_{22} & X_{13} \\ 0 & A^{-1}_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \\ X_{41} & X_{42} & X_{43} \end{pmatrix}; \quad A\{1,3\} = \begin{pmatrix} A^{-1}_{11} & -A^{-1}_{11}A_{12}A^{-1}_{22} & 0 \\ 0 & A^{-1}_{22} & 0 \\ X_{31} & X_{32} & X_{33} \\ X_{41} & X_{42} & X_{43} \end{pmatrix}; \quad A\{1,4\} = \begin{pmatrix} A^{-1}_{11} & -A^{-1}_{11}A_{12}A^{-1}_{22} & X_{13} \\ 0 & A^{-1}_{22} & X_{23} \\ 0 & 0 & X_{33} \\ 0 & 0 & X_{43} \end{pmatrix};$$

$$A\{1,2,3\} = \begin{pmatrix} A^{-1}_{11} & -A^{-1}_{11}A_{12}A^{-1}_{22} & 0 \\ 0 & A^{-1}_{22} & 0 \\ X_{31} & X_{32} & 0 \\ X_{41} & X_{42} & 0 \end{pmatrix}; \; A\{1,2,4\} = \begin{pmatrix} A^{-1}_{11} & -A^{-1}_{11}A_{12}A^{-1}_{22} & X_{13} \\ 0 & A^{-1}_{22} & X_{23} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \; A\{1,3,4\} = \begin{pmatrix} A^{-1}_{11} & -A^{-1}_{11}A_{12}A^{-1}_{22} & 0 \\ 0 & A^{-1}_{22} & 0 \\ 0 & 0 & X_{33} \\ 0 & 0 & X_{43} \end{pmatrix};$$

$$A\{1,2\} = \begin{pmatrix} A^{-1}_{11} & -A^{-1}_{11}A_{12}A^{-1}_{22} & X_{13} \\ 0 & A^{-1}_{22} & X_{23} \\ X_{31} & X_{32} & X_{31}A_{11}X_{13} + (X_{31}A_{12} + X_{32}A_{22})X_{23} \\ X_{41} & X_{42} & X_{41}A_{11}X_{13} + (X_{41}A_{12} + X_{42}A_{22})X_{23} \end{pmatrix}; \quad A^\dagger = A\{1,2,3,4\} = \begin{pmatrix} A^{-1}_{11} & -A^{-1}_{11}A_{12}A^{-1}_{22} & 0 \\ 0 & A^{-1}_{22} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $X_{m,n}$, $m = 1,2,3,4$, $n = 1,2,3$ are arbitrary.

(ii) *The generalized inverses $B\{K\}$ have the representations:*

$$B\{1\} = \begin{pmatrix} B_{11}^{-1} - Y_{13}B_{31}B_{11}^{-1} & Y_{12} & Y_{13} & Y_{14} \\ -Y_{23}B_{31}B_{11}^{-1} & Y_{22} & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & Y_{33} & Y_{34} \end{pmatrix} \quad \textit{with } Y_{13}B_{32} = 0, \quad Y_{23}B_{32} = I;$$

$$B\{1,2\} = \begin{pmatrix} B_{11}^{-1} - Y_{13}B_{31}B_{11}^{-1} & Y_{12} & Y_{13} & Y_{14} \\ -Y_{23}B_{31}B_{11}^{-1} & Y_{22} & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & Y_{33} & Y_{34} \end{pmatrix} \quad \textit{with} \begin{cases} Y_{13}B_{32} = 0, \quad Y_{23}B_{32} = I, \\ Y_{32} = (Y_{31}B_{11} + Y_{33}B_{31})Y_{12} + Y_{33}B_{32}Y_{22}, \\ Y_{33} = (Y_{31}B_{11} + Y_{33}B_{31})Y_{13} + Y_{33}B_{32}Y_{23}, \\ Y_{34} = (Y_{31}B_{11} + Y_{33}B_{31})Y_{14} + Y_{33}B_{32}Y_{24}; \end{cases}$$

$$B\{1,4\} = \begin{pmatrix} B_{11}^{-1} - Y_{13}B_{31}B_{11}^{-1} & Y_{12} & Y_{13} & Y_{14} \\ -Y_{23}B_{31}B_{11}^{-1} & Y_{22} & Y_{23} & Y_{24} \\ -Y_{33}B_{31}B_{11}^{-1} & Y_{32} & Y_{33} & Y_{34} \end{pmatrix} \quad \textit{with } Y_{13}B_{32} = 0, \quad Y_{23}B_{32} = I, \quad Y_{33}B_{32} = 0;$$

$$B\{1,2,4\} = \begin{pmatrix} B_{11}^{-1} - Y_{13}B_{31}B_{11}^{-1} & Y_{12} & Y_{13} & Y_{14} \\ -Y_{23}B_{31}B_{11}^{-1} & Y_{22} & Y_{23} & Y_{24} \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \textit{with } Y_{13}B_{32} = 0, \quad Y_{23}B_{32} = I,$$

where $Y_{m,n}$, $m = 1,2,3$, $n = 1,2,3,4$ are arbitrary. Let Δ be defined by (4). Then

$$B\{1,3\} = \begin{pmatrix} \Delta B_{11}^{*} & 0 & \Delta B_{31}^{*}(I - B_{32}B_{32}^{\dagger}) & 0 \\ -B_{32}^{\dagger}B_{31}\Delta B_{11}^{*} & 0 & B_{32}^{\dagger} - B_{32}^{\dagger}B_{31}\Delta B_{31}^{*}(I - B_{32}B_{32}^{\dagger}) & 0 \\ Y_{31} & Y_{32} & Y_{33} & Y_{34} \end{pmatrix};$$

$$B\{1,3,4\} = \begin{pmatrix} \Delta B_{11}^{*} & 0 & \Delta B_{31}^{*}(I - B_{32}B_{32}^{\dagger}) & 0 \\ -B_{32}^{\dagger}B_{31}\Delta B_{11}^{*} & 0 & B_{32}^{\dagger} - B_{32}^{\dagger}B_{31}\Delta B_{31}^{*}(I - B_{32}B_{32}^{\dagger}) & 0 \\ -Y_{33}B_{31}B_{11}^{-1} & Y_{32} & Y_{33} & Y_{34} \end{pmatrix} \quad \textit{with } Y_{33}B_{32} = 0;$$

$$B\{1,2,3\} = \begin{pmatrix} \Delta B_{11}^{*} & 0 & \Delta B_{31}^{*}(I - B_{32}B_{32}^{\dagger}) & 0 \\ -B_{32}^{\dagger}B_{31}\Delta B_{11}^{*} & 0 & B_{32}^{\dagger} - B_{32}^{\dagger}B_{31}\Delta B_{31}^{*}(I - B_{32}B_{32}^{\dagger}) & 0 \\ Y_{31} & 0 & Y_{33} & 0 \end{pmatrix} \quad \textit{with} \begin{cases} Y_{33}(I - B_{32}B_{32}^{\dagger})\left(I - B_{31}\Delta B_{31}^{*}\right)(I - B_{32}B_{32}^{\dagger}) \\ \qquad = Y_{31}B_{11}\Delta B_{31}^{*}(I - B_{32}B_{32}^{\dagger}); \end{cases}$$

$$B^{\dagger} = \begin{pmatrix} \Delta B_{11}^{*} & 0 & \Delta B_{31}^{*}(I - B_{32}B_{32}^{\dagger}) & 0 \\ -B_{32}^{\dagger}B_{31}\Delta B_{11}^{*} & 0 & B_{32}^{\dagger} - B_{32}^{\dagger}B_{31}\Delta B_{31}^{*}(I - B_{32}B_{32}^{\dagger}) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $Y_{3,n}$, $n = 1,2,3,4$ are arbitrary.

(iii) *The generalized inverses $(AB)\{K\}$ have the representations:*

$$(AB)\{1\} = \begin{pmatrix} B_{11}^{-1}A_{11}^{-1} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{pmatrix}; \quad (AB)\{1,3\} = \begin{pmatrix} B_{11}^{-1}A_{11}^{-1} & 0 & 0 \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{pmatrix}; \quad (AB)\{1,4\} = \begin{pmatrix} B_{11}^{-1}A_{11}^{-1} & Z_{12} & Z_{13} \\ 0 & Z_{22} & Z_{23} \\ 0 & Z_{32} & Z_{33} \end{pmatrix};$$

$$(AB)\{1,2,3\} = \begin{pmatrix} B_{11}^{-1}A_{11}^{-1} & 0 & 0 \\ Z_{21} & 0 & 0 \\ Z_{31} & 0 & 0 \end{pmatrix}; \; (AB)\{1,2,4\} = \begin{pmatrix} B_{11}^{-1}A_{11}^{-1} & Z_{12} & Z_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \; (AB)\{1,3,4\} = \begin{pmatrix} B_{11}^{-1}A_{11}^{-1} & 0 & 0 \\ 0 & Z_{22} & Z_{23} \\ 0 & Z_{32} & Z_{33} \end{pmatrix};$$

$$(AB)\{1,2\} = \begin{pmatrix} B_{11}^{-1}A_{11}^{-1} & Z_{12} & Z_{13} \\ Z_{21} & Z_{21}A_{11}B_{11}Z_{12} & Z_{21}A_{11}B_{11}Z_{13} \\ Z_{31} & Z_{31}A_{11}B_{11}Z_{12} & Z_{31}A_{11}B_{11}Z_{13} \end{pmatrix}; \quad (AB)^{\dagger} = (AB)\{1,2,3,4\} = \begin{pmatrix} B_{11}^{-1}A_{11}^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $Z_{m,n}$, $m = 1,2,3$, $n = 1,2,3$ are arbitrary.

Proof. The results in items (i) and (iii) can be gotten by using the definition of K-inverses. Here, we only show that the results in item (ii) hold. Let $Y = (Y_{ij})_{1 \le i \le 3, 1 \le j \le 4}$. By solving the operator equations $BYB = B$, $YBY = Y$, $(BY)^* = BY$ and $(YB)^* = YB$, respectively, one gets that $B\{1\}$, $B\{1,2\}$, $B\{1,4\}$, $B\{1,2,4\}$ have the representations as in item (ii) and

$$B\{1,3\} = \begin{pmatrix} B_{11}^{-1} - Y_{13}B_{31}B_{11}^{-1} & 0 & Y_{13} & 0 \\ -Y_{23}B_{31}B_{11}^{-1} & 0 & Y_{23} & 0 \\ Y_{31} & Y_{32} & Y_{33} & Y_{34} \end{pmatrix} \text{ with } \begin{cases} Y_{13}B_{32} = 0, \quad Y_{23}B_{32} = I, \\ B_{31}Y_{13} + B_{32}Y_{23} \text{ is selfadjoint}, \\ B_{11}Y_{13} = (B_{11}^*)^{-1}B_{31}^*(I - B_{31}Y_{13} - B_{32}Y_{23}), \end{cases}$$

$$B\{1,3,4\} = \begin{pmatrix} B_{11}^{-1} - Y_{13}B_{31}B_{11}^{-1} & 0 & Y_{13} & 0 \\ -Y_{23}B_{31}B_{11}^{-1} & 0 & Y_{23} & 0 \\ -Y_{33}B_{31}B_{11}^{-1} & Y_{32} & Y_{33} & Y_{34} \end{pmatrix} \text{ with } \begin{cases} Y_{13}B_{32} = 0, \quad Y_{23}B_{32} = I, \quad Y_{33}B_{32} = 0, \\ B_{31}Y_{13} + B_{32}Y_{23} \text{ is selfadjoint}, \\ B_{11}Y_{13} = (B_{11}^*)^{-1}B_{31}^*(I - B_{31}Y_{13} - B_{32}Y_{23}), \end{cases}$$

$$B\{1,2,3\} = \begin{pmatrix} B_{11}^{-1} - Y_{13}B_{31}B_{11}^{-1} & 0 & Y_{13} & 0 \\ -Y_{23}B_{31}B_{11}^{-1} & 0 & Y_{23} & 0 \\ Y_{31} & 0 & Y_{33} & 0 \end{pmatrix} \text{ with } \begin{cases} Y_{13}B_{32} = 0, \quad Y_{23}B_{32} = I, \\ B_{31}Y_{13} + B_{32}Y_{23} \text{ is selfadjoint}, \\ B_{11}Y_{13} = (B_{11}^*)^{-1}B_{31}^*(I - B_{31}Y_{13} - B_{32}Y_{23}), \\ Y_{31}B_{11}Y_{13} = Y_{33}(I - B_{31}Y_{13} - B_{32}Y_{23}). \end{cases}$$

Since $B_{31}Y_{13} + B_{32}Y_{23}$ is selfadjoint, $Y_{13}B_{32} = 0$ and $Y_{23}B_{32} = I$, we get

$$B_{32} = (B_{31}Y_{13} + B_{32}Y_{23})B_{32} = \left[B_{32}^*B_{31}Y_{13} + B_{32}^*B_{32}Y_{23}\right]^*,$$

i.e., $B_{32}^*B_{32}Y_{23} = B_{32}^* - B_{32}^*B_{31}Y_{13}$. Since B_{32}^* is surjective, $B_{32}^*B_{32}$ is invertible and $B_{32}^\dagger = (B_{32}^*B_{32})^{-1}B_{32}^*$. It follows that $Y_{23} = B_{32}^\dagger - B_{32}^\dagger B_{31}Y_{13}$. Since $B_{11}Y_{13} = (B_{11}^*)^{-1}B_{31}^*(I - B_{31}Y_{13} - B_{32}Y_{23})$, we get

$$B_{11}^*B_{11}Y_{13} = B_{31}^* - B_{31}^*B_{31}Y_{13} - B_{31}^*B_{32}Y_{23} = B_{31}^* - B_{31}^*B_{31}Y_{13} - B_{31}^*B_{32}B_{32}^\dagger + B_{31}^*B_{32}B_{32}^\dagger B_{31}Y_{13}.$$

Denote by $\Delta = \left[B_{11}^*B_{11} + B_{31}^*(I - B_{32}B_{32}^\dagger)B_{31}\right]^{-1}$. Then

$$Y_{13} = \Delta B_{31}^*(I - B_{32}B_{32}^\dagger), \quad Y_{23} = B_{32}^\dagger - B_{32}^\dagger B_{31}\Delta B_{31}^*(I - B_{32}B_{32}^\dagger).$$

Hence,

$$B_{11}^{-1} - Y_{13}B_{31}B_{11}^{-1} = B_{11}^{-1} - \Delta B_{31}^*(I - B_{32}B_{32}^\dagger)B_{31}B_{11}^{-1} = \Delta B_{11}^*,$$
$$Y_{23}B_{31}B_{11}^{-1} = B_{32}^\dagger B_{31}B_{11}^{-1} - B_{32}^\dagger B_{31}\Delta B_{31}^*(I - B_{32}B_{32}^\dagger)B_{31}B_{11}^{-1} = B_{32}^\dagger B_{31}\Delta B_{11}^*.$$

We get

$$B\{1,3\} = \begin{pmatrix} \Delta B_{11}^* & 0 & \Delta B_{31}^*(I - B_{32}B_{32}^\dagger) & 0 \\ -B_{32}^\dagger B_{31}\Delta B_{11}^* & 0 & B_{32}^\dagger - B_{32}^\dagger B_{31}\Delta B_{31}^*(I - B_{32}B_{32}^\dagger) & 0 \\ Y_{31} & Y_{32} & Y_{33} & Y_{34} \end{pmatrix}$$

and

$$B\{1,3,4\} = \begin{pmatrix} \Delta B_{11}^* & 0 & \Delta B_{31}^*(I - B_{32}B_{32}^\dagger) & 0 \\ -B_{32}^\dagger B_{31}\Delta B_{11}^* & 0 & B_{32}^\dagger - B_{32}^\dagger B_{31}\Delta B_{31}^*(I - B_{32}B_{32}^\dagger) & 0 \\ -Y_{33}B_{31}B_{11}^{-1} & Y_{32} & Y_{33} & Y_{34} \end{pmatrix} \text{ with } Y_{33}B_{32} = 0.$$

Note that $Y_{31}B_{11}Y_{13} = Y_{31}B_{11}\Delta B_{31}^*(I - B_{32}B_{32}^\dagger)$ and

$$\begin{aligned} Y_{33}(I - B_{31}Y_{13} - B_{32}Y_{23}) &= Y_{33}\left[I - B_{31}\Delta B_{31}^*(I - B_{32}B_{32}^\dagger) - B_{32}B_{32}^\dagger + B_{32}B_{32}^\dagger B_{31}\Delta B_{31}^*(I - B_{32}B_{32}^\dagger)\right] \\ &= Y_{33}(I - B_{32}B_{32}^\dagger)\left(I - B_{31}\Delta B_{31}^*\right)(I - B_{32}B_{32}^\dagger). \end{aligned}$$

We get

$$B\{1,2,3\} = \begin{pmatrix} \Delta B_{11}^* & 0 & \Delta B_{31}^*(I - B_{32}B_{32}^\dagger) & 0 \\ -B_{32}^\dagger B_{31}\Delta B_{11}^* & 0 & B_{32}^\dagger - B_{32}^\dagger B_{31}\Delta B_{31}^*(I - B_{32}B_{32}^\dagger) & 0 \\ Y_{31} & 0 & Y_{33} & 0 \end{pmatrix}$$

with

$$Y_{31}B_{11}\Delta B_{31}^*(I - B_{32}B_{32}^\dagger) = Y_{33}(I - B_{32}B_{32}^\dagger)\left(I - B_{31}\Delta B_{31}^*\right)(I - B_{32}B_{32}^\dagger).$$

Moreover, from $B^\dagger = B\{1,2,4\} \cap B\{1,3,4\}$, one gets

$$B^\dagger = B\{1,2,3,4\} = \begin{pmatrix} \Delta B_{11}^* & 0 & \Delta B_{31}^*(I - B_{32}B_{32}^\dagger) & 0 \\ -B_{32}^\dagger B_{31}\Delta B_{11}^* & 0 & B_{32}^\dagger - B_{32}^\dagger B_{31}\Delta B_{31}^*(I - B_{32}B_{32}^\dagger) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

□

Remark 2.1.

(i) *In Theorem 2.1, for example, we use* $A\{1\} = \begin{pmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} & X_{13} \\ 0 & A_{22}^{-1} & X_{23} \\ X_{31} & X_{32} & X_{33} \\ X_{41} & X_{42} & X_{43} \end{pmatrix}$ *for short to express the set* $A\{1\}$, *i.e.,*

$$\begin{pmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} & X_{13} \\ 0 & A_{22}^{-1} & X_{23} \\ X_{31} & X_{32} & X_{33} \\ X_{41} & X_{42} & X_{43} \end{pmatrix} =: \left\{ \begin{pmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} & X_{13} \\ 0 & A_{22}^{-1} & X_{23} \\ X_{31} & X_{32} & X_{33} \\ X_{41} & X_{42} & X_{43} \end{pmatrix} : \forall\, X_{m,n} \in \mathcal{B}(\mathcal{K}_n, \mathcal{H}_m),\ m = 1,2,3,4,\ n = 1,2,3 \right\}.$$

The matrix forms (2) and (3) can also be applied for the cases $\{K\} \in \{\{2,3\}, \{2,4\}, \{3,4\}, \{2,3,4\}\}$. *The representations are complicated in these cases and we leave it to interested readers.*

(ii) *Note that* B_{32}^* *is surjective. In addition, if* B_{32}^* *is injective, then* B_{32} *is invertible and*

$$B^\dagger = \begin{pmatrix} B_{11}^{-1} & 0 & 0 & 0 \\ -B_{32}^{-1}B_{31}B_{11}^{-1} & 0 & B_{32}^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(iii) *It is interesting that, although the reverse order law has been considered for many types of generalized inverses and from various aspects too, the results in Theorem 2.1 provide an effective method to prove these results. In Section 3, the readers can see the importance of the expressions in Theorem 2.1.*

We next present the necessary and sufficient conditions for the invertibility of B_{32}, for $A_{12} = 0$ and for $B_{31} = 0$, respectively. As we know, B_{32}^* in (3) is surjective and closed range operator, i.e., $B_{32}^\dagger B_{32} = I$. But B_{32}^* is not necessarily injective. The following theorem studies the situation in which B_{32} is invertible. The range relations play an important role in this case.

Theorem 2.2. *Let A and B have the matrix representations (2) and (3), respectively. Then the following are equivalent:*

(i) $\Delta B_{11}^* = B_{11}^{-1}$.

(ii) $(I - B_{32}B_{32}^\dagger)B_{31} = 0$.

(iii) $\mathcal{R}(B_{31}) \subseteq \mathcal{R}(B_{32})$.

(iv) B_{32} *is invertible.*

Proof. (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii) By the representation of B, we know B_{11} is invertible and B_{32}^* is surjective and a closed range operator. By (4), $\Delta B_{11}^* = B_{11}^{-1}$ if and only if $B_{31}^*(I - B_{32}B_{32}^\dagger)B_{31} = 0$ if and only if $(I - B_{32}B_{32}^\dagger)B_{31} = 0$ if and only if $\mathcal{R}(B_{31}) \subseteq \mathcal{R}(B_{32})$.

(iv)$\Longrightarrow$ (iii) It is trivial.

(iii)$\Longrightarrow$ (iv) Since B_{32}^* is surjective, one needs to show that B_{32}^* is injective if $\mathcal{R}(B_{31}) \subseteq \mathcal{R}(B_{32})$. For every $x = (x_1, x_2, x_3, x_4)$, where $x_1 \in \mathcal{H}_1 = \mathcal{R}(A^*) \ominus \mathcal{H}_2$, $x_2 \in \mathcal{H}_2 = \mathcal{R}(A^*) \cap \mathcal{N}(B^*)$, $x_3 \in \mathcal{H}_3 = \mathcal{N}(A) \ominus \mathcal{H}_4$ and $x_4 \in \mathcal{H}_4 = \mathcal{N}(A) \cap \mathcal{N}(B^*)$, then

$$B^*x = \begin{pmatrix} B_{11}^* & 0 & B_{31}^* & 0 \\ 0 & 0 & B_{32}^* & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0$$

implies that $B_{11}^*x_1 + B_{31}^*x_3 = 0$ and $B_{32}^*x_3 = 0$. If $\mathcal{R}(B_{31}) \subseteq \mathcal{R}(B_{32})$, then $\mathcal{N}(B_{32}^*) \subseteq \mathcal{N}(B_{31}^*)$. We get $x_1 = 0$ and $x_3 \in \mathcal{H}_4 = \mathcal{N}(A) \cap \mathcal{N}(B^*)$. Since $x_3 \in \mathcal{H}_3 = \mathcal{N}(A) \ominus \mathcal{H}_4 = \mathcal{N}(A) \ominus (\mathcal{N}(A) \cap \mathcal{N}(B^*))$, we get $x_3 = 0$.
□

Note that, if $\mathcal{R}(B) \cap \mathcal{N}(A) = \{0\}$, then $\mathcal{R}(B^*) \cap \mathcal{N}(AB) = \{0\}$. In fact, for every $x \in \mathcal{R}(B^*) \cap \mathcal{N}(AB)$, $x = B^\dagger Bx = 0$ since $ABx = 0$ and $Bx \in \mathcal{R}(B) \cap \mathcal{N}(A)=\{0\}$.

On the other hand, if $\mathcal{R}(B^*) \cap \mathcal{N}(AB) = \{0\}$, then $\mathcal{R}(B) \cap \mathcal{N}(A) = \{0\}$. In fact, if $x \in \mathcal{R}(B) \cap \mathcal{N}(A)$, then $Ax = 0$ and $BB^\dagger x = x$. So, $ABB^\dagger x = Ax = 0$ and $B^\dagger x \in \mathcal{R}(B^*) \cap \mathcal{N}(AB) = \{0\}$. It follows that $x = BB^\dagger x = 0$. Hence,

$$\mathcal{R}(B) \cap \mathcal{N}(A) = \{0\} \Longleftrightarrow \mathcal{R}(B^*) \cap \mathcal{N}(AB) = \{0\}.$$

This result had been pointed out in [17].

Note also that $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$ if and only if $(I - BB^\dagger)A^*AB = 0$.

As for further relations among $\mathcal{R}(A^*AB)$, $\mathcal{R}(B)$ and $\mathcal{R}(B) \cap \mathcal{N}(A)$, one has the following results.

Theorem 2.3. *Let A and B have the matrix representations (2) and (3), respectively. Then the following are equivalent:*

(i) $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$.

(ii) $\mathcal{R}(A^*AB) = \mathcal{R}(B) \ominus (\mathcal{R}(B) \cap \mathcal{N}(A))$.

(iii) $A_{12} = 0$ *and* $\mathcal{R}(B_{31}) \subseteq \mathcal{R}(B_{32})$.

Proof. (i) $\Longleftrightarrow$ (iii) Note that A_{11} and B_{11} are invertible and

$$A^*AB = \begin{pmatrix} A_{11}^*A_{11}B_{11} & 0 & 0 \\ A_{12}^*A_{11}B_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $\mathcal{R}(A_{12}^*A_{11}B_{11}) \subseteq \mathcal{H}_2 = \mathcal{R}(A^*) \cap \mathcal{N}(B^*) \subseteq \mathcal{R}(B)^\perp$, we get $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$ if and only if $A_{12} = 0$ and $\mathcal{R}(A_{11}^*A_{11}B_{11}) \subseteq \mathcal{R}(B)$. Note that $\mathcal{R}(A_{11}^*A_{11}B_{11}) = \mathcal{R}(B_{11})$ and

$$\mathcal{R}(B) = \left\{ \begin{pmatrix} B_{11}x \\ 0 \\ B_{31}x + B_{32}y \\ 0 \end{pmatrix} : x \in \mathcal{L}_1, y \in \mathcal{L}_2 \right\}.$$

We derive that $\mathcal{R}(B_{11}) \subseteq \mathcal{R}(B)$ if and only if $\mathcal{R}(B_{31}) \subseteq \mathcal{R}(B_{32})$.

(ii) $\Longrightarrow$ (iii) If $\mathcal{R}(A^*AB) = \mathcal{R}(B) \ominus (\mathcal{R}(B) \cap \mathcal{N}(A))$, then $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$. We get (iii) holds since (i) $\Longleftrightarrow$ (iii).

(iii) $\Longrightarrow$ (ii). If (iii) holds, then $\mathcal{R}(A^*AB) = \mathcal{R}(A_{11}^*A_{11}B_{11}) = \mathcal{R}(B_{11})$ and $\mathcal{R}(B) = \mathcal{R}(B_{11}) \oplus \mathcal{R}(B_{32})$. Since $\mathcal{R}(B_{32}) \subseteq \mathcal{N}(A)$, we get $\mathcal{R}(A^*AB) = \mathcal{R}(B) \ominus (\mathcal{R}(B) \cap \mathcal{N}(A))$.
□

Note that $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$ if and only if $(I - A^\dagger A)BB^*A^* = 0$.

Theorem 2.4. *Let A and B have the matrix representations (2) and (3), respectively. Then the following are equivalent:*

(i) $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$.

(ii) $B^*(\mathcal{R}(B) \cap \mathcal{N}(A)) = B^\dagger(\mathcal{R}(B) \cap \mathcal{N}(A))$.

(iii) $\mathcal{R}(A^*) = \mathcal{R}(BB^*A^*) \oplus [\mathcal{R}(A^*) \cap \mathcal{N}(B^*)]$.

(iv) $B_{31} = 0$.

Proof. (i) $\Longleftrightarrow$ (iv) Note that

$$BB^*A^* = \begin{pmatrix} B_{11}B_{11}^*A_{11}^* & 0 & 0 \\ 0 & 0 & 0 \\ B_{31}B_{11}^*A_{11}^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{R}(A) \ominus \mathcal{K}_2 \\ (A^*)^\dagger \mathcal{H}_2 \\ \mathcal{N}(A^*) \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(A^*) \ominus (\mathcal{R}(A^*) \cap \mathcal{N}(B^*)) \\ \mathcal{R}(A^*) \cap \mathcal{N}(B^*) \\ \mathcal{N}(A) \ominus (\mathcal{N}(A) \cap \mathcal{N}(B^*)) \\ \mathcal{N}(A) \cap \mathcal{N}(B^*) \end{pmatrix} \tag{5}$$

with A_{11} and B_{11} being invertible by (2) and (3). Since $\mathcal{R}(B_{31}B_{11}^*A_{11}^*) \subseteq \mathcal{H}_3 \subseteq \mathcal{N}(A) = \mathcal{R}(A^*)^\perp$, we get $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$ if and only if $B_{31} = 0$ and $\mathcal{R}(B_{11}B_{11}^*A_{11}^*) \subseteq \mathcal{R}(A^*)$. Note that $\mathcal{R}(B_{11}B_{11}^*A_{11}^*) = \mathcal{R}(A_{11}^*)$ and $\mathcal{R}(A^*) = \mathcal{R}(A_{11}^*) \oplus \mathcal{R}(A_{22}^*)$. Hence, $\mathcal{R}(B_{11}B_{11}^*A_{11}^*) \subseteq \mathcal{R}(A^*)$ is trivial.

(iii) $\Longleftrightarrow$ (iv) By (5), $\mathcal{R}(BB^*A^*) = [\mathcal{R}(A^*) \ominus (\mathcal{R}(A^*) \cap \mathcal{N}(B^*))] \oplus [\mathcal{N}(A) \ominus (\mathcal{N}(A) \cap \mathcal{N}(B^*))] = \mathcal{R}(B_{11}) \oplus \mathcal{R}(B_{31})$, i.e.,

$$B_{31} = 0 \Longleftrightarrow \mathcal{R}(A^*) = \mathcal{R}(BB^*A^*) \oplus [\mathcal{R}(A^*) \cap \mathcal{N}(B^*)] .$$

(iv) $\Longrightarrow$ (ii) Note that $\mathcal{R}(B_{32}^*) = \mathcal{R}(B_{32}^\dagger)$ and $\mathcal{R}(B) \cap \mathcal{N}(A) = \mathcal{H}_3$. By Theorem 2.1, item (ii),

$$B^\dagger = \begin{pmatrix} \Delta B_{11}^* & 0 & \Delta B_{31}^*(I - B_{32}B_{32}^\dagger) & 0 \\ -B_{32}^\dagger B_{31}\Delta B_{11}^* & 0 & B_{32}^\dagger - B_{32}^\dagger B_{31}\Delta B_{31}^*(I - B_{32}B_{32}^\dagger) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, if $B_{31} = 0$, then $B^*(\mathcal{R}(B) \cap \mathcal{N}(A)) = B^\dagger(\mathcal{R}(B) \cap \mathcal{N}(A))$.

(ii) $\Longrightarrow$ (iv) Note that

$$BB^* = \begin{pmatrix} B_{11}B_{11}^* & 0 & B_{11}B_{31}^* & 0 \\ 0 & 0 & 0 & 0 \\ B_{31}B_{11}^* & 0 & B_{31}B_{31}^* + B_{32}B_{32}^* & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

If $B^*(\mathcal{R}(B) \cap \mathcal{N}(A)) = B^\dagger(\mathcal{R}(B) \cap \mathcal{N}(A))$, then $BB^*(\mathcal{H}_3) = BB^*(\mathcal{R}(B) \cap \mathcal{N}(A)) = BB^\dagger(\mathcal{R}(B) \cap \mathcal{N}(A)) = \mathcal{R}(B) \cap \mathcal{N}(A) = \mathcal{H}_3$. One gets that $\mathcal{R}(B_{11}B_{31}^*) = \{0\}$. Hence, $B_{31} = 0$ since B_{11} is invertible.

□

3. Multi-relations of $A\{K\}$, $B\{K\}$ and $(AB)\{K\}$

Throughout this section we will use the notation $*$ to denote the arbitrary operator which is the one suitable entry in the corresponding operator matrices. In Section 2, we have built some equivalent range relations which ensure that $A_{12} = 0$, $B_{31} = 0$ and B_{32} is invertible, respectively. In this section, many new mixed-type generalized inverses relations are established and some mixed-type reverse-order laws results appearing in recent papers are covered.

We first study equivalent conditions for the multi-relations among $A\{K\}$, $B\{K\}$ and $(AB)\{K\}$ which ensure that $A_{12} = 0$.

Theorem 3.1. *Let A and B have the matrix representations (2) and (3), respectively. Then the following statements are equivalent.*

(i) $A_{12} = 0$.

(ii) $\mathcal{R}(AA^*AB) \subseteq \mathcal{R}(AB)$.

(iii) $(TAB)\{1,3,4\} \cdot A\{1,3,4\} = (AB)\{1,3,4\}$ *for* $T = A^\dagger$ *or T belongs to any set of* $A\{1,4\}$, $A\{1,3,4\}$, $A\{1,2,4\}$.

(iv) *Any of the following relations holds:*

$$(TAB)\{1,2,3\} \cdot A\{1,2,3\} = (AB)\{1,2,3\}; \quad (TAB)\{1,2,3\} \cdot A\{1,3,4\} = (AB)\{1,2,3\};$$
$$(TAB)\{1,2,3\} \cdot A\{1,3,4\} \subseteq (AB)\{1,3\}; \quad (TAB)\{1,2,3\} \cdot A\{1,3\} \subseteq (AB)\{1,3\};$$
$$(TAB)\{1,2,3\} \cdot A\{1,3\} = (AB)\{1,2,3\}; \quad (TAB)\{1,2,3\} \cdot A^\dagger = (AB)\{1,2,3\};$$

$$(TAB)\{1,3\} \cdot A\{1,3\} = (AB)\{1,3\}; \quad (TAB)\{1,3\} \cdot A\{1,2,3\} \subseteq (AB)\{1,3\};$$
$$(TAB)\{1,3\} \cdot A\{1,3,4\} = (AB)\{1,3\}; \quad (TAB)\{1,3\} \cdot A^\dagger \subseteq (AB)\{1,3\};$$

$$(TAB)\{1,3,4\} \cdot A\{1,3\} = (AB)\{1,3\}; \quad (TAB)\{1,3,4\} \cdot A\{1,2,3\} \subseteq (AB)\{1,3\}.$$

for $T = A^\dagger$ *or T belongs to any set of* $A\{1,4\}$, $A\{1,3,4\}$, $A\{1,2,4\}$.

(v) $(A^\dagger AB)^\dagger A^\dagger = (AB)^\dagger$ *or* $(A^\dagger AB)^\dagger A^\dagger$ *belongs to any of* $(AB)\{1,3\}$, $(AB)\{1,2,3\}$, $(AB)\{1,3,4\}$.

Proof. (i) $\Longleftrightarrow$ (ii) By (2) and (3), since $AB = A_{11}B_{11} \oplus 0 \oplus 0$ and

$$AA^*AB = \begin{pmatrix} (A_{11}A_{11}^* + A_{12}A_{12}^*)A_{11}B_{11} & 0 & 0 \\ A_{22}A_{12}^*A_{11}B_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ with } A_{11}, B_{11}, A_{22} \text{ are invertible,}$$

we get $\mathcal{R}(AA^*AB) \subseteq \mathcal{R}(AB) \Longleftrightarrow \mathcal{R}(A_{22}A_{12}^*A_{11}B_{11}) = \{0\} \Longleftrightarrow A_{22}A_{12}^*A_{11}B_{11} = 0 \Longleftrightarrow A_{12}^* = 0 \Longleftrightarrow A_{12} = 0$.

(i) $\Longleftrightarrow$ (iii) By Theorem 2.1, items (i) and (iii), for $T = A^\dagger$ or for every $T \in A\{1,4\}$, or $T \in A\{1,3,4\}$, or $T \in A\{1,2,4\}$, we have

$$TAB = \begin{pmatrix} B_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (AB)\{1,3,4\} = \begin{pmatrix} B_{11}^{-1}A_{11}^{-1} & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, \quad (TAB)\{1,3,4\} = \begin{pmatrix} B_{11}^{-1} & 0 & 0 & 0 \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}.$$

Hence,

$$(TAB)\{1,3,4\} \cdot A\{1,3,4\} = \begin{pmatrix} B_{11}^{-1}A_{11}^{-1} & -B_{11}^{-1}A_{11}^{-1}A_{12}A_{22}^{-1} & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$$

It follows that $(TAB)\{1,3,4\} \cdot A\{1,3,4\} = (AB)\{1,3,4\} \Longleftrightarrow A_{12} = 0$.

(i) $\Longleftrightarrow$ (iv) Similar to (i) $\Longleftrightarrow$ (iii).

(i) $\Longleftrightarrow$ (v) By Theorem 2.1, items (i) and (iii),

$$A^\dagger AB = \begin{pmatrix} B_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (A^\dagger AB)^\dagger = \begin{pmatrix} B_{11}^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (A^\dagger AB)^\dagger A^\dagger = \begin{pmatrix} B_{11}^{-1}A_{11}^{-1} & -B_{11}^{-1}A_{11}^{-1}A_{12}A_{22}^{-1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $(AB)^\dagger = B_{11}^{-1}A_{11}^{-1} \oplus 0 \oplus 0$, it follows that $(A^\dagger AB)^\dagger A^\dagger = (AB)^\dagger \Longleftrightarrow A_{12} = 0$. By Theorem 2.1, item (iii), it is easy to get that $(A^\dagger AB)^\dagger A^\dagger$ belongs to any of $(AB)\{1,3\}$, $(AB)\{1,2,3\}$, $(AB)\{1,3,4\} \Longleftrightarrow A_{12} = 0$.

□

Next theorem gives some mixed-type reverse order laws results associated to $A\{K\}$, $B\{K\}$ and $(AB)\{K\}$. These results don't need any additional conditions.

Theorem 3.2. *Let A and B have the matrix representations (2) and (3), respectively.Then the following statements hold.*

(i) $M \cdot (ABN)\{1,3\} \subseteq (AB)\{1,3\}$ *for* $M = B^\dagger$, $N = B^\dagger$ *or* M, N *belong to any set of* $B\{1,3\}$, $B\{1,2,3\}$, $B\{1,3,4\}$.

(ii) $M \cdot (ABN)\{1,2,3\} \subseteq (AB)\{1,2,3\}$ *for* $M = B^\dagger$, $N = B^\dagger$ *or* M, N *belong to any set of* $B\{1,3\}$, $B\{1,2,3\}$, $B\{1,3,4\}$.

(iii) $(EAB)\{1,4\} \cdot A^\dagger \subseteq (AB)\{1,4\}$ *for* $E = A^\dagger$ *or E belongs to any set of* $A\{1,4\}$, $A\{1,2,4\}$, $A\{1,3,4\}\}$.

(iv) $(EAB)\{1,2,4\} \cdot A^\dagger \subseteq (AB)\{1,2,4\}$ *for* $E = A^\dagger$ *or E belongs to any set of* $A\{1,4\}$, $A\{1,2,4\}$, $A\{1,3,4\}\}$.

(v) $(EAB)\{1,4\} \cdot F = (AB)\{1,4\}$ *for* $E = A^\dagger$, *or* E, F *belong to any set of* $A\{1,4\}$, $A\{1,2,4\}$, $A\{1,3,4\}\}$.

(vi) $(EAB)\{1,2,4\} \cdot F = (AB)\{1,2,4\}$ *for* $E = A^\dagger$, *or* E, F *belong to any set of* $A\{1,4\}$, $A\{1,2,4\}$, $A\{1,3,4\}\}$.

Proof. (i) We only show that $B\{1,3\} \cdot (ABN)\{1,3\} \subseteq (AB)\{1,3\}$ when $N = B^\dagger$ or N belong to any set of $B\{1,3\}$, $B\{1,2,3\}$, $B\{1,3,4\}$. The rest of the proof is similar. So we omit it. Note that

$$ABN = \begin{pmatrix} A_{11}B_{11}\Delta B_{11}^* & 0 & A_{11}B_{11}\Delta B_{31}^*(I - B_{32}B_{32}^\dagger) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (AB)\{1,3\} = \begin{pmatrix} B_{11}^{-1}A_{11}^{-1} & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix},$$

$$(ABN)\{1,3\}=\begin{pmatrix}N_{11}&N_{12}&N_{13}\\ *&*&*\\ N_{31}&N_{32}&N_{33}\\ *&*&*\end{pmatrix}\quad\text{with}\begin{cases}\Delta B_{11}^*N_{11}+\Delta B_{31}^*(I-B_{32}B_{32}^\dagger)N_{31}=B_{11}^{-1}A_{11}^{-1},\\ \Delta B_{11}^*N_{12}+\Delta B_{31}^*(I-B_{32}B_{32}^\dagger)N_{32}=0,\\ \Delta B_{11}^*N_{13}+\Delta B_{31}^*(I-B_{32}B_{32}^\dagger)N_{33}=0.\end{cases}$$

Hence,

$$\begin{aligned}B\{1,3\}\cdot(ABN)\{1,3\} &= \begin{pmatrix}\Delta B_{11}^* & 0 & \Delta B_{31}^*(I-B_{32}B_{32}^\dagger) & 0\\ -B_{32}^\dagger B_{31}\Delta B_{11}^* & 0 & B_{32}^\dagger-B_{32}^\dagger B_{31}\Delta B_{31}^*(I-B_{32}B_{32}^\dagger) & 0\\ Y_{31} & Y_{32} & Y_{33} & Y_{34}\end{pmatrix}\begin{pmatrix}N_{11}&N_{12}&N_{13}\\ *&*&*\\ N_{31}&N_{32}&N_{33}\\ *&*&*\end{pmatrix}\\ &= \begin{pmatrix}B_{11}^{-1}A_{11}^{-1} & 0 & 0\\ B_{32}^\dagger N_{31}-B_{32}^\dagger B_{31}B_{11}^{-1}A_{11}^{-1} & B_{32}^\dagger N_{32} & B_{32}^\dagger N_{33}\\ * & * & *\end{pmatrix}\subseteq (AB)\{1,3\}.\end{aligned}$$

(ii) Similar to (i).

(iii)-(vi) We only show that $(EAB)\{1,4\}\cdot A\{1,4\}=(AB)\{1,4\}$ when $E=A^\dagger$ or E belongs to any set of $A\{1,4\}$, $A\{1,2,4\}$, $A\{1,3,4\}\}$. The rest of the proof is similar. So we omit it. For $E=A^\dagger$ or E belongs to any set of $A\{1,4\}$, $A\{1,2,4\}$, $A\{1,3,4\}\}$, we have

$$EAB=\begin{pmatrix}B_{11}&0&0\\0&0&0\\0&0&0\\0&0&0\end{pmatrix},\quad (AB)\{1,4\}=\begin{pmatrix}B_{11}^{-1}A_{11}^{-1}&*&*\\0&*&*\\0&*&*\end{pmatrix},\quad (EAB)\{1,4\}=\begin{pmatrix}B_{11}^{-1}&*&*&*\\0&*&*&*\\0&*&*&*\end{pmatrix}.$$

Hence,

$$(EAB)\{1,4\}\cdot A\{1,4\} = \begin{pmatrix}B_{11}^{-1}&*&*&*\\0&*&*&*\\0&*&*&*\end{pmatrix}\begin{pmatrix}A_{11}^{-1}&-A_{11}^{-1}A_{12}A_{22}^{-1}&X_{13}\\0&A_{22}^{-1}&X_{23}\\0&0&X_{33}\\0&0&X_{43}\end{pmatrix}=\begin{pmatrix}B_{11}^{-1}A_{11}^{-1}&*&*\\0&*&*\\0&*&*\end{pmatrix}.$$

Thus, $(EAB)\{1,4\}\cdot A\{1,4\}=(AB)\{1,4\}$.

□

Using Theorem 2.1, the new equivalent conditions for $B_{31}=0$ are derived.

Theorem 3.3. *Let A and B have the matrix representations (2) and (3), respectively. Then the following statements are equivalent.*

(i) $B_{31}=0$.

(ii) $\mathcal{R}(BB^*A^*)\subseteq\mathcal{R}(A^*)$.

(iii) $B^\dagger A^\dagger\in(AB)\{1,4\}$.

(iv) $B^\dagger A^\dagger\in(AB)\{1,2,4\}$.

(v) $B^\dagger T\subseteq(AB)\{1,4\}$ *for* $T\in\left\{A\{1,4\},\ A\{1,2,4\},\ A\{1,3,4\}\right\}$.

(vi) $B^\dagger\cdot A\{1,2,4\}\subseteq(AB)\{1,2,4\}$.

(vii) $B\{1,4\}\cdot A\{1,4\}\subseteq(AB)\{1,4\}$.

(viii) $B\{1,3,4\}\cdot T\subseteq(AB)\{1,4\}$ *for* $T\in\left\{A^\dagger,\ A\{1,4\},\ A\{1,2,4\},\ A\{1,3,4\}\right\}$.

(ix) $BSAB=TAB$ *for* $S\in\left\{(AB)^\dagger,(AB)\{1,4\},(AB)\{1,2,4\},(AB)\{1,3,4\}\right\}$ *and* $T\in\left\{A^\dagger,A\{1,4\},A\{1,2,4\},A\{1,3,4\}\right\}$.

Proof. (i) $\Longleftrightarrow$ (ii) See Theorem 2.4, items (i) and (iv).

(i) $\Longleftrightarrow$ (iii) Note that, by Theorem 2.1, items (i) and (ii),

$$\begin{aligned} B^\dagger A^\dagger &= \begin{pmatrix} \Delta B_{11}^* & 0 & \Delta B_{31}^*(I - B_{32}B_{32}^\dagger) & 0 \\ -B_{32}^\dagger B_{31}\Delta B_{11}^* & 0 & B_{32}^\dagger - B_{32}^\dagger B_{31}\Delta B_{31}^*(I - B_{32}B_{32}^\dagger) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} & 0 \\ 0 & A_{22}^{-1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \Delta B_{11}^* A_{11}^{-1} & -\Delta B_{11}^* A_{11}^{-1}A_{12}A_{22}^{-1} & 0 \\ -B_{32}^\dagger B_{31}\Delta B_{11}^* A_{11}^{-1} & B_{32}^\dagger B_{31}\Delta B_{11}^* A_{11}^{-1}A_{12}A_{22}^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{6}$$

By Theorem 2.1, (iii), $B^\dagger A^\dagger \in (AB)\{1,4\}$ if and only if $\Delta B_{11}^* A_{11}^{-1} = B_{11}^{-1}A_{11}^{-1}$ and $B_{32}^\dagger B_{31}\Delta B_{11}^* A_{11}^{-1} = 0$ if and only if $B_{32}^\dagger B_{31} = 0$ and $\Delta = B_{11}^{-1}(B_{11}^*)^{-1}$ if and only if $B_{31} = 0$.

(i) $\Longleftrightarrow$ (iv), or (vi), or (vii) Similar to (i) $\Longleftrightarrow$ (iii).

(i) $\Longleftrightarrow$ (viii) For example, we only prove that $B\{1,3,4\} \cdot A\{1,3,4\} \subseteq (AB)\{1,4\}$ if and only if $B_{31} = 0$. By Theorem 2.1, item (iii), $B\{1,3,4\} \cdot A\{1,3,4\} \subseteq (AB)\{1,4\}$ if and only if the first column of $B\{1,3,4\} \cdot A\{1,3,4\}$ is same as the first column of $(AB)\{1,4\}$, i.e.,

$$\Delta B_{11}^* A_{11}^{-1} = B_{11}^{-1}A_{11}^{-1}, \quad B_{32}^\dagger B_{31}\Delta B_{11}^* A_{11}^{-1} = 0, \quad Y_{33}B_{31}B_{11}^{-1}A_{11}^{-1} = 0$$

if and only if $(I - B_{32}B_{32}^\dagger)B_{31} = 0$ and $B_{32}^\dagger B_{31} = 0$ if and only if $B_{31} = 0$ if and only if $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$.

(i) $\Longleftrightarrow$ (ix) By Theorem 2.1, items (i) and (iii), we have

$$BSAB = \begin{pmatrix} B_{11} & 0 & 0 \\ 0 & 0 & 0 \\ B_{31} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad TAB = \begin{pmatrix} B_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

Hence, $BSAB = TAB \Longleftrightarrow B_{31} = 0$.

□

Remark 3.1.

(i) *In [29], authors had pointed out that*

$$B\{1,3,4\} \cdot A\{1,3,4\} \subseteq (AB)\{1,4\} \Longleftrightarrow \mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*).$$

(ii) *Theorem 3.2, items (iii), (iv) and (vii) have been proved in [14, Theorem 2.3]. Our matrix expressions in Theorem 2.1 are brief and efficient ways to study the various generalized inverse relations.*

In Theorem 2.2, we had proved that $\Delta B_{11}^* = B_{11}^{-1} \Longleftrightarrow (I - B_{32}B_{32}^\dagger)B_{31} = 0 \Longleftrightarrow \mathcal{R}(B_{31}) \subseteq \mathcal{R}(B_{32}) \Longleftrightarrow B_{32}$ is invertible. In fact, further properties of B_{32} being invertible can be gotten by using Theorem 2.1.

Theorem 3.4. *Let A and B have the matrix representations (2) and (3), respectively. Then*

$$\mathcal{R}(B_{31}) \subseteq \mathcal{R}(B_{32}) \Longleftrightarrow MN \subseteq (AB)\{1\}$$

for $M \in \left\{B^\dagger,\ B\{1,3\},\ B\{1,2,3\},\ B\{1,3,4\}\right\}$ *and* $N \in \left\{A^\dagger,\ A\{1\},\ A\{1,2\},\ A\{1,3\},\ A\{1,4\},\ A\{1,2,3\},\ A\{1,2,4\},\ A\{1,3,4\}\right\}$.

Proof. By Theorem 2.1, $MN \subseteq (AB)\{1\}$ if and only if $\Delta B_{11}^* A_{11}^{-1} = B_{11}^{-1}A_{11}^{-1}$ or $\Delta B_{11}^* A_{11}^{-1} + \Delta B_{31}^*(I - B_{32}B_{32}^\dagger)X_{31} = B_{11}^{-1}A_{11}^{-1}$ for arbitrary X_{31} if and only if $(I - B_{32}B_{32}^\dagger)B_{31} = 0$. By Theorem 2.2, we prove the result.

□

In Theorem 2.3, we have gotten that $A_{12} = 0$ and $\mathcal{R}(B_{31}) \subseteq \mathcal{R}(B_{32}) \Longleftrightarrow \mathcal{R}(A^*AB) \subseteq \mathcal{R}(B) \Longleftrightarrow \mathcal{R}(A^*AB) = \mathcal{R}(B) \ominus (\mathcal{R}(B) \cap \mathcal{N}(A))$. In fact, there are various different methods to express these relations.

Theorem 3.5. *Let A and B have the matrix representations (2) and (3), respectively. Then the following statements are equivalent.*

(i) $A_{12} = 0$ *and* $\mathcal{R}(B_{31}) \subseteq \mathcal{R}(B_{32})$.

(ii) $B^{\dagger}A^{\dagger} \in (AB)\{1,3\}$.

(iii) $B^{\dagger}A^{\dagger} \in (AB)\{1,2,3\}$.

(iv) $W \cdot A\{1,3\} \subseteq (AB)\{1,3\}$ *for* $W \in \left\{B^{\dagger},\ B\{1,3\},\ B\{1,2,3\},\ B\{1,3,4\}\right\}$.

(v) $B\{1,2,3\} \cdot A\{1,2,3\} \subseteq (AB)\{1,3\}$.

(vi) $W \cdot A^{\dagger} \subseteq (AB)\{1,3\}$ *for* $W \in \left\{B\{1,3\},\ B\{1,2,3\},\ B\{1,3,4\}\right\}$.

(vii) $B\{1,2,3\} \cdot A^{\dagger} \subseteq (AB)\{1,2,3\}$.

(viii) $W \cdot A\{1,3,4\} \subseteq (AB)\{1,3\}$ *for* $W \in \left\{B^{\dagger},\ B\{1,3\},\ B\{1,2,3\},\ B\{1,3,4\}\right\}$.

Proof. (i) $\Longleftrightarrow$ (ii) By (5) and Theorem 2.1, items (iii), $B^{\dagger}A^{\dagger} \in (AB)\{1,3\}$ if and only if $\Delta B_{11}^{*}A_{11}^{-1} = B_{11}^{-1}A_{11}^{-1}$ and $\Delta B_{11}^{*}A_{11}^{-1}A_{12}A_{22}^{-1} = 0$ if and only if $A_{12} = 0$ and $B_{31}^{*}(I - B_{32}B_{32}^{\dagger})B_{31} = 0$ if and only if $A_{12} = 0$ and $(I - B_{32}B_{32}^{\dagger})B_{31} = 0$ if and only if $A_{12} = 0$ and $\mathcal{R}(B_{31}) \subseteq \mathcal{R}(B_{32})$.

(i) $\Longleftrightarrow$ (iii)-(viii) We only show that $B\{1,3\}\cdot A\{1,3\} \subseteq (AB)\{1,3\} \Longleftrightarrow$ (i) $\Longleftrightarrow B\{1,3,4\}\cdot A\{1,3,4\} \subseteq (AB)\{1,3\}$. The rest of the proofs is similar. Note that, by Theorem 2.1, items (i) and (ii),

$$B\{1,3\} \cdot A\{1,3\} = \begin{pmatrix} \Delta B_{11}^{*}A_{11}^{-1} + \Delta B_{31}^{*}(I - B_{32}B_{32}^{\dagger})X_{31} & -\Delta B_{11}^{*}A_{11}^{-1}A_{12}A_{22}^{-1} + \Delta B_{31}^{*}(I - B_{32}B_{32}^{\dagger})X_{32} & \Delta B_{31}^{*}(I - B_{32}B_{32}^{\dagger})X_{33} \\ * & * & * \\ * & * & * \end{pmatrix}$$

and

$$B\{1,3,4\} \cdot A\{1,3,4\} = \begin{pmatrix} \Delta B_{11}^{*}A_{11}^{-1} & -\Delta B_{11}^{*}A_{11}^{-1}A_{12}A_{22}^{-1} & \Delta B_{31}^{*}(I - B_{32}B_{32}^{\dagger})X_{33} \\ * & * & * \\ * & * & * \end{pmatrix}.$$

By Theorem 2.1, item (iii),

$$B\{1,3\} \cdot A\{1,3\} \subseteq (AB)\{1,3\} \quad (\text{resp. } B\{1,3,4\} \cdot A\{1,3,4\} \subseteq (AB)\{1,3\})$$

if and only if the first row of $B\{1,3\} \cdot A\{1,3\}$ (resp. $B\{1,3,4\} \cdot A\{1,3,4\}$) is same as the first row of $(AB)\{1,3\}$, i.e.,

$$\Delta B_{11}^{*}A_{11}^{-1} = B_{11}^{-1}A_{11}^{-1}, \quad \Delta B_{11}^{*}A_{11}^{-1}A_{12}A_{22}^{-1} = 0, \quad \Delta B_{31}^{*}(I - B_{32}B_{32}^{\dagger})X_{33} = 0$$

if and only if $A_{12} = 0$ and $(I - B_{32}B_{32}^{\dagger})B_{31} = 0$ if and only if $\mathcal{R}(A^{*}AB) \subseteq \mathcal{R}(B)$ by Theorem 2.3, item (i). □

Remark 3.2.

(i) *In [29], authors had pointed out that*

$$B\{1,3,4\} \cdot A\{1,3,4\} \subseteq (AB)\{1,3\} \Longleftrightarrow \mathcal{R}(A^{*}AB) \subseteq \mathcal{R}(B).$$

(ii) *Theorem 3.5, items (ii)-(iv) have been proved in [14, Theorem 2.2]. Our proofs are brief and efficient.*

Moreover, we will establish the equivalent conditions for $\mathcal{R}(A^{*}AB) \subset \mathcal{R}(B)$ for the mixed-type reverse order laws.

Theorem 3.6. *Let A and B have the matrix representations (2) and (3), respectively. Then the following statements are equivalent.*

(i) $\mathcal{R}(A^{*}AB) \subset \mathcal{R}(B)$.

(ii) $ABQA = ABW$ *for* $Q \in \left\{(AB)^{\dagger}, (AB)\{1,3\}, (AB)\{1,2,3\}, (AB)\{1,3,4\}\right\}$ *and* $W \in \left\{B^{\dagger}, B\{1,3\}, B\{1,2,3\}, B\{1,3,4\}\right\}$.

Proof. (i) $\Longleftrightarrow$ (ii) By Theorem 2.1, items (ii) and (iii), for $Q \in \left\{(AB)^{\dagger}, (AB)\{1,3\}, (AB)\{1, 2,3\}, (AB)\{1,3,4\}\right\}$, $W \in \left\{B^{\dagger}, B\{1,3\}, B\{1,2,3\}, B\{1,3,4\}\right\}$, we have

$$ABQA = \begin{pmatrix} A_{11} & A_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad ABW = \begin{pmatrix} A_{11}B_{11}\Delta B_{11}^* & 0 & A_{11}B_{11}\Delta B_{31}^*(I - B_{32}B_{32}^{\dagger}) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

Hence, $ABQA = ABW$ if and only if $A_{11}B_{11}\Delta B_{11}^* = A_{11}$, $A_{12} = 0$ and $A_{11}B_{11}\Delta B_{31}^*(I - B_{32}B_{32}^{\dagger})B_{31} = 0$ if and only if $A_{12} = 0$ and $(I - B_{32}B_{32}^{\dagger})B_{31} = 0$ if and only if $\mathcal{R}(A^*AB) \subset \mathcal{R}(B)$.
□

We immediately get the mixed-type reverse order laws associated to the Moore-Penrose inverse and the $\{1,3,4\}$-inverse.

Corollary 3.7. *Let A and B have the matrix representations (2) and (3), respectively. Then the following statements are equivalent.*

(i) (see [13, Theorem 2.2 (c)]) $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$ *and* $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$.

(ii) $B^{\dagger}A^{\dagger} = (AB)^{\dagger}$.

(iii) $A_{12} = 0$ *and* $B_{31} = 0$.

(iv) $B^{\dagger} \cdot A\{1,3,4\} \subseteq (AB)\{1,3,4\}$.

(v) $B^{\dagger} \cdot A^{\dagger} \in (AB)\{1,3,4\}$.

(vi) $B\{1,3,4\} \cdot A^{\dagger} \subseteq (AB)\{1,3,4\}$.

(vii) $B\{1,3,4\} \cdot A\{1,3,4\} \subseteq (AB)\{1,3,4\}$.

(viii) *One of items in Theorem 3.1* & *one of items in Theorems 2.4, 3.3 hold.*

(ix) *One of items in Theorems 2.4, 3.3* & *one of items in Theorems 2.3, 3.5, 3.6 hold.*

Proof. (i) $\Longleftrightarrow$ (ii) See [15] for matrix case and [13, Theorem 2.2 (c)] for the bounded operators case.

(ii) $\Longleftrightarrow$ (iii) By (6) and Theorem 2.1, (iii), $B^{\dagger}A^{\dagger} = (AB)^{\dagger}$ if and only if $A_{12} = 0$, $B_{32}^{\dagger}B_{31} = 0$ and $\Delta = B_{11}^{-1}(B_{11}^*)^{-1}$ if and only if $A_{12} = 0$ and $B_{31} = 0$.

(iii) $\Longleftrightarrow$ (iv) By Theorem 2.1 again, $(AB)\{1,3,4\} = \begin{pmatrix} B_{11}^{-1}A_{11}^{-1} & 0 & 0 \\ 0 & Z_{22} & Z_{23} \\ 0 & Z_{32} & Z_{33} \end{pmatrix}$ and

$$\begin{aligned} B^{\dagger} \cdot A\{1,3,4\} &= \begin{pmatrix} \Delta B_{11}^* & 0 & \Delta B_{31}^*(I - B_{32}B_{32}^{\dagger}) & 0 \\ -B_{32}^{\dagger}B_{31}\Delta B_{11}^* & 0 & B_{32}^{\dagger} - B_{32}^{\dagger}B_{31}\Delta B_{31}^*(I - B_{32}B_{32}^{\dagger}) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} & 0 \\ 0 & A_{22}^{-1} & 0 \\ 0 & 0 & X_{33} \\ 0 & 0 & X_{43} \end{pmatrix} \\ &= \begin{pmatrix} \Delta B_{11}^*A_{11}^{-1} & -\Delta B_{11}^*A_{11}^{-1}A_{12}A_{22}^{-1} & \Delta B_{31}^*(I - B_{32}B_{32}^{\dagger})X_{33} \\ -B_{32}^{\dagger}B_{31}\Delta B_{11}^*A_{11}^{-1} & B_{32}^{\dagger}B_{31}\Delta B_{11}^*A_{11}^{-1}A_{12}A_{22}^{-1} & \left[B_{32}^{\dagger} - B_{32}^{\dagger}B_{31}\Delta B_{31}^*(I - B_{32}B_{32}^{\dagger})\right]X_{33} \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Then $B^{\dagger} \cdot A\{1,3,4\} \subseteq (AB)\{1,3,4\}$ if and only if $A_{12} = 0$, $B_{32}^{\dagger}B_{31} = 0$ and $\Delta = B_{11}^{-1}(B_{11}^*)^{-1}$ if and only if $A_{12} = 0$ and $B_{31} = 0$.

(iii) $\Longleftrightarrow$ (v)-(vii) Similar to the proof of (iii) $\Longleftrightarrow$ (iv).

(iii) $\Longleftrightarrow$ (viii)-(ix) See the proofs of Theorems 2.2-3.6.
□

Remark 3.3. *Let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and $B \in \mathcal{B}(\mathcal{L},\mathcal{H})$ be such that $\mathcal{R}(A)$, $\mathcal{R}(B)$ and $\mathcal{R}(AB)$ are closed. Corollary 3.7, items (i) and (vii) had been pointed out in [21, 29]*

$$B\{1,3,4\}\cdot A\{1,3,4\} \subseteq (AB)\{1,3,4\} \Longleftrightarrow \mathcal{R}(A^*AB) \subseteq \mathcal{R}(B), \quad \mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*).$$

In fact, $(AB)\{1,3,4\} = (AB)\{1,3\} \cap (AB)\{1,4\}$. The result follows immediately by Theorem 3.3, item (viii) and Theorem 3.5, item (viii).

Theorem 3.8. *Let A and B be denoted as the matrix forms (2) and (3), respectively. The following statements are equivalent.*

(i) $B\{1,2,4\}\cdot A\{1,2,4\} \subseteq (AB)\{1,2,4\}$.

(ii) $B_{31} = 0$, $\mathcal{L}_2 = \{0\}$ *or* $\mathcal{H}_2 = \{0\}$ *where* $\mathcal{L}_2 = \mathcal{R}(B^*) \ominus \mathcal{R}(B^*A^*)$ *and* $\mathcal{H}_2 = \mathcal{R}(A^*) \oplus \mathcal{N}(B^*)$ *are defined in (1).*

(iii) $\mathcal{R}(A^*) = \mathcal{R}(BB^*A^*) \oplus [\mathcal{R}(A^*) \cap \mathcal{N}(B^*)]$, $\quad \mathcal{N}(AB) = \mathcal{N}(B)$ *or* $\mathcal{R}(A^*) \oplus \mathcal{N}(B^*) = \{0\}$.

(iv) $A^*\{1,2,3\}\cdot B^*\{1,2,3\} \subseteq (B^*A^*)\{1,2,3\}$.

Proof. By Theorem 2.1, items (i) and (ii),

$$B\{1,2,4\}\cdot A\{1,2,4\} = \begin{pmatrix} B_{11}^{-1}-Y_{13}B_{31}B_{11}^{-1} & Y_{12} & Y_{13} & Y_{14} \\ -Y_{23}B_{31}B_{11}^{-1} & Y_{22} & Y_{23} & Y_{24} \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} & X_{13} \\ 0 & A_{22}^{-1} & X_{23} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} (B_{11}^{-1}-Y_{13}B_{31}B_{11}^{-1})A_{11}^{-1} & -(B_{11}^{-1}-Y_{13}B_{31}B_{11}^{-1})A_{11}^{-1}A_{12}A_{22}^{-1}+Y_{12}A_{22}^{-1} & (B_{11}^{-1}-Y_{13}B_{31}B_{11}^{-1})X_{13}+Y_{12}X_{23} \\ -Y_{23}B_{31}B_{11}^{-1}A_{11}^{-1} & Y_{23}B_{31}B_{11}^{-1}A_{11}^{-1}A_{12}A_{22}^{-1}+Y_{22}A_{22}^{-1} & -Y_{23}B_{31}B_{11}^{-1}X_{13}+Y_{22}X_{23} \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that

$$(AB)\{1,2,4\} = \begin{pmatrix} B_{11}^{-1}A_{11}^{-1} & Z_{12} & Z_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $X_{*,*}$, $Y_{*,*}$, $Z_{*,*}$ are arbitrary by Theorem 2.1.

(i) $\Longleftrightarrow$ (ii) $B\{1,2,4\}\cdot A\{1,2,4\} \subseteq (AB)\{1,2,4\}$ if and only if $Y_{13}B_{31} = 0$, $Y_{23}B_{31} = 0$, $Y_{22}A_{22}^{-1} = 0$ and $Y_{22}X_{23} = 0$ if and only if $B\{1,2,4\}$ in Theorem 2.1 satisfies the additional conditions that

$$Y_{13}B_{31} = 0, \quad Y_{23}B_{31} = 0, \quad Y_{22} = 0.$$

Which is equivalent with $B_{31} = 0$ and the second row or column in $B\{1,2,4\}$ must disappear, that is, $B_{31} = 0$, and $\mathcal{L}_2 = \mathcal{R}(B^*) \ominus \mathcal{R}(B^*A^*) = \{0\}$ or $\mathcal{R}(A^*) \oplus \mathcal{N}(B^*) = \{0\}$.

(ii) $\Longleftrightarrow$ (iii) Since

$$BB^*A^* = \begin{pmatrix} B_{11}B_{11}^*A_{11}^* & 0 & 0 \\ 0 & 0 & 0 \\ B_{31}B_{11}^*A_{11}^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{R}(A)\ominus\mathcal{K}_2 \\ (A^*)^\dagger\mathcal{H}_2 \\ \mathcal{N}(A^*) \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(A^*)\ominus(\mathcal{R}(A^*)\cap\mathcal{N}(B^*)) \\ \mathcal{R}(A^*)\cap\mathcal{N}(B^*) \\ \mathcal{N}(A)\ominus(\mathcal{N}(A)\cap\mathcal{N}(B^*)) \\ \mathcal{N}(A)\cap\mathcal{N}(B^*) \end{pmatrix}$$

with A_{11} and B_{11} being invertible by (2) and (3), we get $\mathcal{R}(BB^*A^*) = [\mathcal{R}(A^*) \ominus (\mathcal{R}(A^*) \cap \mathcal{N}(B^*))] \oplus \mathcal{R}(B_{31})$, i.e.,

$$B_{31} = 0 \Longleftrightarrow \mathcal{R}(A^*) = \mathcal{R}(BB^*A^*) \oplus [\mathcal{R}(A^*) \cap \mathcal{N}(B^*)]$$

and

$$\mathcal{L}_2 = \{0\} \Longleftrightarrow \mathcal{R}(B^*) = \mathcal{R}(B^*A^*) \Longleftrightarrow \mathcal{N}(AB) = \mathcal{N}(B)$$

since $\mathcal{R}(AB)$ and $\mathcal{R}(B)$ are closed.

(i) $\Longleftrightarrow$ (iv) Note that,

$$X \in (AB)\{1,2,4\} \Longleftrightarrow X^* \in (AB)^*\{1,2,3\} \Longleftrightarrow X^* \in (B^*A^*)\{1,2,3\}.$$

□

It worth point out in [18, Theoren 3.2], the authors gave that $B\{1,2,4\}\cdot A\{1,2,4\} \subseteq (AB)\{1,2,4\}$ if and only if $\mathcal{R}(A^*) = \mathcal{R}(BB^*A^*) \oplus [\mathcal{R}(A^*) \cap \mathcal{N}(B^*)]$, $\mathcal{N}(AB) = \mathcal{N}(B)$ when A, B and AB have closed range. But this result does not necessarily hold. We can see from the following example which is provided by the referee.

Example 1 Let $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. By direct computation,

$$B^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B^*A^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$(AB)\{1,2,4\} = \{\begin{pmatrix} 1 & x_{12} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} : x_{12} \in \mathbb{C}\}, \quad B\{1,2,4\}A\{1,2,4\} = \{\begin{pmatrix} 1 & y_{12} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} : y_{12} \in \mathbb{C}\}.$$

So $\mathcal{L}_2 \neq \{0\}$, $\mathcal{N}(B) \neq \mathcal{N}(AB)$ and $B\{1,2,4\}A\{1,2,4\} = (AB)\{1,2,4\}$.

In [26, Theorem 3.1], the necessary and sufficient conditions for

$$B\{1,2,4\} \cdot A\{1,2,4\} \subseteq (AB)\{1,2,4\}, \quad B\{1,2,3\} \cdot A\{1,2,3\} \subseteq (AB)\{1,2,3\}$$

were presented by using the expressions for maximal and minimal ranks of the generalized Schur complement.

In this paper, we study the mutual relationships of mixed-type generalized inverses. The range relations, the properties of matrix entries, the inclusion or equal relationships of mixed-type generalized inverses of corresponding closed range operators are obtained. Meanwhile some new necessary and sufficient conditions for various generalized inverses relations are given, and some recent related results are covered. It is worth pointing out that various relations of mixed-type generalized inverses which we do not discuss can also be treated effectively by using our methods.

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References

[1] R.H. Bouldin, The pseudo-inverse of a product, SIAM J. Appl. Math. 25 (1973), 489-495.
[2] J. Conway, A Course in Functional Analysis, Spring-Verlag, New York, 1990.
[3] D.S. Cvetković-Ilić, New conditions for the reverse order laws for$\{1,3\}$ and $\{1,4\}$-generalized inverses, Electron. J. Linear Algebra 23 (2012), 231-241.
[4] D.S. Cvetković-Ilić and R. Harte, Reverse order laws in C*-algebras, Linear Algebra Appl. 434 (2011), 1388-1394.
[5] D.S. Cvetković-Ilić, M. Djikić, Various solutions to reverse order law problems, Linear & Multilinear Algebra 64(6) (2016), 1207-1209.
[6] D.S. Cvetković-Ilić, V. Pavlović, A comment on some recent results concerning the reverse order law for $\{1,3,4\}$-inverses, Appl. Math. Comp., 217, (2010), 105-109.
[7] D.S. Cvetković-Ilić, Expression of the Drazin and MP-inverse of partitioned matrix and quotient identity of generalized Schur complement, App. Math. Comp. 213(1) (2009), 18-24.
[8] D.S. Cvetković-Ilić, P. Stanimirović, Marko Miladinović, Comments on some recent results concerning 2,3 and 2,4-generalized inverses, Appl. Math. Comp. 217(22) (2011), 9358-9367.
[9] D.S. Cvetković-Ilić, Reverse order laws for$\{1,3,4\}$-genveralized inverses in C^*-algebras, Appl. Math. Lett. 24 (2011), 210-213.
[10] D.S. Cvetković-Ilić, Y. Wei, Algebraic Properties of Generalized Inverses, Series: Developments in Mathematics, Vol. 52, Springer, 2017
[11] C. Deng, R. Liu, X. Wang, Expression for the Multiplicative Perturbation of the Moore-Penrose Inverse, Linear & Multilinear algebra, 66 (2018), 1171-1185.
[12] C. Deng, Reverse order law for the group inverses, J. Math. Anal. Appl. 382 (2011), 663-671.
[13] D.S. Djordjević, N.Č. Dinčić, Reverse order law for the Moore-Penrose inverse, J. Math. Anal. Appl. 361(1) (2010), 252-261.
[14] D.S. Djordjević, Further results on the reverse order law for generalized inverses, SIAM J. Matrix Anal. Appl. 29 (4) (2007), 1242-1246.
[15] T.N.E. Greville, Note on the generalized inverse of a matrix product, SIAM Rev. 8 (1966), 518-521.

[16] S. Izumino, The product of operators with closed range and an extension of the reverse order law, Tohôku Math. J. 34 (1982), 43-52.
[17] X. Liu, S. Huang, D.S. Cvetković-Ilić, Mixed-type reverse-order laws for $\{1,3,4\}$-generalized inverses over Hilbert spaces, Applied Math. Comput. 218 (2012), 8570-8577.
[18] X. Liu, S. Wu and D.S. Cvetković-Ilić, New results on reverse order law for $\{1,2,3\}$- and $\{1,2,4\}$-inverses of bounded operators, Math. Comp. 82 (2013), 1597-1607.
[19] A.E. Taylor, D.C. Lay, Introduction to Functional Analysis, second ed., John Wiley & Sons, New York, Chichester, Brisbane, Toronto, 1980.
[20] Y. Tian, On mixed-type reverse-order laws for a matrix product, Int. J. Math. Sci. 58 (2004), 3103-3116.
[21] J. Wang, H. Zhang, G. Ji, A generalized reverse order law for the products of two operators, J. Shaanxi Normal Univ. 38(4) (2010), 13-17.
[22] M. Wang, M. Wei, Z. Jia, Mixed-type reverse-order law of $(AB)^{13}$, Linear Algebra Appl. 430 (2009), 1691-1699.
[23] L. Wang, J. Chen, Mixed-type reverse-order laws of $(AB)^{(1,3)}$, $(AB)^{(1,2,3)}$ and $(AB)^{(1,3,4)}$, Applied Math. Comput. 222 (2013), 42-52.
[24] M. Wei, Reverse order laws for generalized inverses of multiple matrix products, Linear Algebra App. 293 (1999), 273-288.
[25] M. Wei, W. Guo, Reverse order laws for least squares g-inveses and minimum norm g-inveses of products of two matrices, Linear Algebra Appl. 342 (2002), 117-132.
[26] Z. Xiong, B. Zheng, The reverse order laws for $\{1,2,3\}$- and $\{1,2,4\}$-inverses of a two matrix product, Appl. Math. Lett. 21 (2008) 649-655.
[27] Z. Xiong, The Mixed-type reverse order laws for generalized inverses of the product of two matrices, Filomat, 27(5) (2013), 937-947.
[28] H. Yang, X. Liu, Mixed-type reverse-order laws $(AB)^{(1,2,3)}$ and $(AB)^{(1,2,4)}$, Applied Math. Comput. 217 (2011), 10361-10367.
[29] H.Y. Zhang, C.Y. Deng, Mixed-type reverse order laws associated to $\{1,3,4\}$-inverse, submit.
[30] H.Y. Zhang, F.L. Lu, Mixed-Type Reverse Order Laws for Generalized Inverses over Hilbert Space, Applied Math. 8 (2017), 637-644.